SOS presentation: SOS is not obviously automatizable, even approximately

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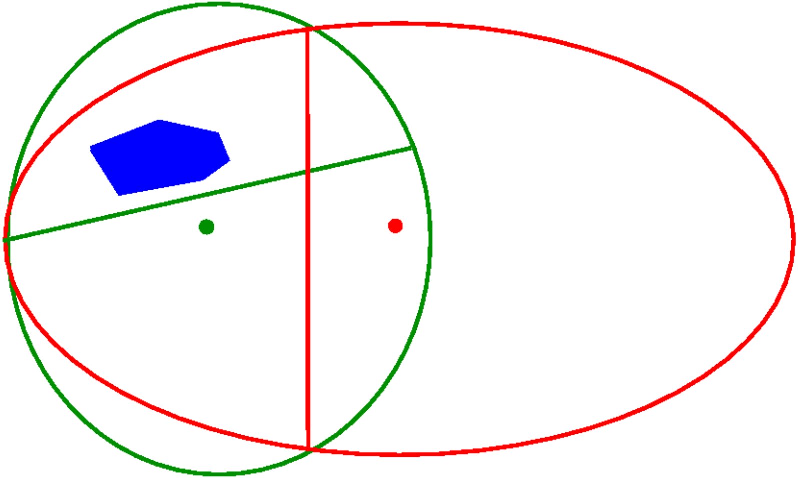
Introduction

We will only look at feasibility, no optimization.
We look at the Ellipsoid algorithm for solving SDP.

- Needs a polynomial time separation oracle for the constraints.
  PSD-ness constraint has a polynomial time separation oracle.
- Needs technical assumptions on solution space.

Ellipsoid algorithm



Ellipsoid algorithm

Technical assumptions for Ellipsoid algorithm

$V =$ feasible region for a given convex optimization problem.
Parameters:

1. R: radius of initial L_2-norm ball containing V
2. r: number such that

$$V \neq \emptyset \iff V \text{ contains some } L_2\text{-norm ball of radius } r$$

Ellipsoid algorithm

Ellipsoid algorithm:

- Start with the ball of size R (initial ellipsoid).
- Repeatedly find a violated constraint for the center, and construct the next ellipsoid based on that.

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Termination:

1. Center of ellipsoid is feasible.
2. Volume gets too small, so there is no solution.

The Ellipsoid algorithm runs in time polynomial in $\log(R/r)$.

Ellipsoid algorithm running time

For linear programming, let

$$L = \text{the total number of bits in all coefficients together}$$

Can always take, without loss of generality, $R = O(2^L)$.
Usually however, $r = 0$.
Can modify the problem by changing

$$a_i x = b_i \implies -\epsilon \leq a_i x - b_i \leq \epsilon$$

for small enough ϵ.
Therefore, running time on LP is polynomial in L.

Degree-d SOS running time

Degree-d SOS can typically be formulated in $n^{O(d)}$ bits.
So $L = n^{O(d)}$, but what are r and R?
The paper gives an example where every SOS proof has very large coefficients:

$$\text{Very large} = 2^{\Omega(2^n)}$$

If we start with an ellipsoid centered at $\mathbf{0}$, then $R = 2^{\Omega(2^n)}$.
It seems that $r = 0$ (?)

Preliminary observation

SDP solutions can need doubly exponential coefficients:

$$x_1 = 2, x_{i+1} = x_i^2 \forall i$$

Solution:

$$x_n = 2^{2^{n-1}}$$

The example with large coefficients

Given the constraints

$$\begin{aligned} 2x_1y_1 &= y_1, & 2x_2y_2 &= y_2, & 2x_3y_3 &= y_3, & 2x_ny_n &= y_n \\ x_1^2 &= x_1, & x_2^2 &= x_2, & x_3^2 &= x_3, & x_n^2 &= x_n \\ y_1^2 &= y_2, & y_2^2 &= y_3, & y_3^2 &= y_4, & y_n^2 &= 0 \end{aligned}$$

Prove that $p_n(x, y) = x_1 + x_2 + x_3 + \ldots + x_n - 2y_1 \geq 0$.

The example with large coefficients

Given the constraints

$$\begin{aligned}
2x_1y_1 &= y_1, & 2x_2y_2 &= y_2, & 2x_3y_3 &= y_3, & 2x_ny_n &= y_n\\
x_1^2 &= x_1, & x_2^2 &= x_2, & x_3^2 &= x_3, & x_n^2 &= x_n\\
y_1^2 &= y_2, & y_2^2 &= y_3, & y_3^2 &= y_4, & y_n^2 &= 0
\end{aligned}$$

Prove that $p_n(x, y) = x_1 + x_2 + x_3 + \ldots + x_n - 2y_1 \geq 0$.

Solution by hand

- Solve the second row to get $x_i \in \{0, 1\} \, \forall i$.
- Solve the third row to get $y_i = 0 \, \forall i$.

The example with large coefficients

Given the constraints

$$\begin{aligned}
2x_1y_1 &= y_1, & 2x_2y_2 &= y_2, & 2x_3y_3 &= y_3, & 2x_ny_n &= y_n \\
x_1^2 &= x_1, & x_2^2 &= x_2, & x_3^2 &= x_3, & x_n^2 &= x_n \\
y_1^2 &= y_2, & y_2^2 &= y_3, & y_3^2 &= y_4, & y_n^2 &= 0
\end{aligned}$$

Prove that $p_n(x, y) = x_1 + x_2 + x_3 + \ldots + x_n - 2y_1 \geq 0$.
How do we show that SOS needs large coefficients?
We focus on degree-2 SOS here.

The example with large coefficients

Given the constraints

$$
\begin{array}{rrrr}
2x_1y_1 = y_1, & 2x_2y_2 = y_2, & 2x_3y_3 = y_3, & 2x_ny_n = y_n \\
x_1^2 = x_1, & x_2^2 = x_2, & x_3^2 = x_3, & x_n^2 = x_n \\
y_1^2 = y_2, & y_2^2 = y_3, & y_3^2 = y_4, & y_n^2 = 0
\end{array}
$$

Prove that $p_n(x, y) = x_1 + x_2 + x_3 + \ldots + x_n - 2y_1 \geq 0$.

Working "mod the ideal"

Solve

$$p_n(x, y) \equiv \sum_j \ell_j(x, y)^2 \mod (K)$$

Where K is the set of equations above and (K) the generated ideal.

The example with large coefficients

Given the constraints

$$\begin{aligned}
2x_1y_1 &= y_1, & 2x_2y_2 &= y_2, & 2x_3y_3 &= y_3, & 2x_ny_n &= y_n\\
x_1^2 &= x_1, & x_2^2 &= x_2, & x_3^2 &= x_3, & x_n^2 &= x_n\\
y_1^2 &= y_2, & y_2^2 &= y_3, & y_3^2 &= y_4, & y_n^2 &= 0
\end{aligned}$$

Prove that $p_n(x,y) = x_1 + x_2 + x_3 + \ldots + x_n - 2y_1 \geq 0$.

Solution

$$p_n(x,y) \equiv \sum_i (x_i - 2^{2^{i-1}} y_i)^2 \mod (K)$$

The example with large coefficients

$$p_n(x, y) \equiv \sum_j \ell_j(x, y)^2 \mod (K)$$

1. We ignore the "cross terms" $x_i x_j$, $x_i y_j$ and $y_i y_j$ $(i \neq j)$.
   They do not "mix" with the rest through the ideal.

The example with large coefficients

$$p_n(x,y) \equiv \sum_j \ell_j(x,y)^2 \mod (K)$$

1. We ignore the “cross terms” $x_i x_j$, $x_i y_j$ and $y_i y_j$ $(i \neq j)$. They do not “mix” with the rest through the ideal.
2. ℓ_j must have zero constant terms.

Proof

The constant term is of the form $\sum_j c_j^2$ and is not reduced by the ideal, and $p_n(x,y)$ has zero constant term.

The example with large coefficients

Therefore, if $\ell_j = \sum_i a_{ij} x_i + \sum_i b_{ij} y_i$,

$$\sum_j \ell_j(x,y)^2 \equiv \sum_i (A_i^2 x_i^2 + 2M_i x_i y_i + B_i^2 y_i^2) \quad \text{mod } (K, \text{crossterms})$$

where $A_i = \sqrt{\sum_j a_{ij}^2}$, $B_i = \sqrt{\sum_j b_{ij}^2}$ and $M_i = \sum_j a_{ij} b_{ij}$.
Note that by Cauchy-Schwarz, $|M_i| \leq A_i B_i$.

The example with large coefficients

The constraints were

$$\begin{aligned} 2x_1y_1 &= y_1, & 2x_2y_2 &= y_2, & 2x_3y_3 &= y_3, & 2x_ny_n &= y_n \\ x_1^2 &= x_1, & x_2^2 &= x_2, & x_3^2 &= x_3, & x_n^2 &= x_n \\ y_1^2 &= y_2, & y_2^2 &= y_3, & y_3^2 &= y_4, & y_n^2 &= 0 \end{aligned}$$

Therefore

$$\sum_i (A_i^2 x_i^2 + 2M_i x_i y_i + B_i^2 y_i^2) \equiv \sum_i (A_i^2 x_i + M_i y_i + B_i^2 y_{i+1}) \mod (K)$$

where $y_{n+1} = 0$.

The example with large coefficients

$$\sum_i x_i - 2y_1 \equiv \sum_i (A_i^2 x_i + M_i y_i + B_i^2 y_{i+1}) \mod (K, \text{crossterms})$$

Now we can drop the "mod".
This implies

1. $A_i = 1$ for all i.
2. $M_1 = -2$.
3. $M_{i+1} = -B_i^2$.

Combining this with $|M_i| \leq A_i B_i$, we get $B_1 \geq 2$ and $B_{i+1} \geq B_i^2$.
Therefore, $B_n \geq 2^{2^{n-1}}$.
So, the largest coefficient is doubly exponential.

Part 2: Even approximately

Degree-2 SOS proofs of the approximate version

$$p_n(x, y) \geq -o_n(1)$$

needs coefficients of size $2^{\Omega(2^n)}$.
It turns out, we can look at $p_n(x, y) \geq -0.01$.

Analysis of approximate case

We can still disregard cross-terms $x_k x_{k'}$, $x_k y_{k'}$ and $y_k y_{k'}$ $(k \neq k')$.
But linear functions may have non-zero constant terms.
Therefore, if $\ell_j = \sum_i a_{ij} x_i + \sum_j b_{ij} y_i + c_j$, $\sum \ell_j(x,y)^2$ becomes

$$\sum_i (A_i^2 x_i^2 + 2M_i x_i y_i + B_i^2 y_i^2 + 2U_i x_i + 2V_i y_i) + C^2$$

where A_i, B_i and M_i are as before, $U_i = \sum a_{ij} c_j$, $V_i = \sum b_{ij} c_j$,
and $C = \sqrt{\sum_j c_j^2}$.

Analysis of approximate case

$\sum \ell_j(x,y)^2$ becomes

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where A_i, B_i and M_i are as before, $U_i = \sum a_{ij} c_j$, $V_i = \sum b_{ij} c_j$, and $C = \sqrt{\sum_j c_j^2}$.

1. By Cauchy-Schwarz, $|U_i| \leq A_i C$ and $|V_i| \leq B_i C$.
2. Reducing modulo the ideal, we get

$$\sum_i (A_i^2 x_i^2 + 2M_i x_i y_i + B_i^2 y_i^2 + 2U_i x_i + 2V_i y_i) + C^2 =$$

$$\sum_i ((A_i^2 + 2U_i) x_i + (M_i + 2V_i) y_i + B_i^2 y_{i+1}) + C^2$$

Analysis of approximate case

$$\sum_i (A_i^2 x_i^2 + 2M_i x_i y_i + B_i^2 y_i^2 + 2U_i x_i + 2V_i y_i) + C^2 =$$
$$\sum_i ((A_i^2 + 2U_i) x_i + (M_i + 2V_i) y_i + B_i^2 y_{i+1}) + C^2 =$$
$$\sum x_i - 2y_1 + 0.01$$

1. $C^2 = 0.01$,
2. $A_i^2 + 2U_i = 1 \,\forall i$,
3. $M_1 + 2V_1 = -2$,
4. $M_{i+1} + 2V_{i+1} = -B_i^2 \,\forall i$.

Analysis of approximate case

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4. $M_{i+1} + 2V_{i+1} = -B_i^2 \, \forall i$.
5. $C = 0.1$ and $|U_i| \leq 0.1A_i$, $|V_i| \leq 0.1B_i$.

Analysis of approximate case

1. $C^2 = 0.01$,
2. $A_i^2 + 2U_i = 1\,\forall i$,
3. $M_1 + 2V_1 = -2$,
4. $M_{i+1} + 2V_{i+1} = -B_i^2\,\forall i$.
5. $C = 0.1$ and $|U_i| \leq 0.1A_i$, $|V_i| \leq 0.1B_i$.
6. $A_i^2 - 0.2A_i \leq 1 \implies A_i \leq 1.2\,\forall i$.

Analysis of approximate case

1. $C^2 = 0.01$,
2. $A_i^2 + 2U_i = 1 \, \forall i$,
3. $M_1 + 2V_1 = -2$,
4. $M_{i+1} + 2V_{i+1} = -B_i^2 \, \forall i$.
5. $C = 0.1$ and $|U_i| \leq 0.1A_i$, $|V_i| \leq 0.1B_i$.
6. $A_i^2 - 0.2A_i \leq 1 \implies A_i \leq 1.2 \, \forall i$.
7. $|M_1| \geq 2 - 0.2B_1$.

Analysis of approximate case

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8. $|M_{i+1}| \geq B_i^2 - 0.2B_{i+1} \, \forall i$.

Analysis of approximate case

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5. $C = 0.1$ and $|U_i| \leq 0.1A_i$, $|V_i| \leq 0.1B_i$.
6. $A_i^2 - 0.2A_i \leq 1 \implies A_i \leq 1.2 \, \forall i$.
7. $|M_1| \geq 2 - 0.2B_1$.
8. $|M_{i+1}| \geq B_i^2 - 0.2B_{i+1} \, \forall i$.

Now combine with $|M_i| \leq A_i B_i \leq 1.2B_i$ to get

1. $1.2B_1 \geq 2 - 0.2B_1$.
2. $1.2B_{i+1} \geq B_i^2 - 0.2B_{i+1} \, \forall i$.

So $B_i \geq 1.4(2/1.4^2)^{2^{i-1}}$, which is doubly exponential.

Analysis of approximate case: Archimedean constraints

The paper goes on to show that after adding the constraints $x_i^2 \leq 1$ and $y_i^2 \leq 1$, we still need doubly-exponential coefficients.

SOS with only booleanness constraints

If the *only* constraints are that *all* variables must be boolean ($x_i^2 = x_i$), then Ellipsoid runs in $n^{O(d)}$ time on the SOS problem. (up to additive error 2^{-n^c} for arbitrary constant c)