Further results on multidimensional rational covariance extension with application to texture generation

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Abstract—The rational covariance extension problem is a moment problem with several important applications in systems and control as, for example, in identification, estimation, and signal analysis. Here we consider the multidimensional counterpart and present new results for the well-posedness of the problem. We apply the theory to texture generation by modeling the texture as the output of a Wiener system. The static nonlinearity in the Wiener system is assumed to be a thresholding function and we identify both the linear dynamical system and the thresholding parameter.

I. INTRODUCTION

Moment problems with rationality constraints are ubiquitous in the areas of systems, control and signal processing. One important example is the rational covariance extension problem. First posed by R.E. Kalman [32] in 1981, this problem can be stated as follows: Given a finite covariance sequence \( c := (c_0,\ldots,c_n) \), determine all infinite extensions \( c_{n+1},c_{n+2},\ldots \) such that

\[
\Phi(e^{i\theta}) = \sum_{k=-\infty}^{\infty} c_k e^{-ik\theta} \tag{1}
\]

is a positive rational function of degree bounded by \( 2n \). The reason for calling this a covariance extension problem is that a function \( \Phi \) of the form (1) can be regarded as the spectral density of a zero-mean, stationary stochastic process \( \{y_t: t \in \mathbb{Z}\} \) with covariance lags \( c_k = E[y_{t+k}y_t] \) [41, Sections 3.2-3.3], [50, Section 1.3]. Here \( \bar{c} \) denotes complex conjugation. Moreover, it is well-known that the rational covariance extension problem is equivalent to a truncated trigonometric moment problem with a certain complexity constraint on the solution [41, Section 12.5]. In fact, the problem amounts to determining all coercive spectral densities \( \Phi(e^{i\theta}) = P(e^{i\theta})/Q(e^{i\theta}) \) such that

\[
c_k = \int_{T} e^{ik\theta} \Phi(e^{i\theta}) \frac{d\theta}{2\pi},
\]

where \( T = [-\pi,\pi] \), and where \( P \) and \( Q \) are trigonometric polynomials of degree at most \( n \).

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The rational covariance extension problem was partially solved in 1983 by T.T. Georgiou [22], [23], who proved that to each positive covariance sequence \( c \) and positive numerator polynomial \( P \) there is a corresponding denominator polynomial \( Q \) such that \( \Phi = P/Q \) matches the covariances and conjectured that this correspondence is unique. This conjecture was then proved in [12], where it was also established that this (complete) parameterization is smooth, i.e., a diffeomorphism.

Since then, this and similar problems has been extensively studied in the literature [7], [8], [10], [17], [18], [24], [40], [43], [44], [53], and this research has provided the stimulus for research in the general theory for scalar moment problems [9], [11], [27]. Moreover a number of multivariate counterparts, i.e., when \( \Phi \) is a matrix-valued spectral density, have also been solved [5], [20], [26], [39], [45], [47]. All this work is connected to dynamical systems that depends on one variable, typically representing time. However, many problems in spectral estimation, signal processing, system identification, and image processing are inherently multidimensional [6]. Multidimensional systems theory has been applied to many different problems, for example pollution models [21], agricultural models [3], [52], texture modeling [35], and image processing [16]. Therefore, more recently, interest has also been directed towards a multidimensional version of the rational covariance extension problem [25], [27], [34], [48], [49], a problem which is linked to earlier work on maximum entropy solutions [36]–[38].

The focus of the present paper is on the multidimensional rational covariance extension problem, which can be posed as a complexity-constrained, multidimensional, truncated, trigonometric moment problem. In Section II we define the problem and present results from the literature, mainly from [49]. Section III is devoted to well-posedness of this inverse problem and contains some new results reported here for the first time. Finally, in Section IV we consider an example related to Wiener system identification and texture generation.

II. THE MULTIDIMENSIONAL RATIONAL COVARIANCE EXTENSION PROBLEM

The multidimensional rational covariance extension problem is an inverse problem. To formally define it, let \( \Lambda \subset \mathbb{Z}^d \) be a finite index set such that \( 0 \in \Lambda \) and \( -\Lambda = \Lambda \), and let

\[
c := \{c_k \mid k := (k_1,\ldots,k_d) \in \Lambda\} \tag{2}
\]

be a set of known covariances, which are complex numbers with the symmetry \( c_{-k} = \bar{c}_k \). Also let \( |\Lambda| \) denote
the cardinality of the index set $\Lambda$. The problem amounts to parametrizing a certain family of nonnegative bounded measures $d\mu$ on $\mathbb{T}^d$ such that

$$c_k = \int_{\mathbb{T}^d} e^{i(k, \theta)} d\mu(\theta) \quad \text{for all } k \in \Lambda,$$

(3)

where $(k, \theta) := \sum_{j=1}^d k_j \theta_j$. Like in the one-dimensional case, the reason for referring to this problem as a covariance extension problem is that the $c$ in (2) can be interpreted as a set of covariance lags $c_k := E[y_{k+1} y_k]$ of a discrete-time, zero-mean, homogeneous\footnote{That is, $E[y_{k+1} y_k]$ is independent of $y$ for all $k$. Homogeneity generalizes stationarity for $d = 1$ to the multidimensional case $d > 1$.} stochastic process $\{y_t; t \in \mathbb{Z}^d\}$.

Now, let

$$\bar{d}\mu(\theta) = \Phi(e^{i\theta}) d\mu(\theta) + d\bar{\mu}(\theta),$$

(4)

where $\Phi d\mu$ is the absolutely continuous and $d\bar{\mu}$ the singular part in the Lebesgue decomposition of $d\mu$ and $d\bar{\mu}(\theta) := (1/2\pi)^d \prod_{j=1}^d d\theta_j$ is the (normalized) Lebesgue measure on $\mathbb{T}^d$. In general, if a solution to (3) exists, there are infinitely many measures $d\mu$ satisfying the equation. We wish to parametrize the family of measures for which the spectral density takes the form

$$\Phi(e^{i\theta}) = \frac{P(e^{i\theta})}{Q(e^{i\theta})}, \quad p, q \in \mathbb{P}_+ \setminus \{0\}.$$

(5)

Here $\mathbb{P}_+$ is the set of coefficients $p := [p_k \mid k \in \Lambda]$ corresponding to trigonometric polynomials

$$P(e^{i\theta}) = \sum_{k \in \Lambda} p_k e^{-i(k, \theta)},$$

(6)

that are positive for all $\theta \in \mathbb{T}^d$. The set $\mathbb{P}_+$ is in fact a convex cone, and by $\mathbb{P}_+$ and $\partial \mathbb{P}_+$ we will denote the closure and the boundary $\mathbb{P}_+$, $\partial \mathbb{P}_+$, respectively. It is then easily verified that $\partial \mathbb{P}_+$ is the subset of all $p \in \mathbb{P}_+$ such that the corresponding nonnegative trigonometric polynomial $P(e^{i\theta})$ is zero in at least one point. In this context we also introduce the dual cone of $\mathbb{P}_+$, called $\mathcal{C}_+$, the interior of which is given by

$$\mathcal{C}_+ := \{ c \mid \langle c, p \rangle > 0, \quad \text{for all } p \in \mathbb{P}_+ \setminus \{0\} \}.$$

Here, $\langle c, p \rangle = \sum_{k \in \Lambda} c_k p_k$ denotes the complex inner product. The boundary $\partial \mathcal{C}_+$ of $\mathcal{C}_+$ consists of all $c \in \mathcal{C}_+$ such that $\langle c, p \rangle = 0$ for some $p \in \mathbb{P}_+ \setminus \{0\}$. The cone $\mathcal{C}_+$ plays an important role in the theory. In particular, it is easily seen that, for any $c$ satisfying (3) for some nonnegative measure $d\mu$, we have

$$\langle c, p \rangle = \int_{\mathbb{T}^d} P(e^{i\theta}) d\mu(\theta) \geq 0$$

for all $p \in \mathbb{P}_+$, and hence $c \in \mathcal{C}_+$ (cf. [34, Proposition 2.2]). In fact, the converse statement is also true [34, Theorem 2.3].

In view of the complexity constraint (5), parametrizing the rational family of solutions to (3) is a non-convex problem. However, formulating the inverse problem as an optimization problem, we can use a regularization function that turns out to promote rational solutions. To this end, let $p \in \mathbb{P}_+ \setminus \{0\}$ and consider the functional

$$\Pi_p(d\mu) = \int_{\mathbb{T}^d} P(e^{i\theta}) \log \frac{P(e^{i\theta})}{Q(e^{i\theta})} d\mu(\theta),$$

(7)

which is the Kullback-Leibler divergence between the two measures $Pd\mu$ and $d\bar{\mu} = \Phi d\mu + d\bar{\mu}$ [27], [49]. The primal problem then amounts to minimizing (7) subject to (3), i.e.,

$$\min_{d\mu \geq 0} \int_{\mathbb{T}^d} P(e^{i\theta}) \log \frac{P(e^{i\theta})}{\Phi(e^{i\theta})} d\mu(\theta),$$

subject to $c_k = \int_{\mathbb{T}^d} e^{i(k, \theta)} d\bar{\mu}(\theta)$ for all $k \in \Lambda$.

(8)

From this, one can readily derive the Lagrangian dual functional obtained by relaxing the equality constraint, which takes the form

$$J_p(q) = \langle c, q \rangle - \int_{\mathbb{T}^d} P(e^{i\theta}) \log Q(e^{i\theta}) d\mu,$$

(9)

and the corresponding dual optimization problem is

$$\min_q \langle c, q \rangle - \int_{\mathbb{T}^d} P(e^{i\theta}) \log Q(e^{i\theta}) d\mu,$$

subject to $q \in \mathbb{P}_+$.

(10)

Using the primal-dual pair of optimization problems (8) and (10), we get the following (complete) characterization of the multidimensional rational covariance extension problem.

**Theorem 1** ([49, Theorem 2.1]): For every $(c, p) \in \mathbb{C}_+ \times (\mathbb{P}_+ \setminus \{0\})$ the functional (9) is strictly convex, and (10) has a unique minimizer $\hat{q} \in \mathbb{P}_+ \setminus \{0\}$. Moreover, there exists a unique $\hat{c} \in \partial \mathcal{C}_+$ and a nonnegative singular measure $d\hat{\mu}$ with support $\text{supp}(d\hat{\mu}) \subseteq \{ \theta \in \mathbb{T}^d \mid Q(e^{i\theta}) = 0 \}$ such that

$$c_k = \int_{\mathbb{T}^d} e^{i(k, \theta)} \left( \frac{P}{Q} d\mu + d\hat{\mu} \right) \quad \text{for all } k \in \Lambda,$$

and

$$\hat{c}_k = \int_{\mathbb{T}^d} e^{i(k, \theta)} d\hat{\mu}, \quad \text{for all } k \in \Lambda.$$

For any such $d\hat{\mu}$, the measure

$$d\mu(\theta) = \frac{P(e^{i\theta})}{Q(e^{i\theta})} d\mu(\theta) + d\hat{\mu}(\theta)$$

is an optimal solution to (8). Moreover, $d\hat{\mu}$ can be chosen with support in at most $|\Lambda| - 1$ points. In view of Theorem 1, the optimality conditions for (10) can be summarized as follows.

**Corollary 2** ([34, Corollary 5.3]): Let $c \in \mathcal{C}_+$. Then $\hat{q}$ is an optimal solution to (10) if and only if

$$\hat{q} \in \mathbb{P}_+, \quad \hat{c} \in \partial \mathcal{C}_+, \quad \langle \hat{c}, \hat{q} \rangle = 0$$

$$c_k = \int_{\mathbb{T}^d} e^{i(k, \theta)} \frac{P}{Q} d\mu + \hat{c}_k \quad \text{for all } k \in \Lambda.$$


III. WELL-POSEDNESS OF THE PROBLEM

Since (8) is an inverse problem, we are not only interested in the existence of a solution but also in the question of whether (8) is well-posed. Theorem 1 guarantees that for $(c, p) \in \mathcal{C}_+ \times (\mathcal{P}_+ \setminus \{0\})$ a unique solution $(\hat{q}, \hat{c})$ exists. It remains to investigate how this solution depends on the parameters of the problem, i.e., on the tuple $(c, p)$. To this end, from Propositions 7.3 and 7.4 in [34], we have the following result.

Proposition 3: Let $c$, $p$ and $\hat{q}$ be as in Theorem 1. Then the map $(c, p) \mapsto \hat{q}$ is continuous.

Next, consider the continuity of the map $(c, p) \mapsto \hat{c}$. For $d \leq 2$, note that $\int_{\mathcal{T}_d} Q^{-1} dm = \infty$ for all $q \in \partial \mathcal{P}_+$. Hence, when $p \in \mathcal{P}_+$ and thus $P$ is strictly positive on all of $\mathcal{T}_d$, the gradient of (9) will be $-\infty$ on the boundary $\partial \mathcal{P}_+$. Therefore, the optimal solution $\hat{q}$ is pushed into $\mathcal{P}_+$, and in view of Theorem 1 we can thus guarantee that $\hat{c} = 0$.

Proposition 4 ([49, Corollary 2.3]): Suppose that $d \leq 2$. Then, for any $c \in \mathcal{C}_+$ and $p \in \mathcal{P}_+$ there exists a $q \in \mathcal{P}_+$ such that $d \mu = (P/Q) dm$ satisfies (3). Moreover this $q$ is the unique solution to (10).

Hence, since $\hat{c} = 0$ for $d \leq 2$ and $P \in \mathcal{P}_+$, the continuity of $\mathcal{C}_+ \times \mathcal{P}_+ \ni (c, p) \mapsto \hat{c}$ is trivial in this case. For later reference we formulate this result in the following proposition.

Proposition 5: Let $c$, $p$, $\hat{q}$ and $\hat{c}$ be as in Theorem 1. Then, for $d \leq 2$ and all $(c, p) \in \mathcal{C}_+ \times \mathcal{P}_+$, the mapping $(c, p) \mapsto (\hat{q}, \hat{c})$ is continuous.

Remark 6: In the case $d \leq 2$ in turns out that the result of Proposition 3 can be strengthened. In fact, for a fixed $p \in \mathcal{P}_+$, the map $\mathcal{C}_+ \ni c \mapsto \hat{q} \in \mathcal{Q}_+ \subset \mathcal{P}_+$ is a diffeomorphism onto its image $\mathcal{Q}_+$ [49, Theorem 4.4]. Moreover, for fixed $c \in \mathcal{C}_+$ the map $\mathcal{P}_+ \ni p \mapsto \hat{q} \in \mathcal{Q}_+$ is also a diffeomorphism [49, Theorem 4.5].

However, the case $d \geq 3$ turns out to be trickier, since we can have $\hat{c} \neq 0$ although $p \in \mathcal{P}_+$. In the following subsection we will investigate the continuity of the map $(c, p) \mapsto \hat{c}$ for the case $d \geq 3$ under certain conditions.

A. New results on well-posedness

In view of Propositions 3 and 5, the problem (8) is well-posed for $d \leq 2$ and $p \in \mathcal{P}_+$. However, the well-posedness in $\hat{c}$ is in some sense trivial, since in this case we have $\hat{c} = 0$ for all $p \in \mathcal{P}_+$ and $c \in \mathcal{C}_+$. For $d \geq 3$ this is not the case, as we can have $\hat{q} \in \partial \mathcal{P}_+$ even when $p \in \mathcal{P}_+$ and thus might have $\hat{c} \neq 0$. In this subsection we extend the well-posedness result of Proposition 5 to hold also in some cases where $d \geq 3$.

However, we begin by noting that the condition $p \in \mathcal{P}_+$ in general cannot be weakened to $p \in \mathcal{P}_+ \setminus \{0\}$, which the following one-dimensional example illustrates (cf. [34, Example 3.8]).

Example 7: Let

$$c = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + \int_{\mathcal{T}} \begin{bmatrix} e^{-i\theta} \\ 0 \\ e^{i\theta} \end{bmatrix} (2dm + dv_0),$$

where $dm = d\theta/2\pi$, and $dv_0$ is the singular measure $\delta_0(\theta)d\theta$ with support in $\theta = 0$. Since $d\mu := 2dm + dv_0$ is positive, $c \in \mathcal{C}_+$. Moreover, since the Toeplitz matrix $T_c := \begin{bmatrix} c_j - \overline{c_k} & j, k \end{bmatrix}$ is positive definite, i.e.,

$$T_c = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} > 0,$$

we have $c \in \mathcal{C}_+$ (see, e.g., [40, p. 2853]). Thus we know that for each $p \in \mathcal{P}_+$ we have a unique $\hat{q} \in \mathcal{P}_+$ such that $P/Q$ matches $c$, and hence $\hat{c} = 0$ (Proposition 4). However, for $p = 2(-1, 2, -1)^T$ we have that $\hat{q} = (-1, 2, -1)^T$ and $\hat{c} = (1, 1, 1)^T$ (Corollary 2). Then, for the sequence $(p_k)$, where $p_k = 2(-1, 2, 1/k, -1)^T \in \mathcal{P}_+$, we have $c_k = 0$, so

$$\lim_{k \to \infty} c_k = \lim_{k \to \infty} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T,$$

which shows that the mapping $p \mapsto \hat{c}$ is not continuous.

One way to try to establish continuity of the map $(c, p) \mapsto \hat{c}$ is to try to use the already established continuity from $(c, p)$ to $\hat{q}$ in Proposition 3. From the KKT conditions in Corollary 2 we have in particular that

$$c_k = \int_{\mathcal{T}} e^{i(k, \theta)} \frac{P}{Q} dm + \hat{c}_k$$

for all $k \in \Lambda$, and hence continuity of $\hat{c}$ would follow if $\int_{\mathcal{T}} P\hat{Q}^{-1} dm$ is continuous in $(c, p, \hat{q})$, whenever the integral is finite. If $p \in \mathcal{P}_+$, this follows from the continuity of the map $p \mapsto \hat{Q}^{-1}$ in $L_1(\mathcal{T}_d)$. Note here that we know from Example 7 that the condition $p \in \mathcal{P}_+$ is actually needed. In fact, if $p \in \partial \mathcal{P}_+$ we may have “pole-zero cancellations” in $P/Q$ (cf. [34, Example 5.10]), and then $\int_{\mathcal{T}} P/Q dm$ may be finite even if $\hat{Q}^{-1} \notin L_1(\mathcal{T}_d)$.

For the case $d \leq 2$, the continuity of the map $\hat{q} \mapsto \hat{Q}^{-1}$ in $L_1(\mathcal{T}_d)$ is trivial, since, if $\int_{\mathcal{T}} \hat{Q}^{-1} dm$ is finite, then $\hat{q} \in \mathcal{P}_+$ and $\hat{Q}$ is bounded away from zero (cf. Propositions 4 and 5). However, for the case $d \geq 3$ the optimal $\hat{q}$ may belong to the boundary $\partial \mathcal{P}_+$, i.e., $\hat{Q}$ is zero in some point. Here we show $L_1$ continuity of $\hat{q} \mapsto \hat{Q}^{-1}$ for certain cases. The proof is deferred to the appendix.

Proposition 8: For $d \geq 3$, let $\hat{q} \in \mathcal{P}_+$, and suppose that the Hessian $\nabla_{\theta\theta} \hat{Q}$ is positive definite in each point where $\hat{Q}$ is zero. Then $\hat{Q}^{-1} \in L_1(\mathcal{T}_d)$ and the mapping from the coefficient vector $q \in \mathcal{P}_+$ to $Q^{-1}$ is $L_1$ continuous in the point $\hat{q}$.

From Propositions 8 and 5 the following continuity result follows directly.

Corollary 9: Let $c$, $p$, $\hat{q}$, and $\hat{c}$ be as in Theorem 1. For all $c \in \mathcal{C}_+$ and all $p \in \mathcal{P}_+$, the mapping $(c, p) \mapsto (\hat{q}, \hat{c})$ is continuous in any point $(c, p)$ in which $\hat{Q}$ is strictly positive or in which the Hessian $\nabla_{\theta\theta} \hat{Q}$ is positive definite in each point where $\hat{Q}$ is zero.

IV. EXAMPLE IN TEXTURE GENERATION

Wiener systems form a class of nonlinear dynamical systems that consist of a linear dynamic part composed with a static nonlinearity, as in Figure 1. They belong to a class of so called block-oriented systems, which has a long history
For some constant $b$. In order to determine $b$, first note that $ho = 0$ implies that $x_1$ and $x_2$ are uncorrelated, and hence independent, since the joint distribution is Gaussian. This in turn means that $y_1$ and $y_2$ are independent, since $f$ is a static function, and hence we get

$$b = R(0) = E[y_1y_2] = E[y_1]E[y_2].$$

Therefore $r$ can be expressed as

$$r = R(\rho) - E[y_1]E[y_2]$$

$$= \int_{-\infty}^{\rho} \frac{1}{2\pi\sqrt{1 - s^2}} \exp \left( -\frac{\tau^2}{1 + s} \right) ds.$$  \hspace{1cm} (13)

The integrand is well-defined for $-1 < \rho < 1$, and the integral converges for all values in the closed interval $[-1, 1]$. Moreover, the integrand is strictly positive on $(-1, 1)$ and by the inverse function theorem this transformation is invertible.

### B. Estimating the linear part of the Wiener system

By using the inverse of (13) we can estimate the covariances $c_k := E[x_{t+k}x_t]$ from estimates of the covariances $r_k := E[y_{t+k}y_t] - E[y_t]E[y_1]$. Note however that (13) depends on the threshold parameter $\tau$, which is assumed to be unknown. In order to estimate $\tau$ we use (12), which gives $\tau_{\text{est}} = \phi^{-1}(1 - E[y_1])$. Having estimates of the covariances $c_k$, we can now appeal to Theorem 1 in order to estimate a rational spectrum for $x_t$.

Given this rational spectral density we want to recover a linear dynamical system corresponding to the spectrum. In the one-dimensional case, $d = 1$, this is always possible by spectral factorization, since the spectral density can be written as a sum-of-one-square

$$\Phi(\omega) = \frac{P(e^{i\omega})}{Q(e^{i\omega})} = \frac{|b(e^{i\omega})|^2}{|a(e^{i\omega})|^2}.$$  

However in higher dimensions this is in general not possible [15]. For a strictly positive spectrum, a factorization as a sum-of-several-squares is always possible [14]

$$\Phi(\omega) = \frac{P(e^{i\omega})}{Q(e^{i\omega})} = \sum_{k=1}^m \frac{|h_k(e^{i\omega})|^2}{\sum_{k=1}^m |a_k(e^{i\omega})|^2}.$$  

However, for $m > 1$ the interpretation of this in terms of a dynamical system is not clear to the authors. We therefore resort to a heuristic and apply the factorization procedure in [28, Theorem 1.1.1] although some of the conditions required to ensure the existence of a spectral factor may not be met (cf. [49, Section 7]).

The complete procedure for identifying the Wiener system with thresholding as static nonlinearity is summarized in Algorithm 1.

### C. Simulation results

Next we test the procedure outlined above on simulated data. To this end, we consider the two-dimensional recursive filter with transfer function given by

$$b(e^{i\theta_1}, e^{i\theta_2}) = \sum_{k \in \Lambda_+} b_k e^{-i(k, \theta)}$$

$$a(e^{i\theta_1}, e^{i\theta_2}) = \sum_{k \in \Lambda_+} a_k e^{-i(k, \theta)}.$$
Algorithm 1

Input: \( y_t \)

1. Estimate threshold parameter: \( \tau_{\text{est}} = \phi^{-1}(1 - E[y_k]) \)
2. Estimate covariances: \( r_k := E[y_{t+k}y_t] - E[y_{t+k}]E[y_t] \)
3. Compute covariances \( c_k := E[x_{t+k}x_t] \) by using (13)
4. Estimate a rational spectrum using Theorem 1
5. Apply the factorization procedure in [28, Theorem 1.1.1]

Output: \( \tau_{\text{est}}, \) coefficients for the linear dynamical system

Where \( \Lambda_+ = \{(k_1, k_2) \in \mathbb{Z}^2 \mid 0 \leq k_1 \leq 2, 0 \leq k_2 \leq 2 \} \)

and

\[ B = \begin{bmatrix} 0.75 & -0.2 & 0.05 \\ 0.2 & 0.3 & 0.05 \\ -0.05 & -0.05 & 0.1 \end{bmatrix}, \quad A = \begin{bmatrix} 3.6623 & -4.0222 & 0.9987 \\ -4.0939 & 4.8705 & -1.1913 \\ 1.2018 & -1.3539 & 0.2155 \end{bmatrix}. \]

The threshold parameter in (11) is set to \( \tau = 0.06 \).

The system is simulated with Gaussian white noise as input, and \( 500 \times 500 \) samples are taken as output. These samples are used to estimate the threshold parameter, which gives the estimate \( \tau_{\text{est}} = 0.0570 \). Moreover, they are used to estimate covariances \( r_k \) on a grid \( \Lambda = \{(k_1, k_2) \in \mathbb{Z}^2 \mid |k_1| \leq 3, |k_2| \leq 3 \} \). Note that this grid \( \Lambda \) does not agree with the true degree of the linear system. From the estimated covariances \( (r_k) \) we determine the covariances \( (c_k) \), which are then used in the optimization problem (10). We compute the solution with two optimization problems; the first one being \( P = 1 \), which corresponds to the maximum entropy (ME) solution, and the second one being \( P = P_{\text{true}} \), i.e., the trigonometric polynomial corresponding to the filter \( b \). The optimization problems are solved using the CVX toolbox in Matlab [29], [30]. The corresponding spectra obtained are shown in Figure 2. As can be seen in the figure, using the true \( P \) gives a better agreement with the true spectrum, shown in Figure 2a, which indicates that an appropriate tuning of \( p \) can improve the fit. Although there are methods in the literature on how to do simultaneously estimation of \( p \) and \( q \) [7], [17], [33], [48], [49], the question on how to best select \( p \) is still open.

After estimating the spectra, we compute estimates of filter coefficients for the autoregressive part of the linear system, and the corresponding estimates are

\[ A_{\text{ME}} = \begin{bmatrix} 4.1270 & -3.8799 & 0.3572 & 0.2297 \\ -5.4210 & 4.0752 & 0.4412 & -0.2174 \\ 2.4057 & -0.0926 & -1.7157 & 0.1816 \\ -0.4199 & -0.6931 & 0.9018 & -0.1010 \end{bmatrix} \]

\[ A_{\text{true}} = \begin{bmatrix} 3.7207 & -4.3079 & 1.3210 & -0.0861 \\ -4.2527 & 5.4070 & -1.6585 & 0.0364 \\ 1.3381 & -1.6108 & 0.1836 & 0.2351 \\ -0.0562 & 0.0019 & 0.2183 & -0.2145 \end{bmatrix} \]

Using these filter coefficients, together with the corresponding filter coefficients for the moving-average part, we simulate the estimated Wiener system. The corresponding generated textures are shown in Figure 3. Visually, the generated textures seem to have similar structures. However, by comparing the covariances, which are shown in Figure 4, it can be seen that the texture generated by the filter obtained using the true \( p \) matches the higher order covariances considerably better.
Fig. 3. Output of the true and the identified systems. Figures 3a - 3c show 500 × 500 samples, and Figures 3d - 3f show 100 × 100 samples.

Fig. 4. Covariances and covariance errors for the textures. Here $k = (k_1, k_2)$ where the x-axis corresponds to $k_1$ and the y-axis corresponds to $k_2$.

V. CONCLUSION AND FUTURE WORK

In this paper we continue our work on the multidimensional rational covariance extension initiated in [49]. We develop theory for identification of Wiener systems with applications to texture generation, and present new results on the well-posedness of the problem in dimension $d \geq 3$. However, a complete such characterization of well-posedness is still missing (see, e.g., Example 12 in the Appendix).

Another remaining issue is that spectral factorization typically is not possible for multidimensional spectral densities, and therefore we have resorted to an heuristic approach to approximate factorization. An alternative framework that avoids this problem is to model the texture as an output of an one-dimensional vector valued process as in [13] and [45]. This framework is not symmetric with respect to the coordinate-axes since it assumes stationarity only in one direction, a feature which may or may not be desirable depending on the application at hand.

APPENDIX

Proof of Proposition 8: The proof will be carried out using a sequence of lemmas. First we bound the integral of
$Q^{-1}$ over the ball $B_\rho(\theta_0):=\{\theta\in\mathbb{T}^d \mid \|\theta-\theta_0\|_2 \leq \rho\}$, where the bound only depends on the Hessian of $Q$.

**Lemma 10**: Let $d \geq 3$ and $q \in \mathbb{Q}^+$. If the Hessian $\nabla_{\theta\theta} Q(e^{i\varphi}) \geq \gamma I > 0$, for $\theta \in B_\rho(\theta_0)$, then

$$\int_{B_\rho(\theta_0)} Q(e^{i\varphi})^{-1}d\mu(\theta) \leq \frac{\rho^{d-2}}{\gamma}.$$  

Proof: By integrating the inequality twice, we see that $Q(e^{i\varphi}) \geq \|\theta-\theta_0\|^2/2$ for $\theta = \arg\min_{\theta \in B_\rho(\theta_0)} Q(e^{i\varphi})$. By radial symmetry, the integral of $(\|\theta-\theta_0\|/2)^{-2}$ is in turn bounded by the integral of $(\|\theta-\theta_0\|/2)^{-1}$. Therefore,

$$\int_{B_\rho(\theta_0)} Q(e^{i\varphi})^{-1}d\mu(\theta) \leq \int_{B_\rho(\theta_0)} \frac{2}{\|\theta-\theta_0\|^2}d\mu(\theta) \leq \frac{\rho^{d-2}}{\gamma},$$

using basic approximations in spherical coordinates.

Secondly we show that if $q$ satisfies the condition in Lemma 10, then any polynomial sufficiently close to $q$ is well-behaved.

**Lemma 11**: Let $d \geq 3$, $q \in \mathbb{Q}^+$, and assume that the Hessian $\nabla_{\theta\theta} Q(e^{i\varphi})$ is positive definite in the zero $\theta_0$ of $Q$. Further, let $q_k \in \mathbb{Q}^+$ for $k \in \mathbb{N}$ such that $q_k \to q$ as $k \to \infty$. Then for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ and a $\rho > 0$ such that

$$\int_{B_\rho(\theta_0)} Q(e^{i\varphi})^{-1}d\mu(\theta) \leq \varepsilon$$

for all $k \geq N$.

Proof: Let $\gamma$ be such that $\nabla_{\theta\theta} Q(e^{i\varphi})|_{\theta=\theta_0} \geq 3\gamma I$, and let $\rho_1 > 0$ be such that $\nabla_{\theta\theta} Q(e^{i\varphi}) \geq 2\gamma I$ for $\theta \in B_{\rho_1}(\theta_0)$. This is always possible since $Q$ is $C^\infty$ and hence the second derivatives are continuous. Next, let $\rho = \min(\rho_1, (\varepsilon\gamma)^{1/(d-2)})$ and select an $N$ such that $\nabla_{\theta\theta} Q_k(e^{i\varphi}) \geq \gamma I$ for $\theta \in B_\rho(\theta_0)$ holds for all $k \geq N$. Such an $N$ exists since $\nabla_{\theta\theta} Q_k(e^{i\varphi}) \to \nabla_{\theta\theta} Q(e^{i\varphi}) > 2\gamma I$ on $B_\rho(\theta_0)$. From Lemma 10 it then follows that

$$\int_{B_\rho(\theta_0)} Q(e^{i\varphi})^{-1}d\mu(\theta) \leq \rho^{d-2}/\gamma \leq \varepsilon$$

which proves the lemma.

Next, continuing the proof of Proposition 8, we use the fact that the integrals of $Q^{-1}$ and $Q_k^{-1}$ in a neighborhood of the zero set of $Q$ can be made arbitrarily small. Further, the convergence is uniform on the complement of this set, and hence convergence of the integrals will follow. To this end, let the sequence $(q_k) \subset \mathbb{Q}^+$ converge to $q$. For any $\varepsilon > 0$ we need to show that there is an $N \in \mathbb{N}$ such that $\|Q^{-1}_k-\hat{Q}^{-1}\|_1 < \varepsilon$ for all $k > N$. Also note that $Q$ has finitely many zeros. To see this, assume that this is not so. By compactness of $\mathbb{T}^d$ the zeros have an accumulation point. However, this is contradicted by the fact that the Hessian is positive definite in each zero of $Q$ and hence any zero of $Q$ is isolated. Using Lemma 11 there is a $\rho > 0$ and an $N_1 \in \mathbb{N}$ such that

$$\int_{B_\rho(\theta_0)} Q_k(e^{i\varphi})^{-1}d\mu(\theta) \leq \varepsilon/3$$

for all $k > N_1$. Since $Q_k \to \hat{Q}$ uniformly and $\hat{Q} > 0$ on $\mathbb{T}^d \setminus \cup_k B_\rho(\theta_0)$, there is an $N_2$ such that $\|Q^{-1}_k-\hat{Q}^{-1}\|_{L_1(\mathbb{T}^d \setminus \cup_k B_\rho(\theta_0))} < \varepsilon/3$ for all $k > N_2$. The result now follows since, for $k > N := \max(N_1, N_2)$, we have

$$\|Q_k^{-1}-\hat{Q}^{-1}\|_1 \leq \|Q_k^{-1}-\hat{Q}^{-1}\|_{L_1(\mathbb{T}^d \setminus \cup_k B_\rho(\theta_0))} + \|\hat{Q}^{-1}\|_{L_1(\cup_k B_\rho(\theta_0))} \leq \varepsilon,$$

which shows the continuity in the point $\hat{q}$.

Finally, we note that there are cases, even for $d = 3$, where the conditions in Corollary 9 are not satisfied and hence the corollary does not apply. In these cases the question of well-posedness is still open. The following is an example of this.

**Example 12**: Let $d = 3$, $q \in \mathbb{Q}^+$, and let the integer $n \geq 2$. Then, assuming that

$$Q(e^{i\varphi}) \geq \theta_1^2 + \theta_2^2 + \theta_3^2,$$

we have $\|Q^{-1}\|_{L_1(\mathbb{T}^3)} < \infty$. To see this, first note that $Q^{-1}$ is unbounded only at the origin. Therefore $\|Q^{-1}\|_{L_1(\mathbb{T}^3)} < \infty$ if $\|Q^{-1}\|_{L_1(B_\rho)} < \infty$ for some $1 > \rho_1 > 0$, where $B_\rho := B_\rho(0)$. A variable change into spherical coordinates gives

$$\int_{B_\rho} Q(e^{i\varphi})^{-1}d\mu(\theta) = \int_{-\pi/2}^{\pi/2} \int_0^{\rho} \frac{\sin(\varphi_1)d\varphi_1}{2\pi} \int_0^{\rho/2} \frac{\sin(\varphi_2)d\varphi_2}{2\pi} d\varphi_3 + \frac{1}{2} \log(1 + \frac{\rho^2}{r^{2(n-1)}}) dr.$$  

Note that the integrand is uniformly bounded outside the set $|\varphi_1| < \varepsilon$ for any $1 > \varepsilon > 0$. However due to symmetry it is enough to consider the set $S = \{(r, \varphi_1) \mid 0 \leq \varphi_1 \leq \varepsilon, r \in [0, \rho]\}$. Moreover, inside $S$ we have $\alpha_1 \varphi_1 \leq \sin(\varphi_1) \leq \alpha_2 \varphi_1$ and $\cos(\varphi_1) \geq \alpha_3$ for some positive constants $\alpha_1, \alpha_2, \alpha_3$. Therefore $\|Q^{-1}\|_{L_1(B_\rho)}$ is finite if the following integral is finite:

$$\int_0^{\rho} \left(\int_0^{\rho} \frac{\varphi_1 d\varphi_1}{\varphi_1^2 + r^{2(n-1)}} dr + \int_0^{\rho/2} \frac{1}{2} \log(1 + \frac{\rho^2}{r^{2(n-1)}}) dr \right) dr \leq \int_0^{\rho} \left(\int_0^{\rho} \frac{1}{2} \log(\frac{1}{r^{2(n-1)}}) dr \right) dr \leq \int_0^{\rho} \left(\int_0^{\rho} \frac{1}{2} \log(2 - 2(n-1) \log r) dr \right) dr < \infty.$$  

This shows that the integral of $Q^{-1}$ over $\mathbb{T}^3$ is finite.

**References**


$^2$Let $(\theta_1, \theta_2, \theta_3) = (r \sin(\varphi_1) \cos(\varphi_2), r \sin(\varphi_1) \sin(\varphi_2), r \cos(\varphi_1))$ and hence $dm = (r^2 \sin(\varphi_1))/((2\pi)^3)dr d\varphi_1 d\varphi_2.$


