Lower bounds on the maximum delay margin by analytic interpolation

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19th of December 2018
CDC 2018, Miami Beach

Outline

- Introduction
  - The maximum delay margin problem
  - Upper and lower bounds for the maximum delay margin
- Lower bound using analytic interpolation
- An improved algorithm
- Numerical experiment

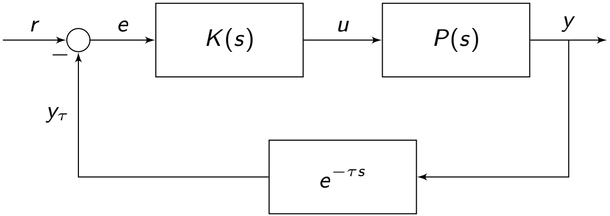
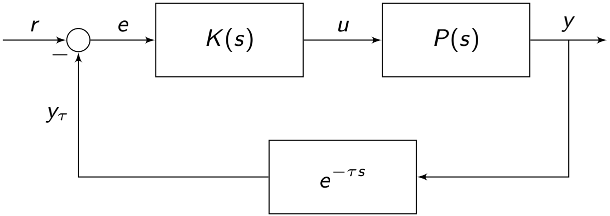


Figure: Block diagram representation of an LTI SISO system with time delay.

Notation:

- $P(s)$ is a (unstable) causal linear time-invariant (LTI) single-input-single-output (SISO) system,
- $K(s)$ is a causal LTI SISO controller,
- $e^{-\tau s}$ is a time delay of length τ.



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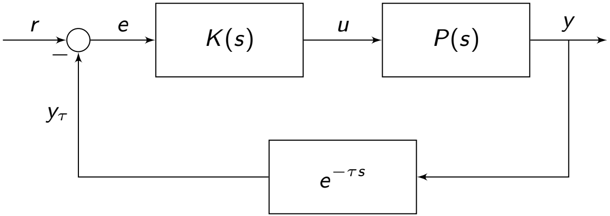

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For general systems, computing $\tau_{\max}$ is an unsolved problem.

Upper bounds

- If P has a real unstable pole p, then $\tau_{\max} \leq 2/p$ [1]. First paper showing an upper bound.
  - Tight if p is the only unstable pole, and if there are no nonminimum phase zeros.
  - Also extended to tight bound for a few other cases.

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- Problem can be cast in a robust control framework [4, 5].

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- Analyzed using analytic interpolation and rational approximation [7]. We build on this approach.

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Lower bound using analytic interpolation

Tools and notation:

- $\mathcal{H}_\infty$ is the space of bounded analytic functions on $\mathbb{C}_+$,
  $\rightsquigarrow$ "stable transfer functions".
- $T(s)$ is the complementary sensitivity function of the system without delay

$$T(s) := \frac{P(s)K(s)}{1 + P(s)K(s)}.$$

- a system is called well-posed if $1 + P(s)K(s) \neq 0$ for all $s \in \bar{\mathbb{C}}_+ := \{s \in \mathbb{C} \mid s = a + ib,\ a \geq 0\}$ [1], i.e., no poles in closed right half-plane.
- Internally stable if, in addition, there is no pole-zero cancellation between K and P in $\bar{\mathbb{C}}_+$ [1].

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The condition on stability can equivalently be reformulated as follows [2]:

- let $p_1, \ldots, p_n$ be the unstable poles and $z_1, \ldots, z_m$ the nonminimum phase zeros of P.
- then necessary and sufficient coinditions for stability is that $T \in \mathcal{H}_\infty$ and

$$T(p_j) = 1, \quad j = 1, \ldots, n,$$
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This is a NevanlinnaPick interpolation problem.

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Lower bound using analytic interpolation

In [1], the authors note that the system is well-posed if

$$1 + P(s)K(s)e^{-\tau s} \neq 0, \quad \text{for all } s \in \bar{\mathbb{C}}_+$$

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$$1 + P(s)K(s)e^{-\tau s} \neq 0, \quad \text{for all } s \in \bar{\mathbb{C}}_+ \qquad \Longleftrightarrow \qquad 1 + T(s)(e^{-\tau s} - 1) \neq 0, \quad \text{for all } s \in \bar{\mathbb{C}}_+.$$

since K must stabilize P without delay.

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Together with $T \in \mathcal{H}_\infty$ and

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these are necessary and sufficient conditions for stability of the time-delay system.

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This is equivalent to

$$\sup_{\tau \in [0, \bar{\tau}]} \inf_{T \in \mathcal{H}_\infty} \| T(i\omega)(e^{-\tau i\omega} - 1) \|_{L_\infty} < 1.$$

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In this case one can show that sup inf = inf sup, so the condition holds whenever

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where

$$\phi_{\bar{\tau}}(\omega) = \sup_{\tau \in [0,\bar{\tau}]} |e^{-i\tau\omega} - 1| = \sup_{\tau \in [0,\bar{\tau}]} 2 \left|\sin(\frac{\bar{\tau}\omega}{2})\right| = \begin{cases} 2\left|\sin(\frac{\bar{\tau}\omega}{2})\right| & \text{for } |\omega\bar{\tau}| \le \pi \\ 2 & \text{for } |\omega\bar{\tau}| > \pi. \end{cases}$$

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In [1] they now use a rational approximation of $\phi_{\bar{\tau}}(\omega)$, and derive a method for finding the largest $\bar{\tau}$ so that the above condition holds.

Instead, we tackle the above problem directly.

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Method for checking solvability of the above problem:

- replace $\phi_{\bar{\tau}}$ by the outer function $W_{\bar{\tau}}(s) \in \mathcal{H}_\infty$ given by

$$W_{\bar{\tau}}(s) = \exp\left[\frac{1}{\pi}\int_{-\infty}^{\infty} \log\left(\phi_{\bar{\tau}}(\omega)\right) \frac{\omega s + i}{\omega + is} \frac{1}{1+\omega^2}\, d\omega\right],$$

Outer function ≈ generalization of finite-dimensional stable minimum phase transfer function.

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- Introduce $\tilde{T} = TW_{\bar{\tau}}$, which gives

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  This is a Nevanlinna-Pick interpolation problem! $W_{\bar{\tau}}$ is outer, so we get $T = \tilde{T} W_{\bar{\tau}}^{-1}$.

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- The Nevanlinna-Pick interpolation problem has a solution if and only if the Pick matrix

$$\text{Pick}(v, w) := \left[\frac{1 - w_j \bar{w}_k}{v_j + \bar{v}_k} \right]_{j,k=1}^{n+m} \succeq 0,$$

  where $v := [p_1, \dots, p_n, z_1, \dots, z_m]$ and $w := [W_{\bar{\tau}}(p_1), \dots, W_{\bar{\tau}}(p_n), 0, \dots, 0]$.

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- Introduce $\tilde{T} = TW_{\bar{\tau}}$, which gives

$$\|\tilde{T}\|_{\mathcal{H}_\infty} < 1 \quad \text{and} \quad \begin{cases} \tilde{T}(p_j) = W_{\bar{\tau}}(p_j), & j = 1, \dots, n, \\ \tilde{T}(z_j) = 0, & j = 1, \dots, m, \end{cases}$$

  This is a Nevanlinna-Pick interpolation problem! $W_{\bar{\tau}}$ is outer, so we get $T = \tilde{T} W_{\bar{\tau}}^{-1}$.

- The Nevanlinna-Pick interpolation problem has a solution if and only if the Pick matrix

$$\mathrm{Pick}(v, w) := \left[\frac{1 - w_j \bar{w}_k}{v_j + \bar{v}_k} \right]_{j,k=1}^{n+m} \succeq 0,$$

  where $v := [p_1, \dots, p_n, z_1, \dots, z_m]$ and $w := [W_{\bar{\tau}}(p_1), \dots, W_{\bar{\tau}}(p_n), 0, \dots, 0]$.

Method for computing maximum value of $\bar{\tau}$: bisection algorithm checking positive semi-definiteness of the Pick matrix.

An improved algorithm

Reconsider the sufficient condition

$$\inf_{\substack{T \in \mathcal{H}_\infty \\ T(p_j)=1 \\ T(z_j)=0}} \| T(i\omega)\phi_{\mp}(\omega)\|_{L_\infty} < 1.$$

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$$\| T(i\omega)\phi_{\mp}(\omega)\|_{L_\infty} < 1 \quad \Longleftrightarrow \quad | T(i\omega)\phi_{\mp}(\omega)| < 1,\ \forall\, \omega \in \mathbb{R} \quad \Longleftrightarrow \quad | T(i\omega)| < \frac{1}{|\phi_{\mp}(\omega)|},\ \forall\, \omega \in \mathbb{R}.$$

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Equivalently expressed: $T(i\omega)$ can take values in a circle centered at the origin with radius $1/|\phi_{\bar{\tau}}(\omega)|$.

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Equivalently expressed: $T(i\omega)$ can take values in a circle centered at the origin with radius $1/|\phi_{\bar{\tau}}(\omega)|$.

Questions:

- Can we center the circle in some other point?
- How would that be done?

An improved algorithm

Go back to the necessary and sufficient conditions for stability :

$$T(s)\big(e^{-\tau s} - 1\big) \neq -1,\ \forall\, s \in \bar{\mathbb{C}}_+ \qquad \text{and} \qquad \begin{cases} T(p_j) = 1, & j = 1, \dots, n, \\ T(z_j) = 0, & j = 1, \dots, m, \end{cases}$$

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Let $T = \hat{T} + w_0$ where $w_0 \in \mathbb{C}$. This w_0 should be seen as a parameter the we control and can vary. Then the conditions for stability can be rewritten as

$$\hat{T}(s)\left(e^{-\tau s}-1\right) \neq -1 + w_0 - w_0 e^{-\tau s},\ \forall\, s \in \bar{\mathbb{C}}_+ \qquad \text{and} \qquad \begin{cases} \hat{T}(p_j)=1-w_0, & j=1,\dots,n,\\ \hat{T}(z_j)=-w_0, & j=1,\dots,m.\end{cases}$$

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$-1 + w_0 - w_0 e^{-\tau s} \in \mathcal{H}_\infty$ and it can be shown that it is $\neq 0$ in all of $\bar{\mathbb{C}}_+$ if and only if $\Re(w_0) < 1/2$.

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$$\hat{T}(s)\frac{e^{-\tau s}-1}{1-w_0+w_0 e^{-\tau s}} \neq -1, \quad \forall\, s \in \bar{\mathbb{C}}_+.$$

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We can repeat the previous arguments, but where $\phi_{\bar{\tau}}(\omega)$ is replaced by

$$\phi_{\bar{\tau}}(\omega) := \sup_{\tau \in [0,\bar{\tau}]} \left| \frac{e^{-\tau i\omega}-1}{1 - w_0 + w_0 e^{-\tau i\omega}} \right|$$

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$$\phi_{\bar\tau}(\omega) := \sup_{\tau\in[0,\bar\tau]}\left|\frac{e^{-\tau i\omega}-1}{1-w_0+w_0e^{-\tau i\omega}}\right| = \begin{cases}(0.5-\Re(w_0))^{-1}, & \omega \geq \bar\omega_+,\\ (|0.5-i0.5\cot(\omega\bar\tau/2)-w_0|)^{-1}, & \bar\omega_+ > \omega > \bar\omega_-,\\ (0.5-\Re(w_0))^{-1}, & \omega \leq \bar\omega_-,\end{cases}$$ form computable $\bar\omega_\pm$.

An improved algorithm

Summary of improved method:

- Let

$$\phi_{\bar{\tau}}(\omega) := \sup_{\tau \in [0,\bar{\tau}]} \left| \frac{e^{-\tau i \omega} - 1}{1 - w_0 + w_0 e^{-\tau i \omega}} \right|$$

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- Let $W_{\bar\tau}$ be the outer function corresponding to $\phi_{\bar\tau}$, and let $\tilde{T} = \hat{T} W_{\bar\tau}$. The corresponding Nevanlinna-Pick interpolation problem becomes
$$\|\tilde{T}\|_{\mathcal{H}_\infty} < 1 \quad \text{and} \quad \begin{cases} \tilde{T}(p_j) = (1 - w_0) W_{\bar\tau}(p_j), & j = 1, \ldots, n, \\ \tilde{T}(z_j) = -w_0 W_{\bar\tau}(z_j), & j = 1, \ldots, m. \end{cases}$$

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- Compute maximum value of $\bar\tau$ using the bisection algorithm, checking positive semi-definiteness of the Pick matrix.

Numerical experiment

We investigate the performance of the method on a few examples from [1].

The first system is

$$P(s) = \frac{s - z}{s - p},$$

where $z = 2$ and $p > 0$. We estimate the maximum delay margin for different values of p in $[0.3, 4]$.

[1] T. Qi, J. Zhu, & J. Chen. Fundamental limits on uncertain delays: When is a delay system stabilizable by LTI controllers? *IEEE Transactions on Automatic Control*, 62(3):1314–1328, 2017.

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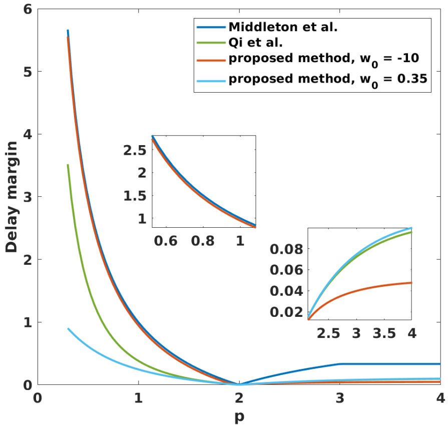

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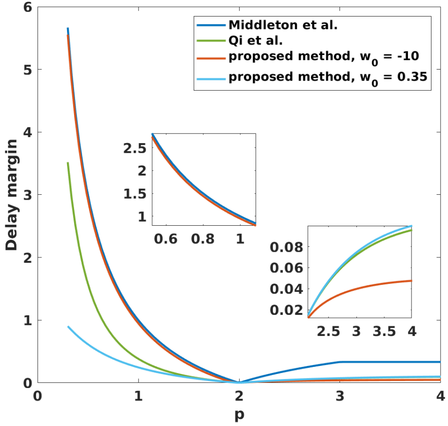
Observations:

- different w_0 gives different performance in different regions of $[0.3, 4]$,
- for $p < 2$, with $w_0 = -10$ we get close to the upper bound from [2], which is tight in this region.



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[2] R.H. Middleton & D.E. Miller. On the achievable delay margin using LTI control for unstable plants. *IEEE Transactions on Automatic Control*, 52(7):1194–1207, 2007.

Numerical experiment

We also consider the system
$$P(s) = \frac{s - z}{(s - re^{i\theta})(s - re^{-i\theta})}.$$

- we compute an estimate of the delay margin for the pairs $(r, \theta) = (1, \pi/4)$, $(1, \pi/3)$, and $(2, \pi/3)$.
- For these values of (r, θ) we vary z in $[0.01, 4]$ and for each value of z we investigate which $w_0 \in [-1.5, 0.5)$ that maximizes the estimated maximum delay margin.

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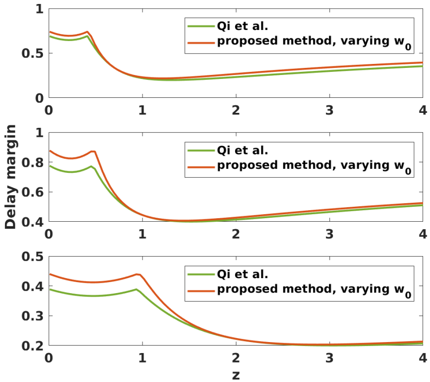

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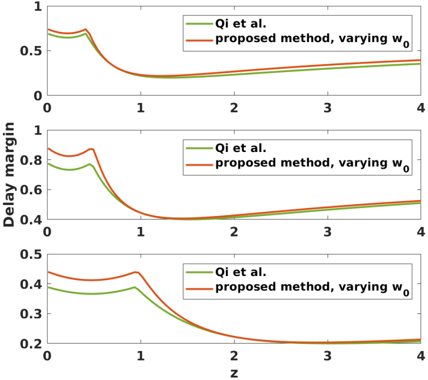

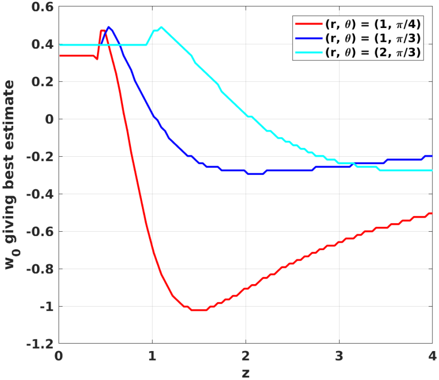

Conclusion and future work

Conclusions:

- Method for computing a lower bound on the maximum delay margin, based on Nevanlinna-Pick interpolation and the bisection algorithm.
- Improved method by introducing a constant shift w_0 in the problem.

Ongoing work:

- Interpretation of the method and of the w_0-shift from a robust control perspective.
  $\rightsquigarrow$ Improved estimation procedure for lower bounds?
- Understanding the maximum delay margin problem from a “Nyquist”-perspective.
  $\rightsquigarrow$ Better understanding of relation to gain and phase margin.
- Control design for time delay robustness, also incorporating gain and phase margin considerations.

Thank you for your attention!

Questions?