Multidimensional Rational Covariance Extension with Approximate Covariance Matching

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Introduction

Inverse problems is a class of problems which informally can be described as seeking the cause of a given effect. For physical systems with a known 'action' this amounts to finding a 'state' that produces the given measurements. In this work we consider the following stochastic system identification problem: *given a stochastic process find a linear stochastic system such that, when driven with white noise, the system has the stochastic process as output.*

Linear stochastic systems

Stochastic processes

In this work we consider discrete time stochastic process $y_t \in \mathbb{C}$, $t \in \mathbb{Z}$, that are

- zero mean: $\mathbb{E}(y_t) = 0$,
- ergodic
- second-order stationary, with covariances

$$c_k = \mathbb{E}(y_t y_{t-k}^*), \quad k \in \mathbb{Z} \quad (\text{note that } c_{-k} = c_k^*).$$

The *power spectrum* of $(y_t)_{t\in\mathbb{Z}}$, which describes the frequency content of the signal, is the positive function $\Phi(e^{i\theta})$ on $(-\pi, \pi] \sim \mathbb{T}$, such that the covariances of y_t are its Fourier coefficients:

$$c_k := \int_{\mathbb{T}} e^{ik\theta}\Phi(e^{i\theta})\frac{d\theta}{2\pi},\ k \in \mathbb{Z} \qquad \Longleftrightarrow \qquad \Phi(e^{i\theta}) \sim \sum_{k=-\infty}^{\infty} c_k e^{-ik\theta}.$$

Linear dynamical systems

Let y_t is produced by passing a white noise process u_t through a linear system $W(z)$.

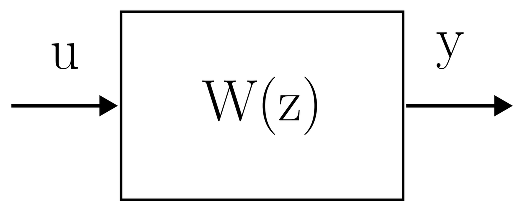

Figure 1 : A linear stochastic system.

Finite-dimensional linear system $\Rightarrow$ W is a rational transfer function:

$$W(z) = \frac{\sum_{k=0}^{n} b_k z^k}{\sum_{k=0}^{n} a_k z^k} = \frac{b(z)}{a(z)}.$$

In steady state, y_t is second-order stationary and with spectral density

$$\Phi(e^{i\theta}) = |W(e^{i\theta})|^2 = \frac{|b(e^{i\theta})|^2}{|a(e^{i\theta})|^2} = \frac{P(e^{i\theta})}{Q(e^{i\theta})},$$

P, Q trigonometric polynomials $P(e^{i\theta}) = \sum_{-n}^{n} p_k e^{-ik\theta}$.

Given nonnegative trigonometric polynomials P and Q, the factors b and a can be obtained by spectral factorization. Therefore, *given a rational spectrum we can identify a corresponding system.*

Rational covariance extension problem [1, 2]

Given covariances $c = (c_{-n}, \dots, c_0, c_1, \dots, c_n)$ find all positive functions $\Phi(e^{i\theta})$ so that

$$\begin{cases} c_k := \int_{\mathbb{T}} e^{ik\theta}\Phi(e^{i\theta})\frac{d\theta}{2\pi}, & k = -n, \dots, 0, 1, \dots, n, \\ \Phi(e^{i\theta}) = \frac{P(e^{i\theta})}{Q(e^{i\theta})}, & P \text{ and } Q \text{ nonnegative trigonometric polynomials of degree } \le n. \end{cases}$$

Multidimensional problem

The problem is now extended into d dimensions. This is done as follows:

- Indices k are changed to multi-indices $\boldsymbol{k} := (k_1, \dots, k_d)$.
  They belong to a grid $\Lambda \subset \mathbb{Z}^d$ such that:
  - Λ contains the origin: $\boldsymbol{0} \in \Lambda$,
  - Λ is symmetric: $-\Lambda = \Lambda$.

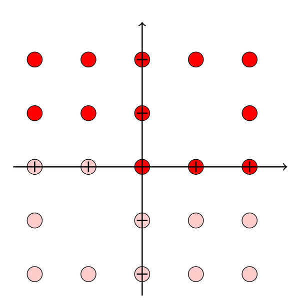

Figure 2 : Example of a two-dimensional gird Λ.

- The trigonometric polynomials are defined based on the grid Λ:

$$P(e^{i\boldsymbol{\theta}}) = \sum_{\boldsymbol{k}\in\Lambda} p_{\boldsymbol{k}} e^{-i(\boldsymbol{k},\boldsymbol{\theta})} = \sum_{\boldsymbol{k}\in\Lambda} p_{\boldsymbol{k}} e^{-i(k_1\theta_1 + \dots + k_d\theta_d)},\ p_{-\boldsymbol{k}} = p_{\boldsymbol{k}}^*.$$

To solve the problem, we enlarge the class of spectra from nonnegative functions to nonnegative measures. The problem can now be written: *given a set of complex numbers* $c = (c_{\boldsymbol{k}})_{\boldsymbol{k}\in\Lambda}$, *find all nonnegative* $d\mu$ *on* $\mathbb{T}^d$ *such that*

$$\begin{cases} c_{\boldsymbol{k}} = \int_{\mathbb{T}^d} e^{i(\boldsymbol{k},\boldsymbol{\theta})} d\mu(\boldsymbol{\theta}), & \text{for all } \boldsymbol{k} \in \Lambda \\ d\mu(\boldsymbol{\theta}) = \Phi(e^{i\boldsymbol{\theta}})dm(\boldsymbol{\theta}),\ \text{where } \Phi(e^{i\boldsymbol{\theta}}) = \frac{P(e^{i\boldsymbol{\theta}})}{Q(e^{i\boldsymbol{\theta}})}, & P \text{ and } Q \text{ are nonnegative trigonometric polynomials.} \end{cases}$$

Here, $dm(\boldsymbol{\theta}) := (1/2\pi)^d \prod_{j=1}^{d} d\theta_j$ is the (normalized) Lebesgue measure.

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Solution with exact matching

The multidimensional rational covariance extension problem can be solved by considering the optimization problem

$$(P) \qquad \min_{d\mu \ge 0} \int_{\mathbb{T}^d} \left(P \log \frac{P}{\Phi} dm + d\mu - P dm \right)$$
$$\text{subject to} \quad c_{\boldsymbol{k}} = \int_{\mathbb{T}^d} e^{i(\boldsymbol{k},\boldsymbol{\theta})} d\mu(\boldsymbol{\theta}), \quad \boldsymbol{k} \in \Lambda.$$

Introducing the notation

$$\bar{\mathfrak{P}}_+ := \{p \in \mathbb{C}^{|\Lambda|} \mid P(e^{i\boldsymbol{\theta}}) \ge 0,\ \forall \boldsymbol{\theta} \in \mathbb{T}^d\}$$
$$\mathfrak{C}_+ := \{c \in \mathbb{C}^{|\Lambda|} \mid c_{-\boldsymbol{k}} = \bar{c}_{\boldsymbol{k}},\ \sum_{\boldsymbol{k}\in\Lambda} c_{\boldsymbol{k}}\bar{p}_{\boldsymbol{k}} > 0,\ \forall p \in \bar{\mathfrak{P}}_+ \setminus \{0\}\},$$

the result can be stated as follows.

Theorem 1 **([3])**

Problem (P) has a solution if and only if $c \in \mathfrak{C}_+$. Moreover, for every $c \in \mathfrak{C}_+$, given a trigonometric polynomial P the solution to (P) is given by

$$d\mu(\boldsymbol{\theta}) = \frac{P(e^{i\boldsymbol{\theta}})}{\hat{Q}(e^{i\boldsymbol{\theta}})} dm(\boldsymbol{\theta}) + d\hat{\mu}(\boldsymbol{\theta}),$$

where $\hat{Q}$ is the unique solution to the dual problem

$$(D) \qquad \min_{q\in\bar{\mathfrak{P}}_+} \langle c, q\rangle - \int_{\mathbb{T}^d} P \log(Q) dm,$$

and $d\hat{\mu}$ is a singular measure (containing for example spectral lines) with $\text{supp}(d\hat{\mu}) \subseteq \{\boldsymbol{\theta} \in \mathbb{T}^d \mid \hat{Q}(e^{i\boldsymbol{\theta}}) = 0\}$. Moreover, if $d \le 2$ and if $P(e^{i\boldsymbol{\theta}}) > 0$, then $d\hat{\mu} \equiv 0$.

Corollary 1 **([3])**

Any $d\mu = (P/Q)dm$ corresponds to a $c \in \mathfrak{C}_+$, and for $c \in \mathfrak{C}_+$ any $d\mu = (P/Q)dm$ that matches c can be obtained by solving (P) and (D).

Solution with approximate matching

But how to check in an efficient way if $c \in \mathfrak{C}_+$? And what if $c \notin \mathfrak{C}_+$? In this case one can consider a solution with approximate covariance matching. This can be done by considering the following optimization problem.

$$(P') \qquad \min_{d\mu\ge 0,\, \tilde{c}} \int_{\mathbb{T}^d} \left(P \log \frac{P}{\Phi} dm + d\mu - P dm \right)$$
$$\text{subject to} \quad \tilde{c}_{\boldsymbol{k}} = \int_{\mathbb{T}^d} e^{i(\boldsymbol{k},\boldsymbol{\theta})} d\mu(\boldsymbol{\theta}), \quad \boldsymbol{k} \in \Lambda,$$
$$\|\tilde{c} - c\|^2 \le \varepsilon^2.$$

Theorem 1 **([4])**

Given any complex sequence c, for ε large enough the primal problem (P') has an optimal solution given by

$$d\mu(\boldsymbol{\theta}) = \frac{P(e^{i\boldsymbol{\theta}})}{\hat{Q}(e^{i\boldsymbol{\theta}})} dm(\boldsymbol{\theta}) + d\hat{\mu}(\boldsymbol{\theta}),$$

where $\hat{Q}$ is the unique solution to the dual problem

$$(D') \qquad \min_{q\in\bar{\mathfrak{P}}_+} \langle c, q\rangle - \int_{\mathbb{T}^d} P \log(Q) dm + \varepsilon\|q - e\|,$$

where $e \in \mathbb{C}^{|\Lambda|}$, $e_{\boldsymbol{0}} = 1$ and $e_{\boldsymbol{k}} = 0$ for $\boldsymbol{k} \in \Lambda \setminus \{\boldsymbol{0}\}$, and $d\hat{\mu}$ is a singular measure with $\text{supp}(d\hat{\mu}) \subseteq \{\boldsymbol{\theta} \in \mathbb{T}^d \mid \hat{Q}(e^{i\boldsymbol{\theta}}) = 0\}$.

Open questions

In one dimension, spectral factorization as a sum-of-one-square $P(e^{i\theta})/Q(e^{i\theta}) = |b(e^{i\theta})|^2/|a(e^{i\theta})|^2$ is always possible. However this is not true in the multidimensional case. Only factorization as sum-of-several squares can be guaranteed:

$$\frac{P(e^{i\boldsymbol{\theta}})}{Q(e^{i\boldsymbol{\theta}})} = \frac{\sum_{k=1}^{\ell} |b_k(e^{i\boldsymbol{\theta}})|^2}{\sum_{k=1}^{m} |a_k(e^{i\theta})|^2}.$$

- How to construct a realization from such a spectrum?
- Is it possible to characterize P for which Q is a sum-of-one-square?

Acknowledgements and references

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[1] C.I. Byrnes, A. Lindquist, S.V. Gusev, and A.S. Matveev. A complete parameterization of all positive rational extensions of a covariance sequence. *IEEE Transactions on Automatic Control*, 40(11):1841–1857, 1995.

[2] A. Lindquist and G. Picci. *Linear Stochastic Systems: A Geometric Approach to Modeling, Estimation and Identification*, volume 1 of *Series in Contemporary Mathematics*. Springer-Verlag Berlin Heidelberg, 2015.

[3] A. Ringh, J. Karlsson, and A. Lindquist. Multidimensional rational covariance extension with applications to spectral estimation and image compression. *Accepted to SIAM Journal on Control and Optimization. Preprint: arXiv:1507.01430*, 2015.

[4] A. Ringh, J. Karlsson, and A. Lindquist. Multidimensional rational covariance extension with approximate covariance matching. In *Proceedings of the 22nd International Symposium on Mathematical Theory of Networks and Systems*, pages 457–460, 2016.