A down-sampled controller to reduce network usage with guaranteed closed-loop performance

José Araújo, André Teixeira, Erik Henriksson and Karl H. Johansson

***Abstract*— We propose and evaluate a down-sampled controller which reduces the network usage while providing a guaranteed desired linear quadratic control performance. This method is based on fast and slow sampling intervals, as the closed-system benefits by being brought quickly to steady-state conditions while behaving satisfactorily when being actuated at a slow rate once at those conditions. This mechanism is shown to provide large savings with respect to network usage when compared to traditional periodic time-triggered control and other aperiodic controllers proposed in the literature.**

I. INTRODUCTION

The decision of when to sample and actuate in a control system while guaranteeing specific conditions of the closed-loop system has been the topic of much research since the early 1960's. This topic is highly relevant in networked control systems where communication and computation efforts at devices operating in large scale networks, possibly battery-operated, must be efficiently utilized. Particularly in process control systems, requirements for sampling and actuation rates have been devised which can range between tens of milliseconds and tens of seconds, depending of the specific process and task at hand [1].

Adaptive sampling for control was firstly proposed by [2] and since then much research has been conducted on topics ranging from adaptive sampling, multi-rate control, control under scheduling constraints and event-based control. Multi-rate control has been proposed due to the need to operate several processes with different sampling rate requirements, where the research on this topic focused mainly on the stability analysis and robust design of such systems [3]. The scheduling of sampling and control, using both offline and online optimization techniques has been proposed in [4]–[6] where the focus has been on scheduling multiple plants competing in a single processor or a single communication channel. Since the work in [7], many researchers have proposed sampling and actuation policies based on events instead of the classic periodically sampled control systems [8]. Optimal event-triggered control has been proposed in, e.g., [7], [9], [10] considering first-order linear systems and reset actuation inputs, while Lyapunov-based methods that guarantee stability of the closed-loop system have been proposed in several works, e.g., [11], [12], [13], [14], [15], [16]. More recently, model-predictive event-triggered control has been proposed in [17], [18]. A broad survey of event-triggered control is found in [19]. However, in all these works, no guarantees are given by design that the achieved trade-off between control performance and transmission rate is improved with respect to the periodic case. In [20], the authors proposed a roll-out event-triggered controller which guarantees better control performance versus transmission rate trade-off than periodic control and are able to quantify this gain. However, the method is computationally intensive and the computation of the best scheduling sequence must be computed in a receding-horizon manner. A self-triggered linear quadratic regulator was proposed in [21] which by design guarantees that the obtained quadratic control cost does not exceed a specific maximum value. The method is numerically evaluated and is shown to achieve lower control costs than periodic control at the same transmission rate. However, no analytical guarantees are provided that the transmission rate does not exceed the periodic control rate.

In this paper, we propose a down-sampling control policy which guarantees the same linear quadratic (LQ) control performance of any periodically sampled LQ controller at a given period, while reducing the rate of sampling/transmission. The technique is developed considering the disturbance-free case, as well as when sporadic impulse disturbances affect the system. The down-sampled control policy is based on a fast and a slow sampling rate, utilized within specific intervals of time in order to guarantee the required control performance. The transient period is performed with fast control updates, while the steady-state control is performed at a slower rate. It was recently shown in [22] that the optimal transmission schedule for a linear system without disturbances and with a limited amount of transmissions on a finite horizon, is to transmit consecutively at the beginning of the interval. This is in line with method we propose in this paper. With the reduction in complexity over the aforementioned existing methods, we develop a sampling and transmission scheme which is able to guarantee by design a specified control performance. Additionally, one is able to specify the slow rate according to the application and/or system characteristics. Numerical examples validate the down-sampled policy where we provide comparisons to the recent work of [20] and [21] for the deterministic case.

The rest of this paper is organized as follows. Sec. II presents the system setup and the problem formulation. The down-sampled controller is presented in Sec. III where its implementation under the absence or presence of disturbances is proposed and analysed. Finally, numerical examples validate the proposed mechanism and illustrate its benefits and Sec. V concludes this paper.

ACCESS Linnaeus Centre, KTH Royal Institute of Technology, Stockholm, Sweden. (e-mail: {araujo, andretei, erik02, kallej}@kth.se)

This work is supported by the Knut and Alice Wallenberg Foundation, the Swedish Research Council (VR 2009-4565) and the HYCON2 EU project.

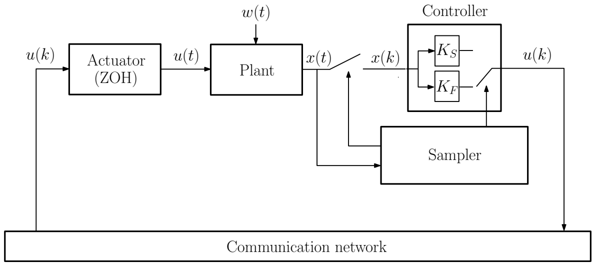


Fig. 1: Down-sampled controller architecture. The sampler is responsible for selecting when a sample is taken, which control law to use and when a control input is sent to the actuator.

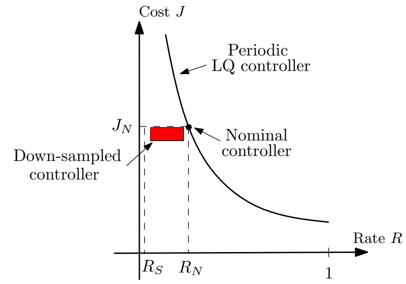


Fig. 2: Problem illustration. The down-sampled controller is designed to achieve a lower or equal cost at a lower sampling rate, than that of any periodic LQ controller sampled at a nominal rate. Solution region represented in red.

II. Problem formulation

Assume the plant is a continuous-time linear and time-invariant system,

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

with a state $x(t) \in \mathbb{R}^n$ and input $u(t) \in \mathbb{R}^m$ given by a state-feedback law and that the control performance is defined by the normalized linear quadratic (LQ) cost

$$\bar{J} = \lim_{T\to\infty} \frac{1}{T} J^{[0,T]}, \tag{2}$$

where

$$J^{[0,T]} = \int_0^T x(t)^T Q_c x(t) + u(t)^T R_c u(t)\, dt,$$

and where $Q_c \succ 0$, $R_c \succ 0$ and let (A, B) be controllable.

We deal with the discrete-time counterpart of (1) and (2) where the sampling of the state and computation of the control input are performed by embedded devices with a fixed operating frequency which is governed by a time-clock. In such systems, the controller design is typically performed by discretizing the continuous-time system (1) for a specific period, and optimally designing the control policy that minimizes the discretized control cost (2) [8].

Sampling (1) with a zero-order-hold for a baseline period $h = 1$, gives the discrete-time system

$$x(k+1) = \Phi_1 x(k) + \Gamma_1 u(k), \tag{3}$$

where $\Phi_h = e^{Ah}$, $\Gamma_h = \int_0^h e^{As} B ds$, where for $h = 1$ we drop the subscript on Φ and Γ. Such system is affected by impulsive disturbances $w(k) \in \mathbb{R}^n$, occurring at times $d_\kappa,\ \kappa \in \mathbb{N}$, perturbing the state as follows

$$x(k+1) = \begin{cases} w(k) & \text{if } k = d_\kappa, \\ \Phi x(k) + \Gamma u(k) & \text{otherwise,} \end{cases}$$

where the time between disturbances $d_{\kappa+1} - d_\kappa \geq \delta_d$ is unknown, but lower bounded by δ_d. Moreover, we assume that the disturbance occurs during the slow sampling interval, i.e., $d_\kappa + \delta_d > t_s$.

A. Scheduling and control

The sampling and control of the system is performed simultaneously and governed by the sampler as depicted in Fig. 1. Whenever a new control input is computed, it is transmitted to the actuator over a communication network and applied to the plant.

Let $q(k) \in \{q_F, q_S\}$ represent the current sampling mode, which is a fast sampling mode q_F or slow sampling mode q_S and its evolution is governed by the transition map

$$q(k) = \Pi\left(x(k), q(k-1)\right), \tag{4}$$

to be designed in Sec. III. The time of the next sampling instant τ_{k+1} is dependent on the current mode and is given by

$$\tau_{k+1} = \begin{cases} \tau_k + 1 & \text{if } q(k) = q_F, \\ \tau_k + \delta_S & \text{if } q(k) = q_S, \end{cases} \tag{5}$$

where $\delta_S > 1$, which defines the sampling and actuation instants. Hence, the system is either sampled consecutively or down-sampled with interval δ_S.

The control input computed at the controller is given by

$$u(k) = \begin{cases} K_F x(k) & \text{if } q(k) = q_F, \\ K_S x(k) & \text{if } q(k) = q_S, \end{cases} \tag{6}$$

where K_F and K_S are the controller gains. Moreover, the switching instant between sampling modes is denoted by t_s and the last switching time as t_s^-.

B. Problem statement

Let the normalized down-sampled controller cost be $\bar{J}_{DS}$ and its transmission rate

$$\bar{R}_{DS} = \lim_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T} 1_{\tau_k \leq T},$$

during the interval of time $[0, \infty)$. In the same manner, let us denote by $\bar{J}_N$ the normalized cost of an LQ controller periodically sampled at a nominal period δ_N, where $1 < \delta_N < \delta_S$, and by $\bar{R}_N$ the average transmission rate of the nominal controller.

Problem 1: Given the slow sampling interval δ_S, design the transition map Π in (4), and the state-feedback controller (6) which guarantees that the down-sampled controller achieves

1) $\bar{J}_{DS} \leq \bar{J}_N$ while
2) $\bar{R}_{DS} < \bar{R}_N$.

This problem corresponds to the design of a down-sampled controller which achieves a control cost lower or equal than that of any periodic LQ controller sampled with nominal period δ_N, while utilizing a lower transmission rate.An illustration of the problem is given in Fig. 2.

C. Preliminaries

We now present a revision of the classic discrete-time LQ control methods which will be essential to the development of the methods proposed in this chapter.

Assuming that system (3), sampled with a baseline period $h = 1$, is controlled periodically with a fixed period δ, the state-feedback control law given by

$$u(k) = \begin{cases} K_\delta x(k) & \text{if } k \in \mathcal{K}_\delta^{[0,\infty)}, \\ u(k-1) & \text{otherwise,} \end{cases} \tag{7}$$

where $\mathcal{K}_\delta^{[l,l']} = \{k \in \mathbb{N} : \mathrm{rem}(k,\delta) = 0 \ \wedge \ l \leq k \leq l'\}$, minimizes the normalized control cost function $\bar{J}_\delta = \lim_{T\to\infty} \frac{1}{T} J_\delta^{[0,T]}$ where

$$J_\delta^{[0,T]} = \sum_{k \in \mathcal{K}_\delta^{[0,T]}} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} Q_\delta & N_\delta \\ N_\delta^T & R_\delta \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}, \tag{8}$$

and

$$\begin{bmatrix} Q_\delta & N_\delta \\ N_\delta^T & R_\delta \end{bmatrix} = \int_0^\delta e^{\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}^T s} \begin{bmatrix} Q_c & 0 \\ 0 & R_c \end{bmatrix} e^{\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} s} ds.$$

Let us denote by $J_\delta^{[l,\infty)}$ the infinite-horizon control cost calculated from time $k = l$, $l \in \mathcal{K}_\delta^{[l,\infty)}$. From [8], it is known that the infinite-horizon control cost given by (8) under the optimal control policy (7) is $J_\delta^{[l,\infty)} = x(l)^T P_\delta x(l)$, where the solution to the infinite-horizon discrete-time Riccati equation, $P_\delta \succ 0$ and the control gain are given by

$$\begin{aligned} P_\delta &= \Phi_\delta^T P_\delta \Phi_\delta + Q_\delta^T - (\Phi_\delta^T P_\delta \Gamma_\delta + N_\delta) \\ &\quad (\Gamma_\delta^T P_\delta \Gamma_\delta + R_\delta)^{-1} (\Gamma_\delta^T P_\delta \Phi_\delta + N_\delta^T) \\ K_\delta &= -(\Gamma_\delta^T P_\delta \Gamma_\delta + R_\delta)^{-1} (\Gamma_\delta^T P_\delta \Phi_\delta + N_\delta^T). \end{aligned} \tag{9}$$

The finite-horizon control cost between time $k \in [l, l']$, for a system actuated with period δ but discretized with baseline period $h = 1$ as in (3), is

$$J_\delta^{[l,l']} = \sum_{k=l}^{l'} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} Q_1 & N_1 \\ N_1^T & R_1 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}, \tag{10}$$

where from now on we drop the subscript in Q, R and N for $h = 1$. Assuming that the controller is given by (7) with K_δ as (9) and that a sampling and control instant takes place at time $k = l$ and $k = l'$, by induction one can show that the cost (10) is expressed as

$$J_\delta^{[l,l']} = J_\delta^{[l,\infty)} - J_\delta^{[l',\infty)} = x(l)^T P_\delta x(l) - x(l')^T P_\delta x(l').$$

Note that using the state-feedback controller (7) with K_δ given by (9) does not minimize (10) and thus yields a sub-optimal for the finite-horizon problem.

III. Down-sampled controller: design and analysis

In this section we propose a down-sampled controller which solves Problem 1. The design is split between the case where disturbances are absent and when disturbances affect the system.

Definition 1: A sampling period δ is well-defined if and only if the pair $(\Phi_\delta, \Gamma_\delta)$ is controllable [8] and if δ is a non-pathologic sampling period [23].uncontrollable unstable poles.

Assumption 1: We assume that δ_N, δ_S and the baseline period $h = 1$ are non-pathologic sampling periods [23], so that the system remains controllable when sampled with these sampling periods. Moreover, we assume that $J_F^{[0,\infty]} < J_N^{[0,\infty]} < J_S^{[0,\infty]}, \forall x \in \mathbb{R}^n$, which implies $P_F \prec P_N \prec P_S$.

We start by solving Problem 1 in the absence of disturbances, i.e., $w(k) = 0, \forall k$.

A. No disturbance case

Without loss of generality, we present the results assuming that the system is initialized at time $k = 0$.

Theorem 1: Consider system (3) initialized at a given initial condition $x(0)$ with sampling and actuation governed by (5). When $w(k) = 0, \forall k$ the down-sampled controller transition map is defined by

$$\begin{aligned} &\Pi\left(x(k), q(k-1) = q_F\right) \triangleq \begin{cases} q_S & \text{if } x(k) \in \mathcal{G}_F \\ q_F & \text{otherwise} \end{cases} \\ &\Pi\left(x(k), q(k-1) = q_S\right) \triangleq q_S \end{aligned} \tag{11}$$

where

$$\mathcal{G}_F \triangleq \{x(k) \in \mathbb{R}^n \mid x(k)^T \Lambda_F x(k) \leq \sigma_F\}, \tag{12}$$

where $\Lambda_F = P_S - P_F$ and $\sigma_F = x(0)^T \left(P_N - P_F\right) x(0)$. If $x(0) \neq 0$ and since no disturbances occur, there is a single switching between the fast and slow mode which occurs at the switching time

$$t_s = \inf\{k \geq 0 : x(k)^T \Lambda_F x(k) \leq \sigma_F\}. \tag{13}$$

The control input $u(k)$ in (6) has K_F and K_S given by (9) for the baseline and slow period δ_S, respectively. Through this design, the down-sampled controller achieves:

1) a cost no larger than a nominal LQ controller and hence stability of the closed-loop system, while
2) utilizing a lower number of samples than the nominal controller.

Proof: We start by deriving the switching conditions, together with the switching instant t_s. Note that since there are no disturbances, no normalization of the costs is required since $J_{DS}^{[0,\infty)}$ and $J_N^{[0,\infty)}$ both converge to a fixed value. Thus, one must guarantee that $J_{DS}^{[0,\infty)} \leq J_N^{[0,\infty)}$ in order to provide a valid solution to Problem 1.

For $x(0) \neq 0$, the system will always initially be in a fast mode since $J_S(x(0)) > J_N(x(0))$. Thus, the control cost of the down-sampled controller from time $k = 0$ is given by

$$\begin{aligned} J_{DS}^{[0,\infty)} &= J_F^{[0,\infty)} - J_F^{[t_s,\infty)} + J_S^{[t_s,\infty)}, \\ &= x(0)^T P_F x(0) - x(t_s)^T P_F x(t_s) + x(t_s)^T P_S x(t_s), \end{aligned} \tag{14}$$

which has to be guaranteed to be smaller or equal than the nominal control cost

$$J_N^{[0,\infty)} = x(0)^T P_N x(0). \tag{15}$$

The region $\mathcal{G}_F$ in (12) as well as the switching instant (13), are achieved by re-arranging the above terms and setting the inequality. Hence, there is a single switch from the fast mode q_F to the slow mode q_S when $x(k)$ reaches region $\mathcal{G}_F$. After this moment, the system stays in mode q_S since no disturbance affects the system. This defines the switching condition (11).

The switching condition guarantees by design the stability of the closed-loop system since the infinite-horizon cost of the down-sampled controller (14) is guaranteed to be bounded by a bounded infinite-horizon cost (15).

We now prove the existence of the switching time given by (13). Due to the fact that the baseline period and δ_S are well defined and $\delta_S > \delta_F$, it holds that $P_S \succ P_F$. Moreover, $x(k) = \bar{\Phi}_F^k x(0)$, where $\bar{\Phi}_F = \Phi_F + \Gamma_F K_F$ is the closed-loop system matrix for the baseline period and since $\bar{\Phi}_F$ is a Schur matrix since K_F is a stabilizing state-feedback gain. By taking the limit of the left-hand side of the switching condition (13) with $\Lambda_F = P_S - P_F \succ 0$, it holds that

$$\lim_{k\to\infty} x(k)^T \Lambda_F x(k) = \lim_{k\to\infty} x(0)^T \bar{\Phi}_F^{k^T} \bar{P} \bar{\Phi}_F^k x(0) = 0,$$

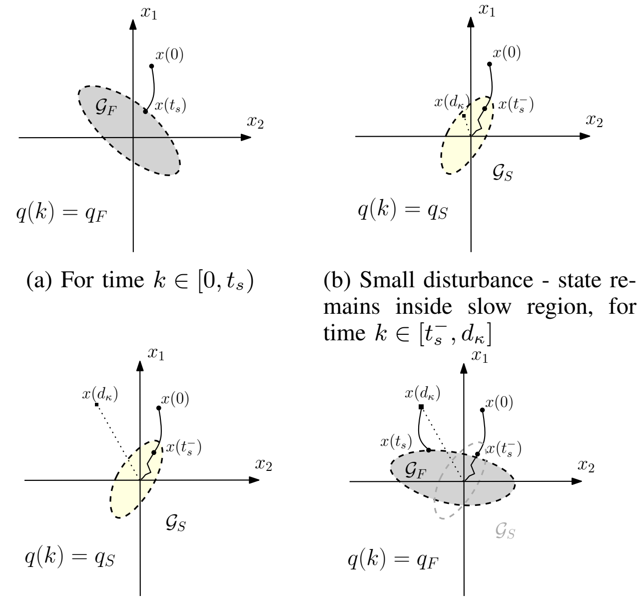
since $\bar{\Phi}_F$ is Schur and $\Lambda_F \succ 0$. Thus, for t_s to exist, $\sigma_s > 0$, which is always true since $P_N \succ P_F$ as $\delta_N > \delta_F$.

The number of transmissions performed by the down-sampled controller over an horizon T is $\Sigma_{DS}^{[0,T]} = t_s + \left\lfloor \frac{T-t_s}{\delta_S} \right\rfloor + 1$, while for the nominal controller is $\Sigma_N^{[0,T]} = \left\lfloor \frac{T}{\delta_N} \right\rfloor + 1$. The rates of both controllers are then given by $R_{DS}^{[0,T]} = \frac{1}{T}\Sigma_{DS}^{[0,T]}$ and $R_N^{[0,T]} = \frac{1}{T}\Sigma_N^{[0,T]}$. Since $\delta_N < \delta_S$, it holds that

$$\begin{aligned} \lim_{T\to\infty} R_N^{[0,T]} - R_{DS}^{[0,T]} &= \lim_{T\to\infty} \frac{1}{T}\left(\Sigma_N^{[0,T]} - \Sigma_{DS}^{[0,T]}\right) \\ &= \frac{1}{\delta_N} - \frac{1}{\delta_S} > 0, \end{aligned}$$

guaranteeing that a lower number of samples are transmitted over the communication network with the down-sampled controller. ∎

An illustration of the behavior of the down-sampled controller until time t_s is given in Fig. 3a. Afterwards, the system state will remain inside $\mathcal{G}_F$ since no disturbances affect the system.



(a) For time $k \in [0, t_s)$

(b) Small disturbance - state remains inside slow region, for time $k \in [t_s^-, d_\kappa]$

(c) Large disturbance, for time $k \in [t_s^-, d_\kappa)$

(d) Large disturbance - trajectory after disturbance, for time $k \in [d_\kappa, t_s)$

Fig. 3: Illustration of the different behaviors of the down-sampled controller with and without disturbances. The evolution of the system from its initial condition until the first mode transition is depicted in a). The disturbance in b) is not large enough to require fast sampling. The case in c) depicts when the disturbance takes the state outside of the slow region, requiring the usage of fast sampling followed by slow sampling which is depicted in d).

B. Disturbance case

We now introduce the design and analysis of the down-sampled controller when disturbances affect the system.

Assumption 2: For simplicity of presentation, we assume that the disturbance occurs synchronously with the nominal controller sampling instants, i.e., $d_\kappa \in \mathcal{K}_{\delta_N}^{[0,\infty)}$. The down-sampled controller without this assumption is given in [24]. For the analysis presented in this section, we only consider the interval of time between disturbances $k \in [d_\kappa, d_{\kappa+1})$, $\forall k$ as the disturbance are impulse disturbances which set the value of $x(k+1) = w(k)$ for $k = d_\kappa$, $\forall\kappa$. Thus, by making sure that the down-sampled controller solves Problem 1 under each interval $k \in [d_\kappa, d_{\kappa+1})$, $\forall\kappa$, one guarantees that the down-sampled controller is a solution to Problem 1 for the interval $k \in [0, \infty)$.

We now introduce the following lemma which will be used for the derivation of the down-sampled controller.

Lemma 1: Since the disturbance interval is lower bounded by δ_d, there exists an $\epsilon > 0$ such that

$$J_N^{(d_{\kappa+1},\infty)} \leq \epsilon x(d_\kappa)^T x(d_\kappa), \tag{16}$$

where $x(d_\kappa)$ is the state at the last disturbance instant. The value of ϵ is given by

$$\epsilon = \lambda_{max}\left(\tilde{\Phi}_N^T P_N \tilde{\Phi}_N\right). \tag{17}$$

Proof: The nominal cost at time $d_\kappa + \delta_d$ is given by

$$J_N^{(d_\kappa+\delta_d,\infty)} = x(d_\kappa)^T \tilde{\Phi}_N^T P_N \tilde{\Phi}_N x(d_\kappa),$$

where $\tilde{\Phi}_N = (\Phi_N - \Gamma_N K_N)^{\frac{\delta_d}{\delta_N}}$. From [25], and since $J_N^{(d_\kappa+\delta_d,\infty)} \geq J_N^{(d_{\kappa+1},\infty)}$, it holds that

$$J_N^{(d_{\kappa+1},\infty)} \leq J_N^{(d_\kappa+\delta_d,\infty)} \leq \lambda_{max}\left(\tilde{\Phi}_N^T P_N \tilde{\Phi}_N\right) x(d_\kappa)^T x(d_\kappa)$$

Thus, one can define ϵ as (17) and bound (16). This completes the proof. ∎

1) **Switching condition:** Next we give the main result of the chapter, which characterizes the transition map (4) illustrated in Fig. 3.

Theorem 2: Assume the system (3) is initialized at a given initial condition $x(0)$ with sampling and actuation governed by (5) and $t_s^- = 0$ at starting time. The transition map is defined as

$$\Pi\left(x(k), q(k-1) = q_F\right) \triangleq \begin{cases} q_S & \text{if } x(k) \in \mathcal{G}_F \\ q_F & \text{otherwise} \end{cases} \tag{18}$$

$$\Pi\left(x(k), q(k-1) = q_S\right) \triangleq \begin{cases} q_F & \text{if } x(k) \in \mathcal{G}_S \wedge k = d_\kappa, \forall\kappa \\ q_S & \text{otherwise} \end{cases} \tag{19}$$

where

$$\mathcal{G}_F \triangleq \left\{x(k) \in \mathbb{R}^n \mid x(k)^T \Lambda_F x(k) \leq \sigma_F\right\}, \tag{20}$$

$$\mathcal{G}_S \triangleq \left\{x(k) \in \mathbb{R}^n \mid \begin{bmatrix} x(k) \\ 1 \end{bmatrix}^T \Lambda_S \begin{bmatrix} x(k) \\ 1 \end{bmatrix} > \sigma_S\right\},$$

and the region parameters are given by

$$\begin{aligned} &\Lambda_F = P_S - P_F, \\ &\sigma_F = x(t_s^-)^T \left(P_N - P_F - \epsilon I\right) x(t_s^-), \\ &\Lambda_S = \begin{cases} \begin{bmatrix} P_S - P_N + \epsilon I & 0 \\ 0 & 0 \end{bmatrix} & \text{if } \mathrm{rem}(k - t_s^-, \delta_S) = 0, \\ \begin{bmatrix} P_F - P_N + \epsilon I & E^T \\ E & G \end{bmatrix} & \text{otherwise,} \end{cases} \\ &\sigma_S = J_N^{[d_{\kappa-1},k]} - J_{DS}^{[d_{\kappa-1},k]}, \end{aligned} \tag{21}$$

where F, E and G are given in the Appendix by (29).

The switching instant t_s under sporadic impulsive disturbances $w(k)$ is given by:

$$t_s = \begin{cases} \inf\{k > t_s^- : \Pi\left(x(k), q(k-1) = q_F\right) = q_S\} \\ \inf\{k > t_s^- : \Pi\left(x(k), q(k-1) = q_S\right) = q_F\} \end{cases} \tag{22}$$

The control input $u(k)$ is defined by (6) with K_F and K_S given by (9) for the baseline period and δ_S, respectively.

Through this design, the down-sampled controller achieves a normalized cost smaller or equal to that of the nominal controller, i.e., $\bar{J}_{DS} \leq \bar{J}_N$.

Proof: The goal of the down-sampled controller is to achieve $\bar{J}_{DS} \leq \bar{J}_N$ under sporadic impulse disturbances $w(k)$. In this case, after entering the slow mode, the state may only be brought to a fast mode by a disturbance and thus a re-switching from slow to fast mode must only occur at the time a disturbance occurs, i.e., $k = d_\kappa$.

Consider the transition map (18) and (19). When the system is in fast sampling mode, its state will be brought to the region $\mathcal{G}_F$ enabling a switch to the slow sampling mode (see Fig. 3a). When the system is in the slow sampling mode, a disturbance must be large enough to bring the system state outside of $\mathcal{G}_S$, as depicted in Figs. 3c and 3d. If the disturbance is not large enough, the system will remain in a slow sampling mode (see Fig. 3b).

The region $\mathcal{G}_F$ and transition map (18) is derived in the same manner of the switching condition in Proposition 1. Rewriting (14) for the interval $[d_\kappa, d_{\kappa+1}]$ we have that

$$J_{DS}^{[d_\kappa,d_{\kappa+1})} = J_F^{[d_\kappa,\infty)} - J_F^{[t_s,\infty)} + J_S^{[t_s,\infty)} - J_S^{[d_{\kappa+1},\infty)}. \tag{23}$$

Given Lemma 1 and (23), we can derive the following inequalities

$$\begin{aligned} J_{DS}^{[d_\kappa,d_{\kappa+1})} &\leq J_F^{[d_\kappa,\infty)} - J_F^{[t_s,\infty)} + J_S^{[t_s,\infty)}, \\ J_N^{[d_\kappa,d_{\kappa+1})} &\geq J_N^{[d_\kappa,\infty)} - \epsilon x(d_\kappa)^T x(d_\kappa). \end{aligned}$$

Thus, to guarantee that $J_{DS}^{[d_\kappa,d_{\kappa+1})} \leq J_N^{[d_\kappa,d_{\kappa+1})}$, one can require that $J_F^{[d_\kappa,\infty)} - J_F^{[t_s,\infty)} + J_S^{[t_s,\infty)} \leq J_N^{[d_\kappa,\infty)} - \epsilon x(d_\kappa)^T x(d_\kappa)$, which is enforced by $\mathcal{G}_F$ in (20) with parameters given by (21).

The transition map for the slow sampling mode is based on the requirement that the slow rate is used from the time the disturbance occurs at time $k = d_\kappa$ onwards, if and only if, the total cost (current cost until time k plus the cost to-go until the next disturbance time $d_{\kappa+1}$) is kept lower than that of the nominal controller. The above condition is formulated as

$$J_{DS}^{[d_{\kappa-1},d_\kappa]} + J_S^{[d_\kappa,d_{\kappa+1})} \leq J_N^{[d_{\kappa-1},d_\kappa]} + J_N^{[d_\kappa,d_{\kappa+1})},\ \forall\kappa. \tag{24}$$

Recall the sampler unit keeps in memory the cost history of both controllers: $J_{DS}^{[d_{\kappa-1},k]}$ and $J_N^{[d_{\kappa-1},k]}$. Using bound (16) from assumption 1, inequality (24) is guaranteed if

$$J_{DS}^{[d_{\kappa-1},k]} + J_S^{[k,\infty)} \leq J_N^{[d_{\kappa-1},k]} + J_N^{[k,\infty)} - \epsilon x(k)^T x(k),\ k = d_\kappa,$$

where $J_S^{[k,\infty)}$ is given by (30). Enforcing this inequality in the form of the region $\mathcal{G}_S$, we arrive to the transition map in (19) with parameters Λ_S and σ_S given by (21).

The switching instant t_s in (22) follows directly from the definitions of the transition map in (18) and (19). This switching condition guarantees by design the stability of the closed-loop system since the normalized cost of the down-sampled controller is guaranteed to be bounded by the normalized nominal cost at each disturbance interval. This completes the proof. ∎

The down-sampled controller designed in Proposition 2 does not give any guarantees w.r.t. the rate of transmission since it is solely designed to guarantee that the performance of the down-sampled controller is no worse than the performance of the nominal controller. We analyze this issue next by providing a minimum allowed disturbance interval which guarantees the fulfillment of the above property.

2) **Minimum allowed disturbance interval $\delta_d^{\min}$:** We now propose a worst-case analysis which finds the minimum allowed disturbance interval $\delta_d^{\min}$ so that for $d_{\kappa+1} - d_\kappa \geq \delta_d^{\min}$ the rate of transmission of the down-sampled controller is lower than the nominal controller, i.e., $\bar{R}_{DS} < \bar{R}_N$.

Hence, the down-sampled controller proposed in Proposition 2 is a solution to Problem 1 under sporadic impulse disturbances.

In order to perform this analysis we require the knowledge of the largest value (worst-case) of the switching time t_s for any initial condition of the state. Such value provides the case where the largest amount of samples/transmissions are utilized by the down-sampled controller and is achieved through the following result.

Lemma 2: Let the switching instant be given by (22) with $q(0) = q_F$ and δ_S and δ_N be fixed. The down-sampled controller has the maximum switching instant $\bar{t}_s$ for a given system (3), for any initial condition $x(0)$, defined by:

$$\bar{t}_s = \inf\left\{k > 0 : \lambda_{max}\left(\bar{\Phi}^{k^T}\Lambda_F\bar{\Phi}^k - P_N + P_F + \epsilon I\right) \leq 0\right\}, \tag{25}$$

where $\bar{\Phi} = \Phi + \Gamma K_F$ is the closed-loop system matrix for the fast period.

Proof: The proof follows from the switching condition and the fact that $x^T Z x \leq 0, \forall x \in \mathbb{R}^n$, if and only if, $\lambda_{max}(Z) \leq 0$ for any symmetric matrix $Z = Z^T$ (see [25]). ■

We can finally characterize the minimum allowed disturbance interval.

Proposition 1: Let δ_N and δ_S be fixed. The minimum allowed disturbance period $\delta_d^{\min}$ which guarantees a solution to Problem 1 for the down-sampled controller designed in Proposition 2 is

$$\begin{aligned} \min_{\delta_d^{\min}} \quad & \delta_d^{\min} \\ \text{s.t.} \quad & \bar{t}_s + \left\lfloor \frac{\delta_d^{\min} - \bar{t}_s}{\delta_S} \right\rfloor + 1 < \left\lfloor \frac{\delta_d^{\min}}{\delta_N} \right\rfloor \end{aligned}$$

where $\bar{t}_s$ is given by (25) in Lemma 2, with ϵ defined in (17) with $\tilde{\Phi}_N = (\Phi_N - \Gamma_N K_N)^{\frac{\delta_d^{\min}}{\delta_N}}$.

Proof: The constraint comes directly from the requirement that during the interval of length $\delta_d^{\min}$,

$$\Sigma_{DS}^{\delta_d^{\min}} < \Sigma_N^{\delta_d^{\min}} \Rightarrow R_{DS}^{\delta_d^{\min}} < R_N^{\delta_d^{\min}},$$

where $\Sigma_{DS}^{\delta_d^{\min}} = \left(\bar{t}_s + \left\lfloor \frac{\delta_d^{\min} - \bar{t}_s}{\delta_S} \right\rfloor + 1\right) + 1$, and $\Sigma_N^{\delta_d^{\min}} = \left\lfloor \frac{\delta_d^{\min}}{\delta_N} \right\rfloor + 1$. An extra transmission is added to the down-sampled controller since due to a disturbance, this controller will switch to the fast period and perform a new actuation. The number of transmissions of the nominal controller is kept the same since this controller does not alter its behavior when a disturbance affects the system. ■

IV. Examples

We now provide an evaluation of the proposed down-sampled controller. We start by analysing the performance of the down-sampled controller when compared to other two methods proposed in the literature [20] for the case when no disturbances affect the system, as in these works no disturbances are considered. Afterwards, we analyse the performance of the down-sampled controller under sporadic impulse disturbances. The evaluation is performed for two different plants, a two-mass and spring system from [20], and a classic double-integrator system [8].

Two-mass and spring: The two-mass and spring system from [26] is modelled as a 4$^{\text{th}}$-order continuous-time system (1) with parameters

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2\pi^2 & 2\pi^2 & 0 & 0 \\ 2\pi^2 & -2\pi^2 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

and with initial condition $x(0) = \begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix}^T$ and control cost matrices $Q_c = \text{diag}(1, 1, 0, 0)$ and $R_c = 0.1$.

Double integrator: The double integrator [8] is modelled as a 2$^{\text{nd}}-$order continuous-time system (1) with parameters

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and control cost matrices $Q_c = \text{diag}(1, 1)$ and $R_c = 1$.

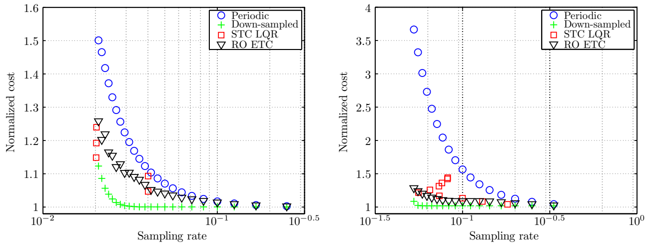
A. No disturbance case

Under this formulation, we compare the down-sampled controller to the traditional periodic controller, the roll-out event-triggered controller (RO-ETC) proposed in [20] and the self-triggered linear quadratic regulator (STC LQR) proposed in [21]

Consider the two-mass and spring system. Fig. 4a presents the results of the control performance of the two-mass and spring system for different average sampling periods, under a traditional periodic controller, the down-sampled controller, the STC LQR [21] and the RO ETC (using the same algorithm parameters as [20]). The control performance for all cases is normalized on the continuous-time control performance which is the lowest possible cost under the given Q_c, R_c and $x(0)$. The displayed down-sampled controller cost is the minimum achievable cost for a nominal period $\delta_N \in]h, \delta]$ and the corresponding δ_S was chosen as the minimum period ensuring an average period of δ over the horizon T. In this case, we set $T = 1150$ s and varied $\delta \in [0.02, 0.48]$ s. As for the STC LQR, we varied the algorithm's tuning parameter $\beta \in \{1.05, 1.1, ..., 1.25\}$. The cost achieved by the down-sampled controller is lower than the other methods.

Consider now the double integrator system. Fig. 4b presents the results of the control performance for this system under the different average sampling periods and controller implementations. In this case, the control cost is averaged over 20 different initial conditions $x(0)$ equally spaced in the unit disk. Moreover, $T = 350$ s, $\delta \in [0.25, 5]$ s and baseline period $h = 0.25$ s. As for the STC LQR, we varied again the algorithm's tuning parameter $\beta \in \{1.05, 1.1, ..., 1.5\}$. As in the previous case, the down-sampled controller has a cost smaller than the other algorithms.

In summary, the down-sampled controller has an advantage over the STC LQR in the fact that, not only a lower



(a) Two-mass and spring (b) Double integrator

Fig. 4: Comparison among different aperiodic control algorithms and the traditional periodic controller in the absence of disturbances.

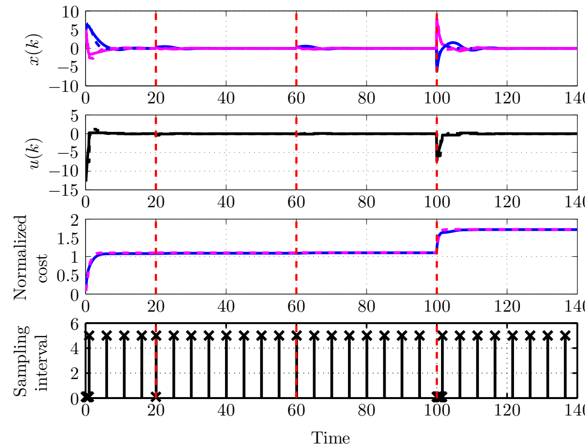


Fig. 5: Time-responses for the down-sampled controller (solid line) and the nominal controller (dashed line) under non-zero disturbances. Sampling intervals are shown for the DS controller, where a cross represents the transmission instant and its height the time since last transmission. Dashed vertical lines denote the instant where a disturbance occurs.

control cost is achieved for the same average sampling period, but we can guarantee a specific cost for a selected average sampling period. This is not achieved in the STC LQR since for a specific β value there is no guarantee what the sampling period will be. As expected, the cost difference to the RO ETC is not as large as to the STC LQR since the RO ETC method is based on a roll-out strategy which is known for being efficient on solving combinatorial optimization problems. Nevertheless, the down-sampled controller requires a very low computational effort and is based on simple switching rules, as opposed to the computationally demanding roll-out method proposed in [20].

B. Disturbance case

The performance of the down-sampled controller under sporadic impulse disturbances is now analyzed on the double integrator system with baseline period $h = 0.1\,\mathrm{s}$, $\delta_N = 10$ and $\delta_S = 50$, during a $T = 140\,\mathrm{s}$ simulation interval. The initial condition is set to $x(0) = \begin{bmatrix}10 & 10\end{bmatrix}^T$ and the disturbance occurs three times at $\{d_1, d_2, d_3\} = \{20, 60, 100\}\,\mathrm{s}$ with values

$$\{w(d_1), w(d_2), w(d_3)\} = \left\{ \begin{bmatrix}0.25\\0.25\end{bmatrix}, \begin{bmatrix}0.25\\0.25\end{bmatrix}, \begin{bmatrix}-6.05\\-7.85\end{bmatrix} \right\}.$$

Using Proposition 1 we obtain that the minimum disturbance interval to guarantee that the down-sampled controller utilizes less transmissions than the nominal controller is $\delta_d^{\min} = 500 = 50\,\mathrm{s}$.

Fig. 5 depicts the time-response of the state and control input of the system under the dual-rate controller (solid line) and nominal controller (dashed line) as well as the sampling instants performed by the dual-rate controller. The normalized control cost of both controllers for the same experiment w.r.t. the continuous-time control cost without disturbances is also depicted in this figure. One can observe that for both controllers the system is stable and that the down-sampled controller cost is always lower or equal than the nominal controller. Moreover, the total number of transmissions during the experiment was $\Sigma_{DS} = 68$ and $\Sigma_N = 141$. As expected, the obtained $\delta_d^{\min}$ is conservative since it is computed for the worst-case scenario as defined in Sec. III. We notice that 9 fast sample and actuation instants occur after the experiment is initialized and at the moment of the first disturbance. No fast sample is required when the second disturbance occurs. This is due to the fact that the disturbance value is small and the down-sampled controller cost is guaranteed to be below the nominal controller cost by the current slow sampling action. When the last and large disturbance affects the system, 22 fast sample and actuation instants are required.

V. Conclusion

In this paper, we have introduced a down-sampled controller which reduces network usage while guaranteeing a specified LQ performance under sporadic impulse disturbances. This method is based on fast and slow sampling intervals with the intuition that the closed-system benefits by being brought quickly to steady-state conditions, while behaving satisfactory when being actuated at a slow rate once at those conditions. Through simulations, we demonstrated the benefits of using the down-sampled policy instead of a fixed-rate one. Moreover, we show that this simple mechanism provides large savings with respect to network usage when compared to the traditional periodic time-triggered controller and state-of-the-art aperiodic controller mechanisms.

VI. Acknowledgments

The authors would like thank D. Quevedo for the useful suggestions and discussions and D. Antunes for providing the implementation files for the algorithm proposed in [20].

References

[1] J. Åkerberg, M. Gidlund, and M. Björkman, “Future research challenges in wireless sensor and actuator networks targeting industrial automation,” in *9th IEEE International Conference on Industrial Informatics (INDIN)*, July 2011.

[2] R. Dorf, M. Farren, and C. Phillips, “Adaptive sampling frequency for sampled-data control systems,” *Automatic Control, IRE Transactions on*, vol. 7, no. 1, pp. 38–47, 1962.

[3] P. Colaneri, R. Scattolini, and N. Schiavoni, "Stabilization of multirate sampled-data linear systems," *Automatica*, vol. 26, no. 2, pp. 377–380, 1990.

[4] B. Lincoln and B. Bernhardsson, "LQR optimization of linear system switching," *Automatic Control, IEEE Transactions on*, vol. 47, no. 10, pp. 1701–1705, 2002.

[5] P. Martí, C. Lin, S. Brandt, M. Velasco, and J. Fuertes, "Optimal state feedback based resource allocation for resource-constrained control tasks," in *Proceedings of the 23rd IEEE Real-Time Systems Symposium*, 2004.

[6] A. Saifullah, C. Wu, P. B. Tiwari, Y. Xu, Y. Fu, C. Lu, and Y. Chen, "Near optimal rate selection for wireless control systems," in *Proceedings of the 2012 IEEE 18th Real Time and Embedded Technology and Applications Symposium*, RTAS '12, pp. 231–240, 2012.

[7] K. Åström and B. Bernhardsson, "Comparison of periodic and event based sampling for first-order stochastic systems," in *14th IFAC World Congress*, 1999.

[8] K. Åström and B. Wittenmark, *Computer controlled systems*. Prentice Hall Englewood Cliffs, NJ, 1990.

[9] M. Rabi and K. H. Johansson, "Scheduling packets for event-triggered control," *Proceedings of 10th European Control Conference*, pp. 3779 – 3784, 2009.

[10] R. Blind and F. Allgower, "Analysis of networked event-based control with a shared communication medium: Part i-pure aloha," in *IFAC World Congress*, 2011.

[11] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks," *IEEE Transactions on Automatic Control*, vol. 52, pp. 1680 – 1685, sept. 2007.

[12] M. Mazo and P. Tabuada, "On event-triggered and self-triggered control over sensor/actuator networks," in *Decision and Control, 2008. CDC 2008. 47th IEEE Conference on*, pp. 435–440, 2008.

[13] M. Velasco, P. Martí, and E. Bini, "On lyapunov sampling for event-driven controllers," in *Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009. Proceedings of the 48th IEEE Conference on*, pp. 6238–6243, IEEE, 2009.

[14] J. Lunze and D. Lehmann, "A state-feedback approach to event-based control," *Automatica*, vol. 46, pp. 211–215, January 2010.

[15] A. Seuret and C. Prieur, "Event-triggered sampling algorithms based on a lyapunov function," in *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*, pp. 6128–6133, IEEE, 2011.

[16] E. Garcia and P. J. Antsaklis, "Optimal model-based control with limited communication," in *19th IFAC World*, August 2014.

[17] A. Eqtami, D. V. Dimarogonas, and K. J. Kyriakopoulos, "Event-based model predictive control for the cooperation of distributed agents," in *American Control Conference (ACC), 2012*, pp. 6473–6478, 2012.

[18] E. Henriksson, D. E. Quevedo, H. Sandberg, and K. H. Johansson, "Self-triggered model predictive control for network scheduling and control," in *IFAC Advanced Control of Chemical Processes*, 2012.

[19] W. Heemels, K. H. Johansson, and P. Tabuada, "An introduction to event-triggered and self-triggered control," in *Decision and Control (CDC), 2012 IEEE 51st Annual Conference on*, pp. 3270–3285, 2012.

[20] D. Antunes and W. Heemels, "Rollout event-triggered control: Beyond periodic control performance," *Automatic Control, IEEE Transactions on*, vol. to appear, 2014.

[21] T. Gommans, D. Antunes, M. Donkers, P. Tabuada, and W. Heemels, "Self-triggered linear quadratic control," *Automatica*, vol. to appear, 2014.

[22] L. Shi, Y. Yuan, and J. Chen, "Finite horizon lqr control with limited controller-system communication," *Automatic Control, IEEE Transactions on*, vol. 58, no. 7, pp. 1835–1841, 2013.

[23] T. Chen and B. A. Francis, *Optimal sampled-data control systems*. Springer Verlag, 1995.

[24] J. Araújo, *Design, Implementation and Validation of Resource-Aware and Resilient Wireless Networked Control Systems*. PhD thesis, KTH Royal Institute of Technology, 2014.

[25] R. A. Horn and C. R. Johnson, *Matrix analysis*. Cambridge university press, 2012.

[26] D. Antunes, W. Heemels, and P. Tabuada, "Dynamic programming formulation of periodic event-triggered control: Performance guarantees and co-design.," in *Decision and Control (CDC), 2012 IEEE 51st Annual Conference on*, pp. 7212–7217, 2012.

APPENDIX

A. Derivation of $J_S^{[l,\infty)}$

The infinite-horizon cost-to-go for the down-sampled controller for $q(k) = q_S$ from time $k = l$, $J_S^{[l,\infty)}$ is defined as

$$J_S^{[l,\infty)} = \begin{cases} x(l)^T P_S x(l) & \text{if } \mathrm{rem}(l - t_s^-, \delta_S) = 0 \\ \hat{J}_S^{[l,\infty)} & \text{otherwise} \end{cases} \tag{26}$$

for the case a slow transmission occurs at time $k = l$ or not, respectively, where cost $\hat{J}^{[l,\infty)}$ is given by

$$\hat{J}_S^{[l,\infty)} = \hat{J}_S^{[l,n_S^+-1]} + x(n_S^+)^T P_S x(n_S^+) \tag{27}$$

where n_S^+ is the next sampling instant of the down-sampled controller after time $k = l$, and given by $n_S^+ = t_s^- + \left(\left\lfloor \frac{l-t_s^-}{\delta_S} \right\rfloor + 1\right)\delta_S$. Note that $\hat{J}_S^{[l,n_S^+-1]}$ is given by (10) iterated from time $k \in [l, n_S^+ - 1]$ where the control input is the last computed control input which we denote by $\bar{u}$. If no switching occurred between time $k = [n_S^+ - \delta_S, l]$, then $\bar{u} = u(n_S^+ - \delta_S)$, otherwise $\bar{u} = u(t_s^-)$. Thus, $\hat{J}_S^{[l,n_S^+-1]}$ is given by

$$\hat{J}_S^{[l,n_S^+-1]} = \sum_{i=0}^{n_S^+ + 1 - l} x(i)^T Q x(i) + 2x(i)^T N \bar{u} + \bar{u}^T R \bar{u}. \tag{28}$$

We can then substitute (28) in (27) and rewriting (27) as a function of $x(l)$ arriving to

$$\hat{J}_S^{[l,\infty)} = \begin{bmatrix} x(l) \\ 1 \end{bmatrix}^T \begin{bmatrix} F & E \\ E^T & G \end{bmatrix} \begin{bmatrix} x(l) \\ 1 \end{bmatrix},$$

where

$$\begin{aligned} F &= \tilde{\Phi}(n_S^+ - l)^T P_S \tilde{\Phi}(n_S^+ - l) + \sum_{i=0}^{n_S^+ + 1 - l} \tilde{\Phi}(i)^T Q \tilde{\Phi}(i) \\ E &= \Bigg(\tilde{\Phi}(n_S^+ - l)^T P_S \tilde{\Gamma}(n_S^+ - l) \\ &\quad + \sum_{i=0}^{n_S^+ + 1 - l} \tilde{\Phi}(i)^T Q \tilde{\Gamma}(i) + \tilde{\Phi}(i)^T N \Bigg) \bar{u} \\ G &= \bar{u}^T \Bigg(\tilde{\Gamma}(n_S^+ - l)^T P_S \tilde{\Gamma}(n_S^+ - l) \sum_{i=0}^{n_S^+ + 1 - l} \tilde{\Gamma}(i)^T Q \tilde{\Gamma}(i) \\ &\quad + 2\tilde{\Gamma}(i) N + R \Bigg) \bar{u} \end{aligned} \tag{29}$$

and $\tilde{\Phi}(n) = \Phi^n$ and $\tilde{\Gamma}(n) = \sum_{j=0}^{n-1} \Phi^j \Gamma$. Finally, we can rewrite (26) as

$$J_S^{[l,\infty)} = \begin{cases} \begin{bmatrix} x(l) \\ 1 \end{bmatrix}^T M_1 \begin{bmatrix} x(l) \\ 1 \end{bmatrix} & \text{if } \mathrm{rem}(l - t_s^-, \delta_S) = 0 \\ \begin{bmatrix} x(l) \\ 1 \end{bmatrix}^T M_2 \begin{bmatrix} x(l) \\ 1 \end{bmatrix} & \text{otherwise} \end{cases} \tag{30}$$

where

$$M_1 = \begin{bmatrix} P_S & 0 \\ 0 & 0 \end{bmatrix}, \; M_2 = \begin{bmatrix} F - P_S & E^T \\ E & G \end{bmatrix}.$$