Multidimensional rational covariance extension with applications to Wiener system identification

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Outline

- Introduction
  - Motivation - texture generation
  - Wiener system identification
- Multidimensional rational covariance extension problem
  - Problem derivation
  - Solution via convex optimization
- Examples in texture generation

- Motivated by the use of thresholded Gaussian random fields to model porous materials [1], we are interested in generating binary textures.

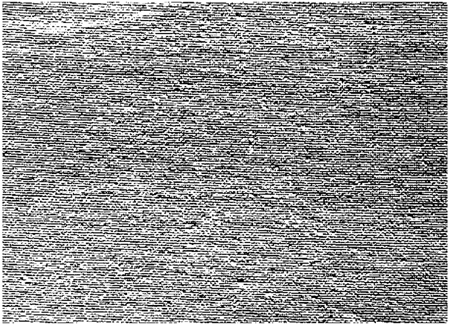

Figure: Example of a texture

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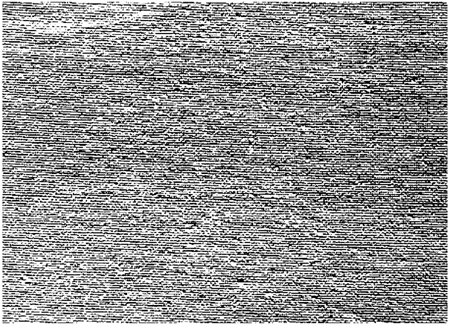
- Motivated by the use of thresholded Gaussian random fields to model porous materials [1], we are interested in generating binary textures.
- We want to estimate a system that can generate similar textures
  ⇝ multidimensional Wiener systems identification.



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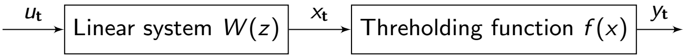
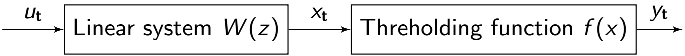


Figure: A Wiener system with thresholding as static nonlinearity.



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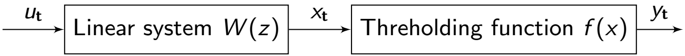
- $\{u_{\mathbf{t}};\ \mathbf{t} \in \mathbb{Z}^d\}$ be a zero-mean Gaussian white noise input.



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- $\{u_{\mathbf{t}};\ \mathbf{t} \in \mathbb{Z}^d\}$ be a zero-mean Gaussian white noise input.
- The linear dynamical system is a strictly causal *autoregressive-moving-average* (ARMA) filter

$$x_{\mathbf{t}} + \sum_{\mathbf{k} \in \Lambda_+ \setminus \{\mathbf{0}\}} a_{\mathbf{k}}\, x_{\mathbf{t}-\mathbf{k}} = \sum_{\mathbf{k} \in \Lambda_+} b_{\mathbf{k}}\, u_{\mathbf{t}-\mathbf{k}} \Leftrightarrow W(\mathbf{z}) = \frac{\sum_{\mathbf{k} \in \Lambda_+} b_{\mathbf{k}}\, \mathbf{z}^{\mathbf{k}}}{\sum_{\mathbf{k} \in \Lambda_+} a_{\mathbf{k}}\, \mathbf{z}^{\mathbf{k}}} = \frac{b(\mathbf{z})}{a(\mathbf{z})}$$

where $\Lambda_+ \subset \mathbb{Z}^2$ is the support of the filter.

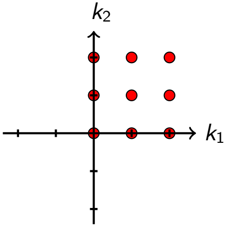
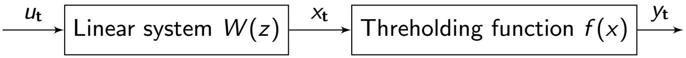


Figure: Example of Λ_+.



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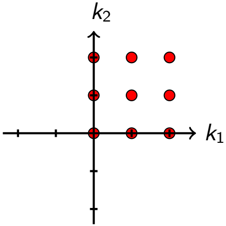
  where $\Lambda_+ \subset \mathbb{Z}^2$ is the support of the filter.
- Thresholding function

$$f(x) = \begin{cases} 1 & x > \tau \\ 0 & \text{otherwise} \end{cases}$$



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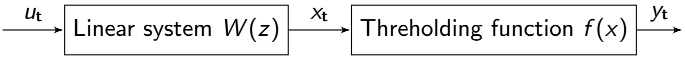

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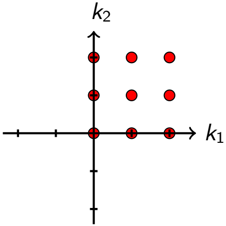

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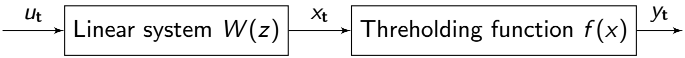
Goal: From samples $(y_{\mathbf{t}})$ we want to identify τ and W.



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Introduction
Wiener system identification

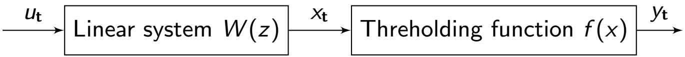
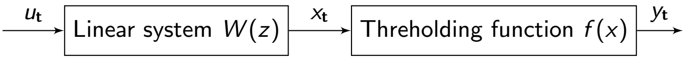


Figure: A Wiener system with thresholding as static nonlinearity.

- Identifying the threshold parameter:
  - Since u_t is zero-mean and Gaussian and $W(\mathbf{z})$ is linear, x_t is zero-mean and Gaussian.
  - $\mathbb{E}[y_t] = P(y_t = 1) = P(x_t > \tau) = 1 - P(x_t \leq \tau) = 1 - \phi(\tau)$, where ϕ is the Gaussian CDF
    $\rightsquigarrow$ we can estimate τ as $\tau_{\text{est}} = \phi^{-1}(1 - \mathbb{E}[y_t])$.



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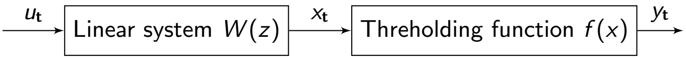
- In steady-state x_t is second order stationary process, i.e., the covariances are independent of the absolute time $\mathbf{t}$: $c_k := \mathbb{E}[x_t x_{t-k}]$. Assume that $c_0 = 1$ (normalization). Estimate covariances c_k:



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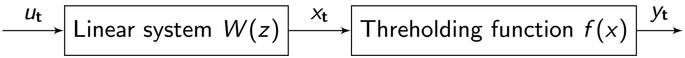
  - Let $r_k := \mathbb{E}[y_{t-k} y_t] - \mathbb{E}[y_{t-k}]\mathbb{E}[y_t]$ be the covariances of the process y_t.
  - Since x_t is Gaussian a theorem by Price [1] gives the following relationship between the covariances

$$r_k = \int_0^{c_k} \frac{1}{2\pi\sqrt{1 - s^2}} \exp\left(-\frac{\tau^2}{1+s}\right) ds.$$

    Since the integrand is positive, the mapping can be inverted (numerically).

[1] R. Price. A useful theorem for nonlinear devices having Gaussian inputs. *IRE Transactions on Information Theory*, 4(2), 69-72, 1958.



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- From the covariances c_k, estimate the linear system $W(\mathbf{z})$.

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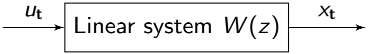
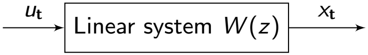


Figure: A linear system.

From covariance data $\{c_{\mathbf{k}}\}_{\mathbf{k}\in\Lambda}$ we want to estimate a linear system $W(\mathbf{z})$.

- The power spectral density $\Phi(e^{i\boldsymbol{\theta}})$ of a stochastic process $\{x_{\mathbf{t}};\ \mathbf{t}\in\mathbb{Z}^d\}$ is defined as the nonnegative function such that

$$c_{\mathbf{k}} := \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} e^{i(\mathbf{k},\boldsymbol{\theta})}\Phi(e^{i\boldsymbol{\theta}})d\boldsymbol{\theta}, \quad \mathbf{k}\in\mathbb{Z}^2 \qquad \Longleftrightarrow \qquad \Phi(e^{i\boldsymbol{\theta}}) = \sum_{\mathbf{k}\in\mathbb{Z}^2} c_{\mathbf{k}} e^{-i(\mathbf{k},\boldsymbol{\theta})}.$$



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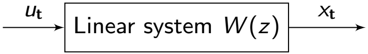
- Moreover,

$$\Phi(e^{i\boldsymbol{\theta}}) = |W(e^{i\boldsymbol{\theta}})|^2\Phi_u(e^{i\boldsymbol{\theta}}) = |W(e^{i\boldsymbol{\theta}})|^2 = \frac{|b(e^{i\boldsymbol{\theta}})|^2}{|a(e^{i\boldsymbol{\theta}})|^2} = \frac{\sum_{\mathbf{k}\in\Lambda} p_{\mathbf{k}} e^{-i(\mathbf{k},\boldsymbol{\theta})}}{\sum_{\mathbf{k}\in\Lambda} q_{\mathbf{k}} e^{-i(\mathbf{k},\boldsymbol{\theta})}} = \frac{P(e^{i\boldsymbol{\theta}})}{Q(e^{i\boldsymbol{\theta}})}.$$

where P and Q are trigonometric polynomials, and $\Lambda = \Lambda_+ - \Lambda_+$ (Minkowski set difference).

Multidimensional rational covariance extension
Derivation of the problem



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- Summarizing, this gives us the following problem:

Problem formulation – Approximate multidimensional rational covariance extension problem

Given a sequence of covariances $c = (c_{\mathbf{k}})_{\mathbf{k}\in\Lambda}$ find a positive function $\Phi(e^{i\boldsymbol{\theta}})$ so that

$$\begin{cases} c_{\mathbf{k}} \approx \dfrac{1}{(2\pi)^2}\displaystyle\int_{\mathbb{T}^2} e^{i(\mathbf{k},\boldsymbol{\theta})}\Phi(e^{i\boldsymbol{\theta}})d\boldsymbol{\theta}, & \mathbf{k}\in\Lambda \\ \Phi(e^{i\boldsymbol{\theta}}) = \dfrac{P(e^{i\boldsymbol{\theta}})}{Q(e^{i\boldsymbol{\theta}})}, & P \text{ and } Q \in \bar{\mathfrak{P}}_+. \end{cases}$$

- Notation for nonnegative trigonometric polynomials:

$$\bar{\mathfrak{P}}_+ = \{p := (p_{\mathbf{k}})_{\mathbf{k}\in\Lambda} \in \mathbb{C}^{|\Lambda|} \mid p_{-\mathbf{k}} = p_{\mathbf{k}}^*,\ P(e^{i\boldsymbol{\theta}}) := \sum_{\mathbf{k}\in\Lambda} p_{\mathbf{k}} e^{-i(\mathbf{k},\boldsymbol{\theta})},\ P(e^{i\boldsymbol{\theta}}) \geq 0 \text{ for all } \boldsymbol{\theta}\in\mathbb{T}^2\}$$

- **Rational covariance extension:**
  - R. Kalman, 1981.
  - T.T. Georgiou, 1983.
  - T.T. Georgiou, 1987.
  - C.I. Byrnes, A. Lindquist, S.V. Gusev, and A.S. Matveev, 1995.
  - C.I. Byrnes, S.V. Gusev, and A. Lindquist, 1998.
  - C.I. Byrnes, T.T. Georgiou, and A. Lindquist, 2000.
  - C.I. Byrnes, P. Enqvist, and A. Lindquist, 2001.
  - P. Enqvist, 2004.
  - H.I. Nurdin, and A. Bagchi, 2006.
  - T.T. Georgiou, and A. Lindquist, 2008.
  - A. Ferrante, M. Pavon, and M. Zorzi, 2012.
  - M. Zorzi, 2014.
- **Periodic/Circulant problem:**
  - A. Lindquist and G. Picci, 2013.
  - A. Lindquist, C. Masiero, and G. Picci, 2013.
  - A. Ringh, and A. Lindquist, 2014.
  - G. Picci, and B. Zhu, 2017.
- **Matrix valued**
  - A. Blomqvist, A. Lindquist, R. Nagamune, 2003.
  - F. Ramponi, A. Ferrante, and M. Pavon, 2009.
  - M. Pavon, and A. Ferrante, 2013.
  - M. Zorzi, 2014.
  - B. Zhu, 2017.
  - B. Zhu, and G. Baggio, 2017.
- **Multidimensional problem:**
  - T.T. Georgiou, 2005.
  - T.T. Georgiou, 2006.
  - A. Ringh, J. Karlsson, and A. Lindquist, 2015.
  - A. Ringh, J. Karlsson, and A. Lindquist, 2016.
  - J. Karlsson, A. Lindquist, and A. Ringh, 2016.
  - A. Ringh, J. Karlsson, and A. Lindquist, 2017.

Notation: let $\mathfrak{C}_+$ be the interior of dual cone of $\bar{\mathfrak{P}}_+$:

$$\mathfrak{C}_+ := \{c \in \mathbb{C}^{|\Lambda|} \mid c_{-\mathbf{k}} = c_{\mathbf{k}}^*,\ \langle c, p \rangle := \sum_{\mathbf{k} \in \Lambda} c_{\mathbf{k}} p_{\mathbf{k}}^* > 0 \text{ for all } p \in \bar{\mathfrak{P}}_+ \setminus \{0\}\}$$

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Theorem

Given a sequence c, the primal problem

$$(P) \qquad \min_{\Phi>0} \int_{\mathbb{T}^2} \left(P \log \frac{P}{\Phi} + \Phi - P\right) \frac{d\boldsymbol{\theta}}{(2\pi)^2}$$

$$\text{subject to} \quad c_{\mathbf{k}} = \int_{\mathbb{T}^2} e^{i(\mathbf{k},\boldsymbol{\theta})} \Phi(e^{i\boldsymbol{\theta}}) \frac{d\boldsymbol{\theta}}{(2\pi)^2}, \quad \mathbf{k} \in \Lambda,$$

has a solution if and only if $c \in \mathfrak{C}_+$. Moreover, for every $c \in \mathfrak{C}_+$ and trigonometric polynomial $P > 0$ the solution to (P) is given by

$$\Phi(e^{i\boldsymbol{\theta}}) = \frac{P(e^{i\boldsymbol{\theta}})}{\hat{Q}(e^{i\boldsymbol{\theta}})}$$

where $\hat{Q}$ is the unique solution to the dual problem

$$(D) \qquad \min_{q\in\bar{\mathfrak{P}}_+} \langle c, q\rangle - \int_{\mathbb{T}^2} P \log(Q) \frac{d\boldsymbol{\theta}}{(2\pi)^2}.$$

[1] A. Ringh, J. Karlsson, and A. Lindquist. Multidimensional rational covariance extension with applications to spectral estimation and image compression. *SIAM Journal on Control and Optimization*, 54(4), 1950-1982, 2016.

Theorem

Given any sequence c, for ε large enough the primal problem

$$(P) \qquad \min_{\Phi>0,\,\tilde{c}} \quad \int_{\mathbb{T}^2} \left(P \log \frac{P}{\Phi} + \Phi - P \right) \frac{d\boldsymbol{\theta}}{(2\pi)^2}$$

$$\text{subject to} \quad \tilde{c}_{\mathbf{k}} = \int_{\mathbb{T}^2} e^{i(\mathbf{k},\boldsymbol{\theta})} \Phi(e^{i\boldsymbol{\theta}}) \frac{d\boldsymbol{\theta}}{(2\pi)^2}, \quad \mathbf{k} \in \Lambda,$$

$$\|\tilde{c} - c\|^2 \le \varepsilon^2,$$

has an optimal solution given by

$$\Phi(e^{i\boldsymbol{\theta}}) = \frac{P(e^{i\boldsymbol{\theta}})}{\hat{Q}(e^{i\boldsymbol{\theta}})},$$

where $\hat{Q}$ is the unique solution to the dual problem

$$(D) \qquad \min_{q \in \bar{\mathfrak{P}}_+} \quad \langle c, q \rangle - \int_{\mathbb{T}^2} P \log(Q) \frac{d\boldsymbol{\theta}}{(2\pi)^2} + \varepsilon \|q - e\|,$$

where $e \in \mathbb{C}^{|\Lambda|}$, $e_{\mathbf{0}} = 1$ and $e_{\mathbf{k}} = 0$ for $\mathbf{k} \in \Lambda \setminus \{\mathbf{0}\}$.

[1] A. Ringh, J. Karlsson, and A. Lindquist. Multidimensional rational covariance extension with approximate covariance matching. Submitted to *SIAM Journal on Control and Optimization*.

- How do we obtain the system from the spectrum?
- In one dimension, spectral factorization as a sum-of-one-square is always possible:

$$\frac{P(e^{i\theta})}{Q(e^{i\theta})} \underset{\text{Trivial}}{\overset{\text{Spectral factorization}}{\overset{\rightarrow}{\underset{\leftarrow}{=}}}} \frac{|b(e^{i\theta})|^2}{|a(e^{i\theta})|^2}$$

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- Not true in the two dimensions - only factorization as sum-of-several-squares can be guaranteed [1, 2]:

$$\frac{P(e^{i\boldsymbol{\theta}})}{Q(e^{i\boldsymbol{\theta}})} = \frac{\sum_{k=1}^{\ell} |b_k(e^{i\boldsymbol{\theta}})|^2}{\sum_{k=1}^{m} |a_k(e^{i\boldsymbol{\theta}})|^2}.$$

- **Open questions**:
  - How to construct a realization from such a spectrum?
  - Is it possible to characterize P for which Q is a sum-of-one-square?

[1] M.A. Dritschel. On factorization of trigonometric polynomials. *Integral Equations and Operator Theory*, 49(1), 11-42, 2004.

[2] J.S. Geronimo, and M.J. Lai. Factorization of multivariate positive Laurent polynomials. *Journal of Approximation Theory*, 139(1-2), 327-345, 2006.

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- We resort to a heuristic, obtained by “abusing” results in [3].

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[3] J.S. Geronimo, and H.J. Woerdeman. Positive extensions, Fejér-Riesz factorization and autoregressive filters in two variables. *Annals of Mathematics*, 839-906, 2004.

Algorithm for Wiener system identification

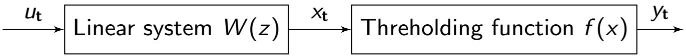


Figure: A Wiener system with thresholding as static nonlinearity.

Algorithm for Wiener system identification with thresholding

Input: (y_t)

1: Estimate threshold parameter: $\tau_{\text{est}} = \phi^{-1}(1 - E[y_t])$ from the data.

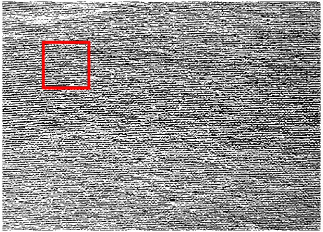
2: Estimate covariances: $r_k := E[y_{t-k}y_t] - E[y_{t-k}]E[y_t]$ from the data.

3: Compute covariances $c_k := E[x_{t-k}x_t]$ by using the relation $r_k = \int_0^{c_k} \frac{1}{2\pi\sqrt{1-s^2}} \exp\left(-\frac{\tau^2}{1+s}\right) ds$

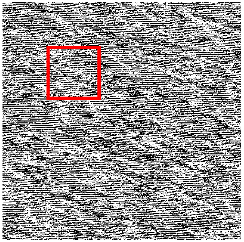
4: Estimate a rational spectrum using the theory developed here.

5: Apply a *heuristic*, approximate factorization procedure.

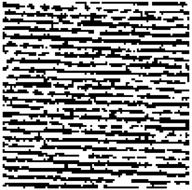
Output: τ_{est}, coefficients for the linear dynamical system



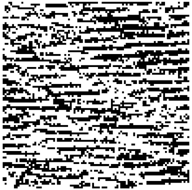
(a) Texture.



(b) Reconstruction.

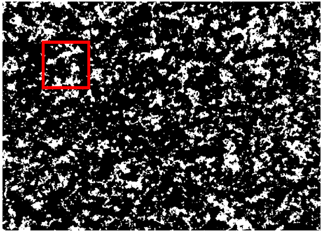


(c) Close-up of the texture.

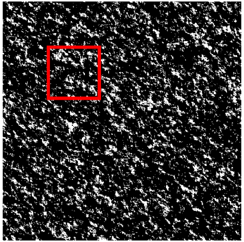


(d) Close-up of the reconstruction.

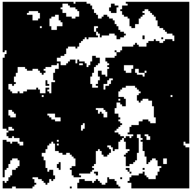
Example in texture generation



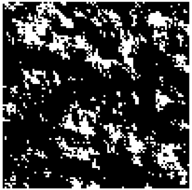
(e) Texture.



(f) Reconstruction.

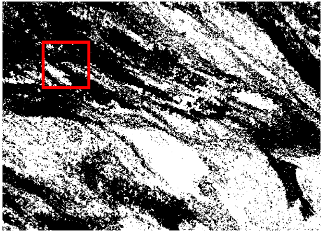


(g) Close-up of the texture.

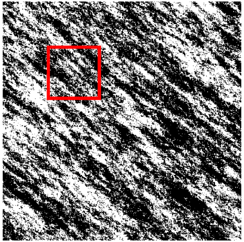


(h) Close-up of the reconstruction.

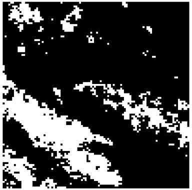
Example in texture generation



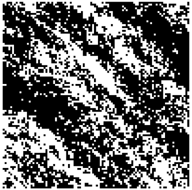
(i) Texture.



(j) Reconstruction.



(k) Close-up of the texture.



(l) Close-up of the reconstruction.

Conclusion

We can do estimation of rational multidimensional spectra.
But we need to better understand what they mean in terms of dynamical systems.

Thank you for your attention!

Questions?