Lecture 11: Graph Matrices

Adapted from the talk at RANDOM 2016

Lecture Outline

- Part I: Graph Matrix Definitions and Examples
- Part II: Rough Norm Bounds on Graph Matrices
- Part III: Open Problems

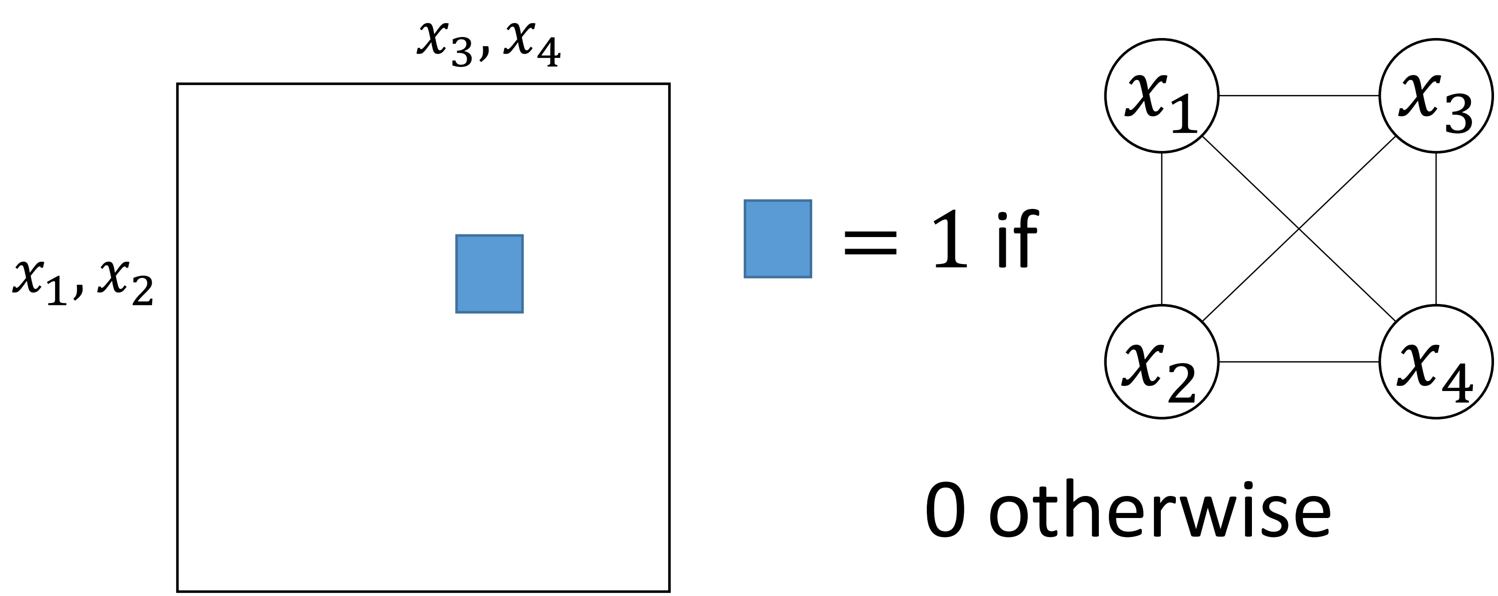
Part I: Graph Matrix Definitions and Examples

Motivation

- Graph matrices appear naturally when analyzing SOS at degree $d \geq 4$
- I have found understanding graph matrices to be very useful in analyzing SOS.

Example Matrix: 4-clique Indicator

- M has rows and columns indexed by pairs of vertices of an input graph G
- $M(\{x_1, x_2\}, \{x_3, x_4\}) = 1$ if x_1, x_2, x_3, x_4 are all distinct and are a clique in G, 0 otherwise

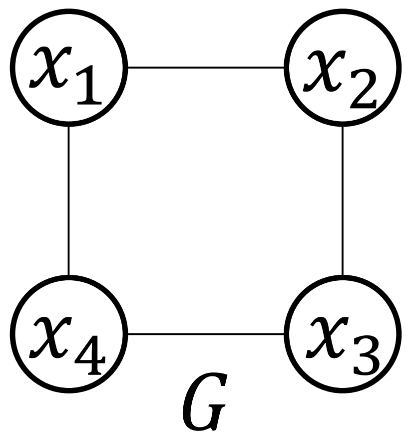


Clique Indicator Properties

- Matrix from previous slide: $M(\{x_1, x_2\}, \{x_3, x_4\}) = 1$ if x_1, x_2, x_3, x_4 are all distinct and are a clique, 0 otherwise
- Entries are random but not independent.
- That said, the entries can be described in terms of a small graph.
- Moreover, the matrix is symmetric under permutations of $[1, n]$ (as a function of the input graph)
- In this lecture, we analyze such matrices.

Fourier Characters χ_E

- Definition: Given a set E of possible edges of G, define $\chi_E(G) = -1^{|E \setminus E(G)|}$



Example: If $E = \{(x_1, x_2), (x_1, x_3), (x_1, x_4)\}$ then $\chi_E(G) = -1$ as $|E \setminus E(G)| = 1$

Structure of R_H

- For each graph H with distinguished ordered sets of vertices U, V, we will define a matrix R_H.
- The rows of R_H are indexed by ordered tuples A of $|U|$ vertices and the columns of R_H are indexed by ordered tuples B of $|V|$ vertices.
- H determines how $R_H(A, B)$ depends on G

Should A, B be in Ascending Order?

- Subtle question: Should we require A and B to be in ascending order?
- Benefit of this requirement: If M is indexed by monomials, we only have one A or B for each monomial, which is simpler.
- Example: $x_1 x_3$ only corresponds to $\mathrm{A} = \{1,3\}$ if A must be in ascending order. Without this requirement, $x_1 x_3$ corresponds to $A = \{1,3\}$ and $A = \{3,1\}$.

Should A, B be in Ascending Order?

- Should we require A and B to be in ascending order?
- Not requiring A, B to be in ascending order makes the combinatorics more complicated but has its own benefits.
- This lecture: A, B need not be in ascending order.

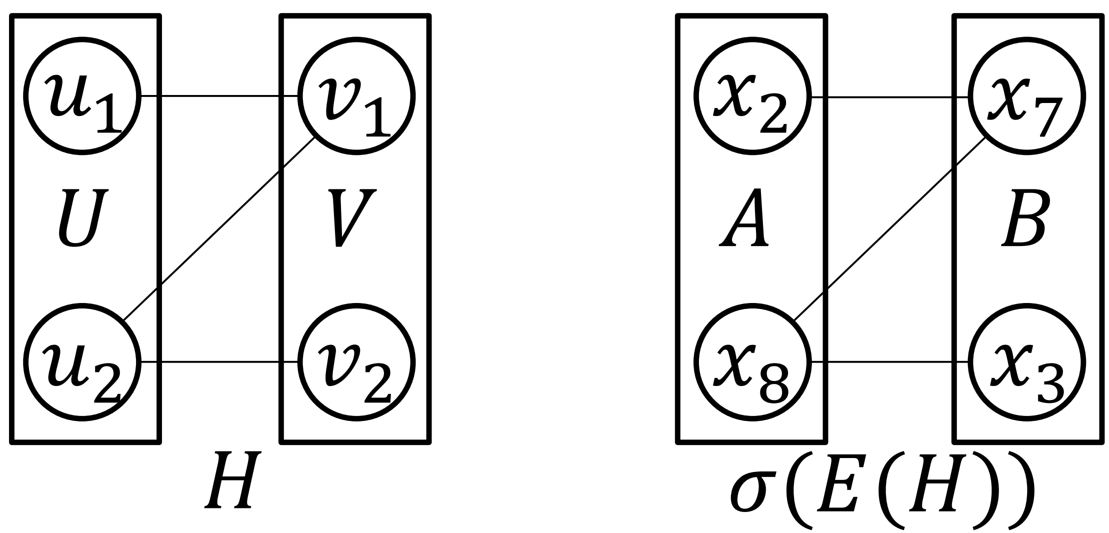
Definition of R_H (no middle vertices)

- We start with the case when $V(H) = U \cup V$
- Definition: If $V(H) = U \cup V$ then define $R_H(A, B) = \chi_{\sigma(E(H))}$ where $\sigma\colon \mathrm{V(H)} \to V(G)$ is the injective map satisfying $\sigma(U) = A, \sigma(V) = B$ and preserving the ordering of U, V.

R_H example (no middle vertices)

- Recall: $R_H(A, B) = \chi_{\sigma(E(H))}$

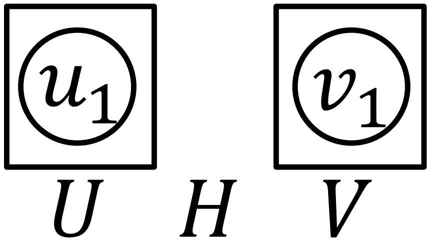
Example:



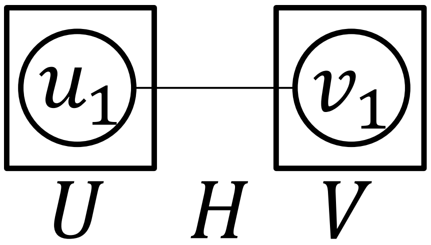
$$R_H(\{x_2, x_8\}, \{x_7, x_3\}) = \chi_{\{(x_2,x_7),(x_7,x_8),(x_3,x_8)\}}$$

Examples:

Example 1: All 1s matrix with 0s on the diagonal



Example 2: Symmetric ± 1 random matrix with 0s on the diagonal



Example: 4-clique indicator

- $M = \frac{1}{2^6}\sum_{H:V(H)=\{u_1,u_2,v_1,v_2\}} R_H$
- If x_1, x_2, x_3, x_4 form a clique, $R_H(\{x_1, x_2\}, \{x_3, x_4\}) = 1$ for all of the H
- If any edge e between x_1, x_2, x_3, x_4 is missing, there is perfect cancellation between H where $e \in E(H)$ and H where $e \notin E(H)$.
- Thus, $M(\{x_1, x_2\}, \{x_3, x_4\}) = 1$ if $x_1, x_2, \mathrm{x}_3, \mathrm{x}_4$ form a clique and is 0 otherwise.

In class exercises Part I

- Express the following matrices which are indexed by pairs of vertices (x_i, x_j) in terms of the matrices R_H:
  1. $M(\{x_1, x_2\}, \{x_3, x_4\}) =$ # of edges between the vertices x_1, x_2, x_3, x_4 if x_1, x_2, x_3, x_4 are distinct and 0 otherwise
  2. $M(\{x_1, x_2\}, \{x_3, x_4\}) = 1$ if there are at least 5 edges between the vertices x_1, x_2, x_3, x_4 and 0 otherwise.

Answers

- $M(\{x_1, x_2\}, \{x_3, x_4\}) =$ # of edges between the vertices x_1, x_2, x_3, x_4 if x_1, x_2, x_3, x_4 are distinct and 0 otherwise:
- Answer: $M = \sum_e \frac{1}{2} \sum_{H:E(H) \subseteq \{e\}} R_H$
- The H with 0 edges has coefficient 3, the Hs with one edge have coefficient $\frac{1}{2}$, and all other coefficients are equal.

Answers

- $M(\{x_1, x_2\}, \{x_3, x_4\}) = 1$ if there are at least 5 edges between the vertices x_1, x_2, x_3, x_4 and 0 otherwise.
- Answer: $M = \sum_e \frac{1}{32}(\sum_{H: e \notin E(H)} R_H) - \frac{5}{64} \sum_H R_H$
- If H has m edges, H appears with coefficient $\frac{7-2m}{64}$.

Discrete Fourier Analysis Equations

- The Fourier character of $M(A,B)$ on a set of edges E is $E_{G\sim G\left(n,\frac{1}{2}\right)}[M(A,B)(G)\chi_E(G)]$
- $M(A,B) = \sum_E E_{G\sim G\left(n,\frac{1}{2}\right)}[M(A,B)(G)\chi_E(G)]\,\chi_E$
- Can use this find the decomposition of M into R_H.

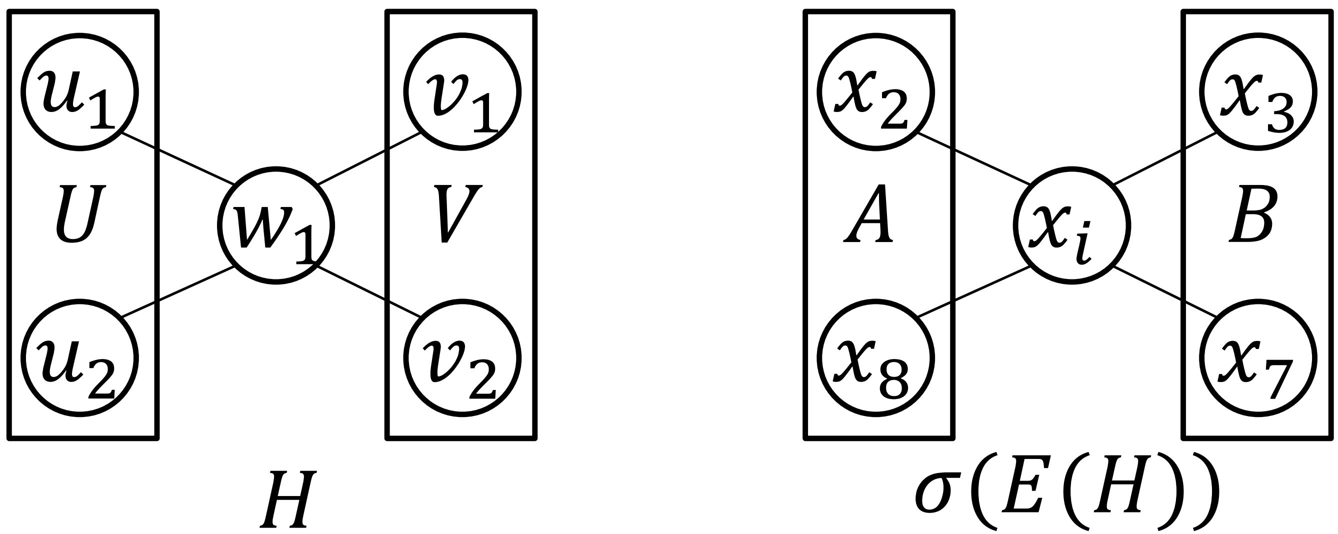
Definition of R_H with middle vertices

- So far: $M(A, B)$ depended only on edges within $A \cup B$.
- Can also have dependence on the rest of G if H has middle vertices not in U or V
- Definition (up to a symmetry related constant): Define $R_H(A, B) = \sum_\sigma \chi_{\sigma(E(H))}$ where we sum over all injective maps $\sigma: \mathrm{V(H)} \rightarrow V(G)$ satisfying $\sigma(U) = A, \sigma(V) = B$ and preserving the ordering of U, V.
- See appendix for an alternate definition.

R_H example with middle vertices

- Recall: $R_H(A,B) = \sum_\sigma \chi_{\sigma(E(H))}$

Example:



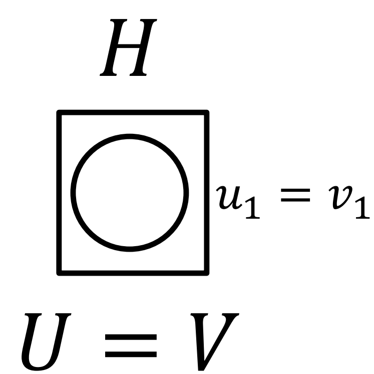
$$R_H(\{x_2,x_8\},\{x_3,x_7\}) = \sum_{i \notin \{2,3,7,8\}} \chi_{\{(x_2,x_i),(x_3,x_i),(x_7,x_i),(x_8,x_i)\}}$$

Example: Counting 5-cliques

- $M = \frac{1}{2^{10}} \sum_{H:V(H)=\{u_1,u_2,v_1,v_2,w_1\}} R_H$
- $M(\{x_1, x_2\}, \{x_3, x_4\})$ = # of 5-cliques containing x_1, x_2, x_3, x_4.

Intersection of U and V

- Thus far, we've only considered examples where U and V are disjoint.
- In general, U and V can intersect arbitrarily, this determines how the indices A and B must intersect in non-zero terms.
- Example: The $n \times n$ identity matrix is

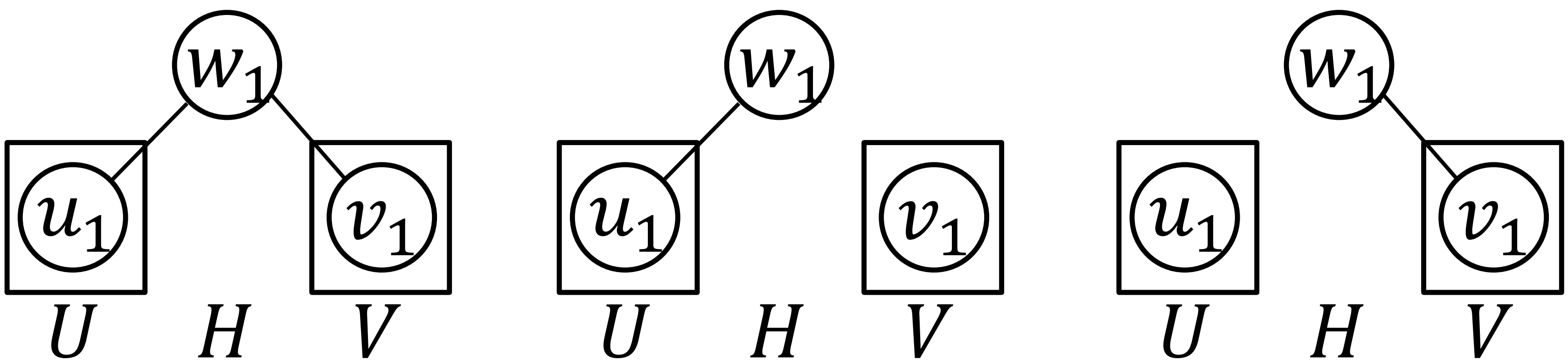


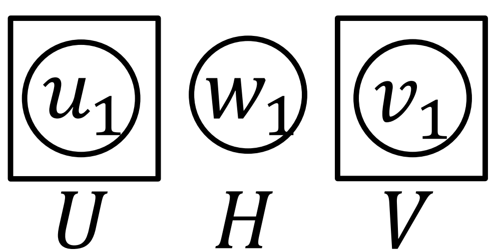
In class exercises Part 2

- Express the following matrices in terms of the matrices R_H:
  1. $M(\{x_1\},\{x_2\}) =$ # of paths of length 2 between x_1 and x_2 if x_1, x_2 are distinct and 0 otherwise.
  2. $M(\{x_1, x_2\},\{x_3, x_4\}) = 1$ for all x_1, x_2, x_3, x_4.

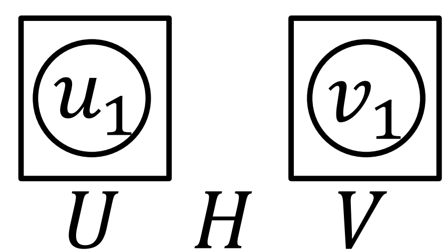
Answers

- $M(\{x_1\},\{x_2\}) =$ # of paths of length 2 between x_1 and x_2 if x_1, x_2 are distinct and 0 otherwise.
- Answer: M is the sum of $\frac{1}{4}$ times the following R_H



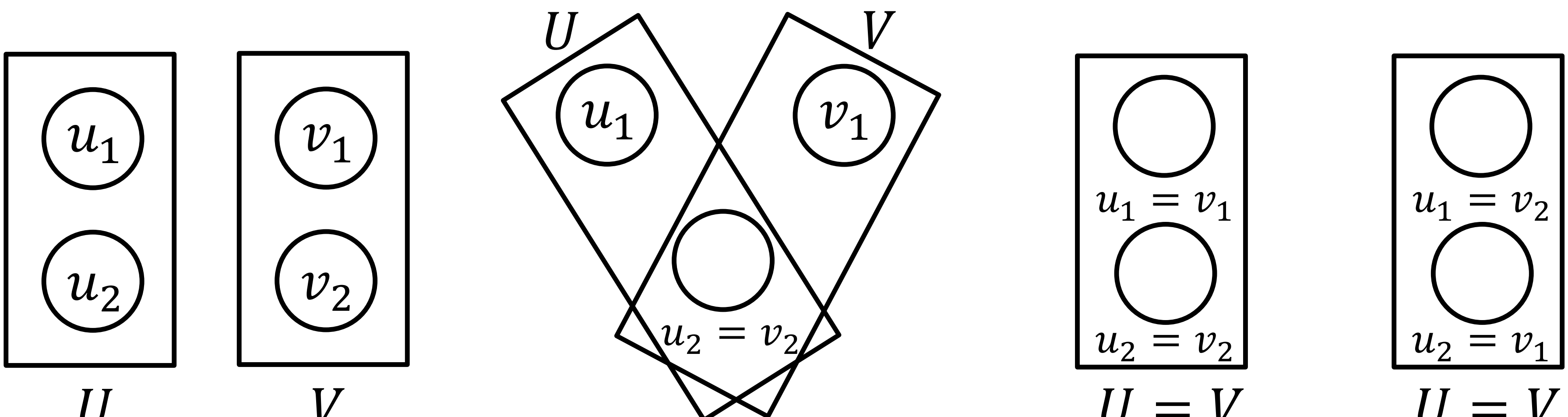


$= (n-2) \times$



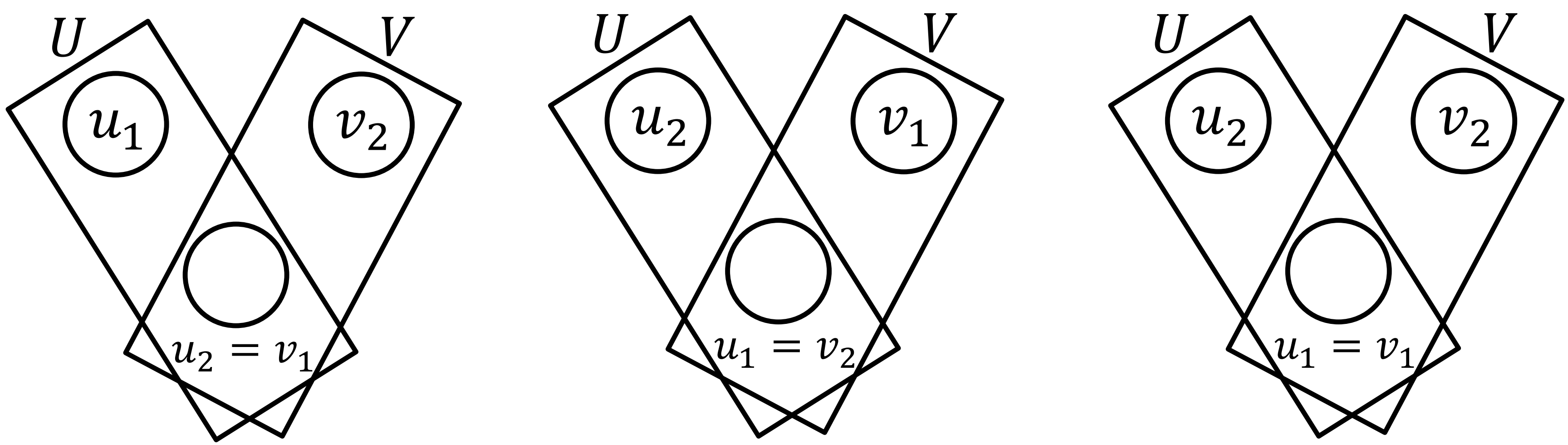
Answers

- $M(\{x_1, x_2\}, \{x_3, x_4\}) = 1$ for all x_1, x_2, x_3, x_4
- Answer: M is the sum of the following R_H (continued on next page)



Answers

- $M(\{x_1, x_2\}, \{x_3, x_4\}) = 1$ for all x_1, x_2, x_3, x_4
- Answer continued:



R_H as a basis

- Claim: The matrices R_H where H has no isolated vertices outside of U, V are a basis for matrices which are symmetric with respect to permutations of [1,n]
- Remark: This is one advantage of not requiring that A, B are in ascending order.
- Good exercise: What is the basis if we do require A, B to be in ascending order?

Part II: Norm Bounds

Rough Norm Bound

- Theorem [MP16]: If H has no isolated vertices then with high probability, $\|R_H\|$ is $\tilde{O}\left(n^{(|V(H)|-s_H)/2}\right)$ where s_H is the minimal size of a vertex separator between U and V (S is a vertex separator of U and V if every path from U to V intersects S)
- Note: The $\tilde{O}$ contains polylog factors and constants related to the size of H.

Techniques

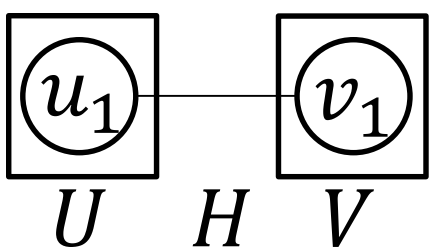
- Use the trace power method:

$$\|M\|^{2q} \leq tr\left(\left(MM^T\right)^q\right)$$

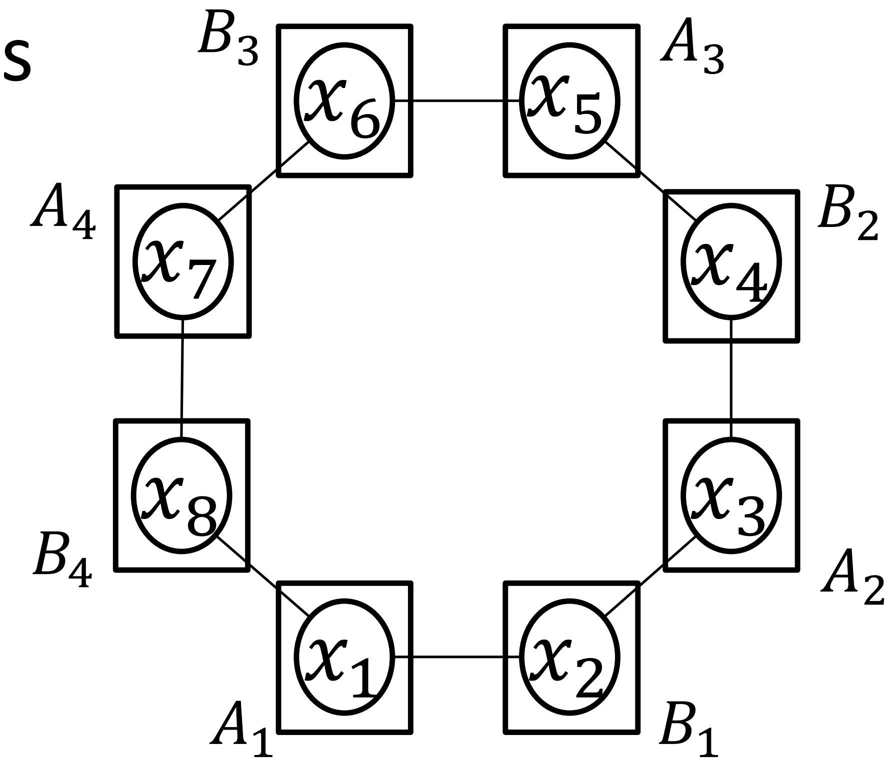
- Bound number of terms in $tr\left(\left(MM^T\right)^q\right)$ with nonzero expected value, use this to bound $E\left[tr\left(\left(MM^T\right)^q\right)\right]$.
- Use Markov's inequality $\Pr[X \geq a] \leq \frac{E[x]}{a}$ (if X is always non-negative) to probabilistically bound $tr\left(\left(MM^T\right)^q\right)$ and thus $\|M\|$.

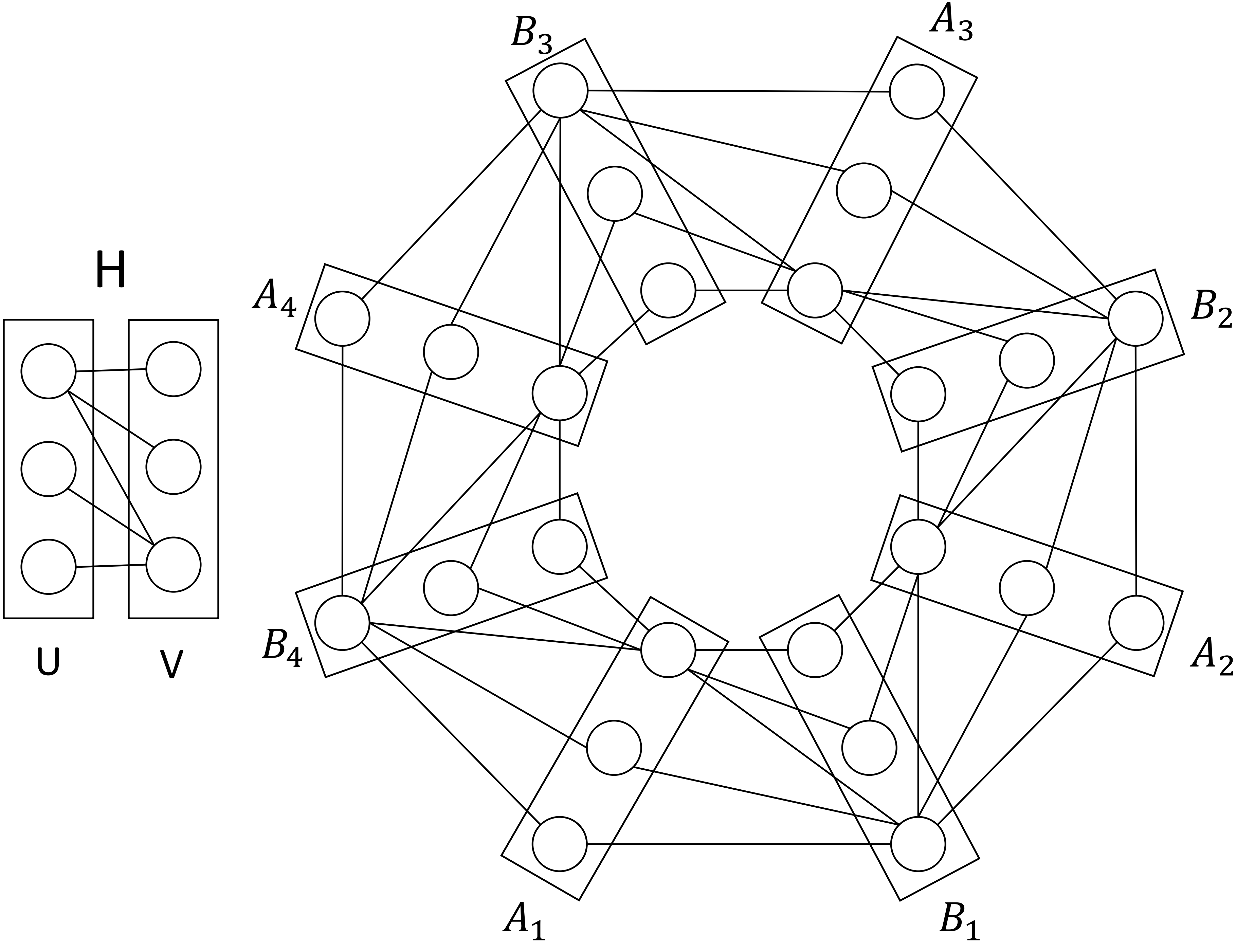
Graphs for Matrix Powers

- $tr\left(\left(MM^T\right)^q\right) = \sum_{A_1,B_1,\ldots A_q,B_q} \prod_{i=1}^{q} M(A_i,B_i)M^T(B_iA_{i+1})$ where $A_{q+1} = A_1$
- Useful to draw graphs for these terms



Example: $q = 4$





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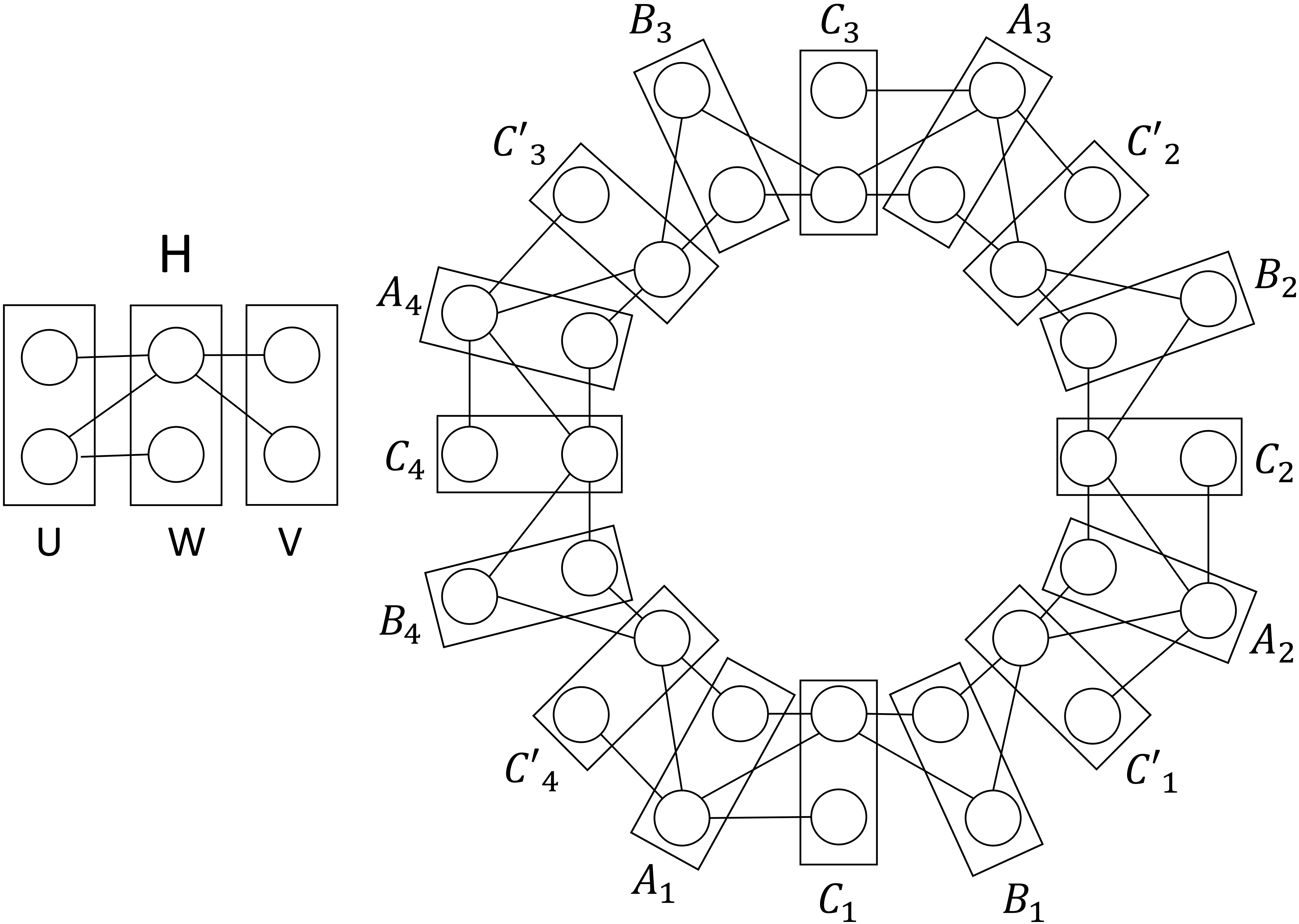

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C'_3
C'_2
B_2
A_4
C_4
C_2
B_4
A_2
C'_4
C'_1
A_1
C_1
B_1

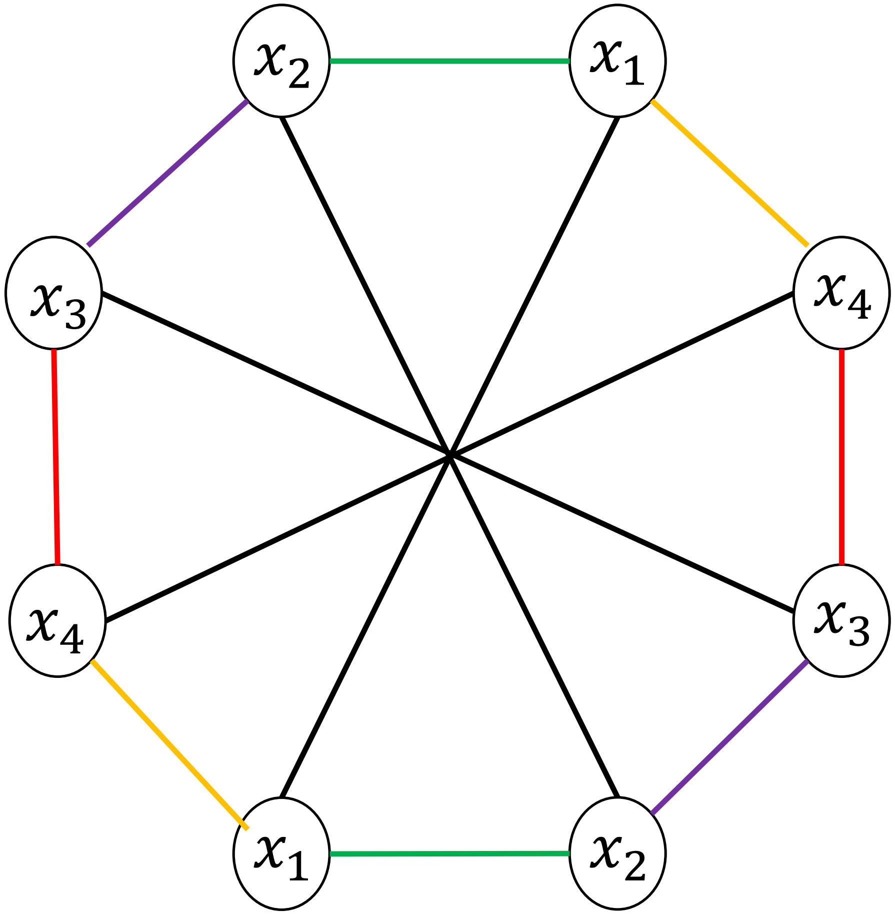
Bounding # of non-zero terms

- Key idea: A given term has zero expected value unless every edge appears an even number of times.
- Key question: For a term with non-zero expected value, what is the maximum possible number of distinct indices?

Cycle Lemma

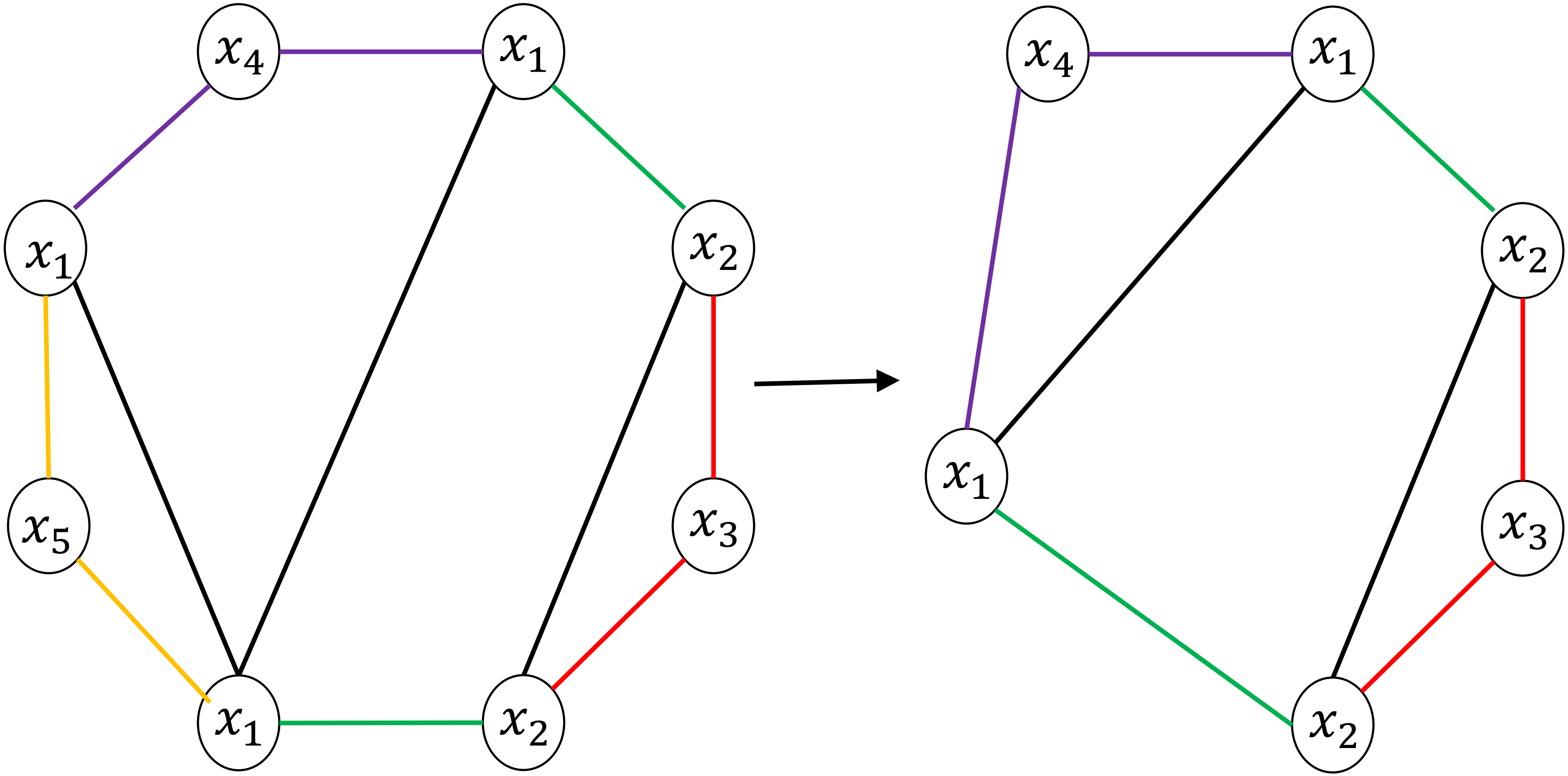
- Lemma: For a cycle of length $2q$, have at most $q + 1$ distinct indices
- Proof: By induction. Base case $q \leq 1$ is immediate.
- If no index is unique, $\leq q$ distinct indices
- If index x_i is unique, its two neighbors must be the same. Contract its two neighbors together and delete x_i, reducing the number of indices by 1 and the cycle length by 2.

Cycle Lemma Picture #1



Case 1: No unique indices

Cycle Lemma Picture #2



Case 2: Unique index

± 1 Random Matrix Norm Bound

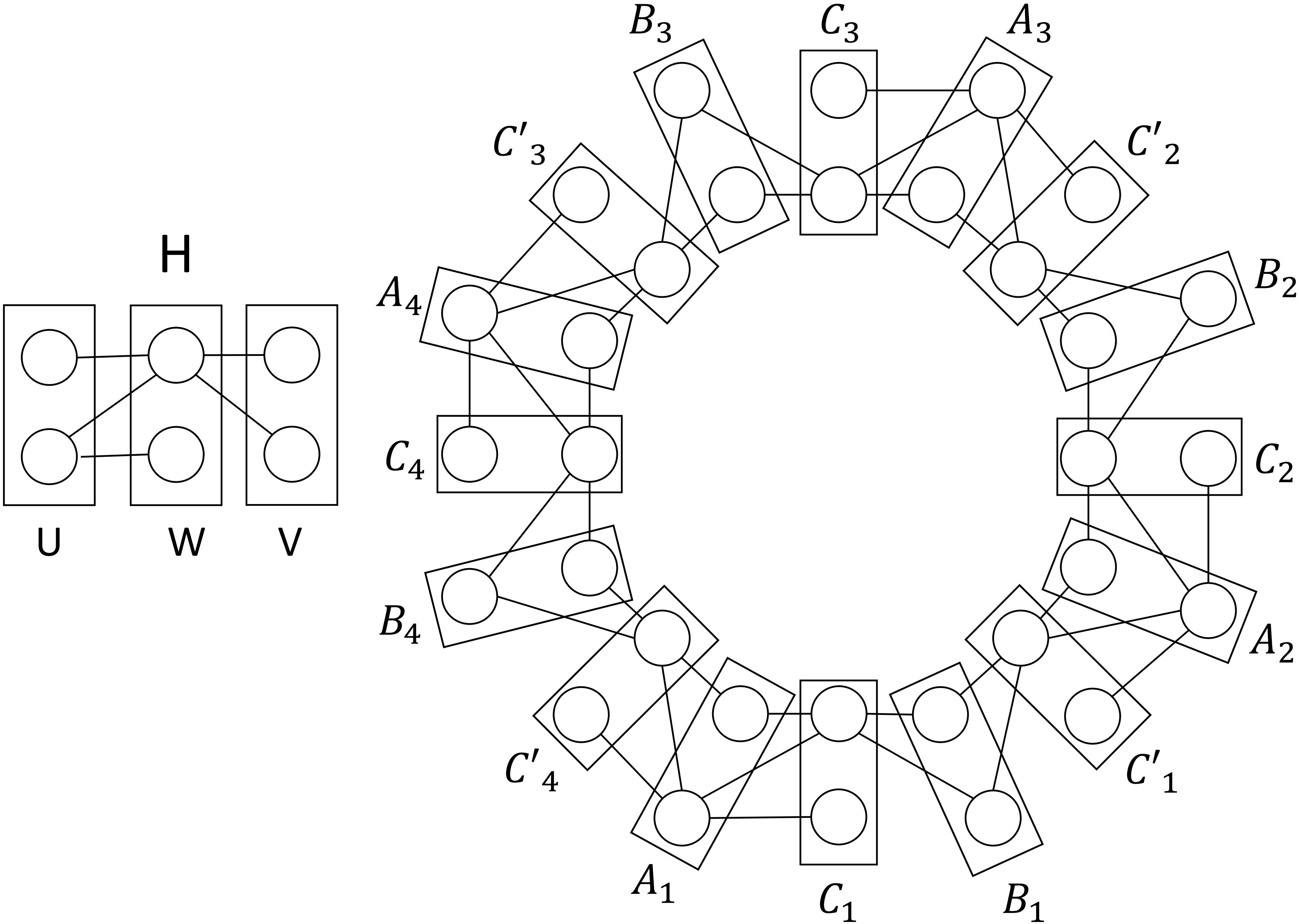
- $E\left[tr\left(\left(R_H {R_H}^T\right)^q\right)\right]$ is $O(n^{q+1})$ (constant depends on q)
- With high probability, $\|R_H\|$ is $O(n^{(q+1)/2q})$
- Taking q to be sufficiently large, w.h.p. $\|R_H\|$ is $\tilde{O}(\sqrt{n})$
- Not as precise as Wigner's semicircle law [Wig55,Wig58], but relatively easy to generalize.

Technical Step: Matrix Preprocessing

- Technical step: For general H, instead of analyzing R_H, we analyze submatrices R'_H where each vertex of H maps into a different subset of $[1, n]$ and these subsets are disjoint
- This allows us to assume that we only have equalities between copies of the same vertex in H, making it easier to prove norm bounds on R'_H
- We then use probabilistic norm bounds on R'_H to prove a probabilistic norm bound on R_H

of Unique Indices: Upper Bound

- Key idea: If we are analyzing $\left(R'_H(R'_H)^T\right)^q$, there are at most q distinct values for any vertex x of H.
- Case 1: If $x \in U$ or $x \in V$ then there are only q copies of x to begin with.
- Case 2: If $x \notin U$ and $x \notin V$, then since x is not isolated, each copy of x must be equal to some other copy of x as otherwise any edge incident to this copy of x would only appear once.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C_2
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A_2
C'_4
C'_1
A_1
C_1
B_1

Cycles

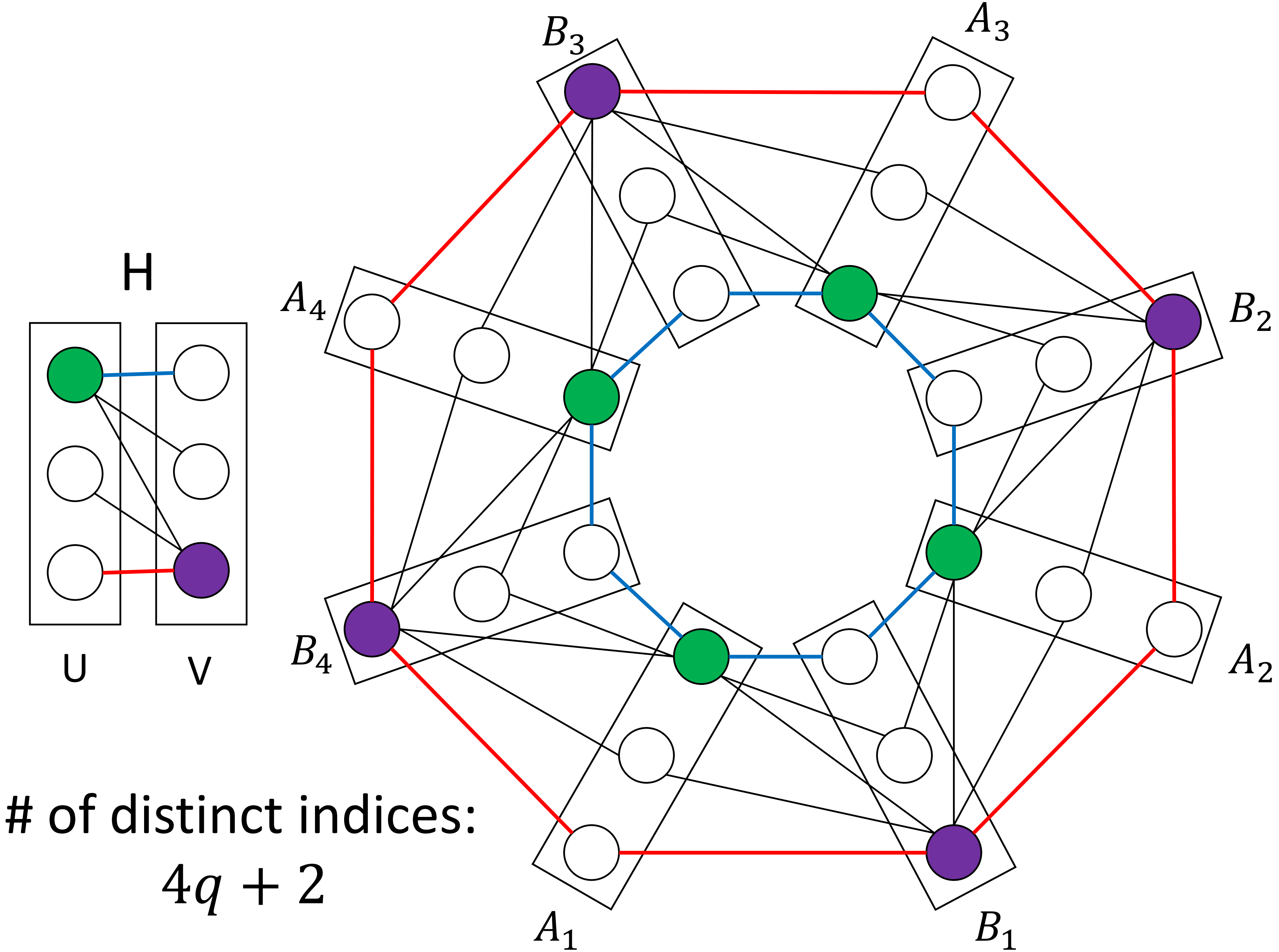
- Each path in H from U to V of length l creates a cycle of length $2ql$.
- Prior bound: There are $l + 1$ distinct vertices of H, each of which could have q distinct values.
- Cycle lemma bound: At most $ql + 1$ distinct values.
- Each disjoint path in H from U to V lowers our bound by $q - 1$

Final Upper Bound

- Maximum # of disjoint paths = s_H (the size of the minimal vertex separator between U and V)
- Final upper bound on # of distinct indices: $q|V(H)| - s_H(q-1) = q(|V(H)| - s_H) + s_H$
- Choosing q appropriately, we can prove our probabilistic norm bound.

Achieving the Upper Bound

- Upper bound is tight
- Can be obtained by choosing a minimal vertex separator, making all copies of the separator the same, and pairing up all remaining vertices which are not in an A or B appropriately.


H
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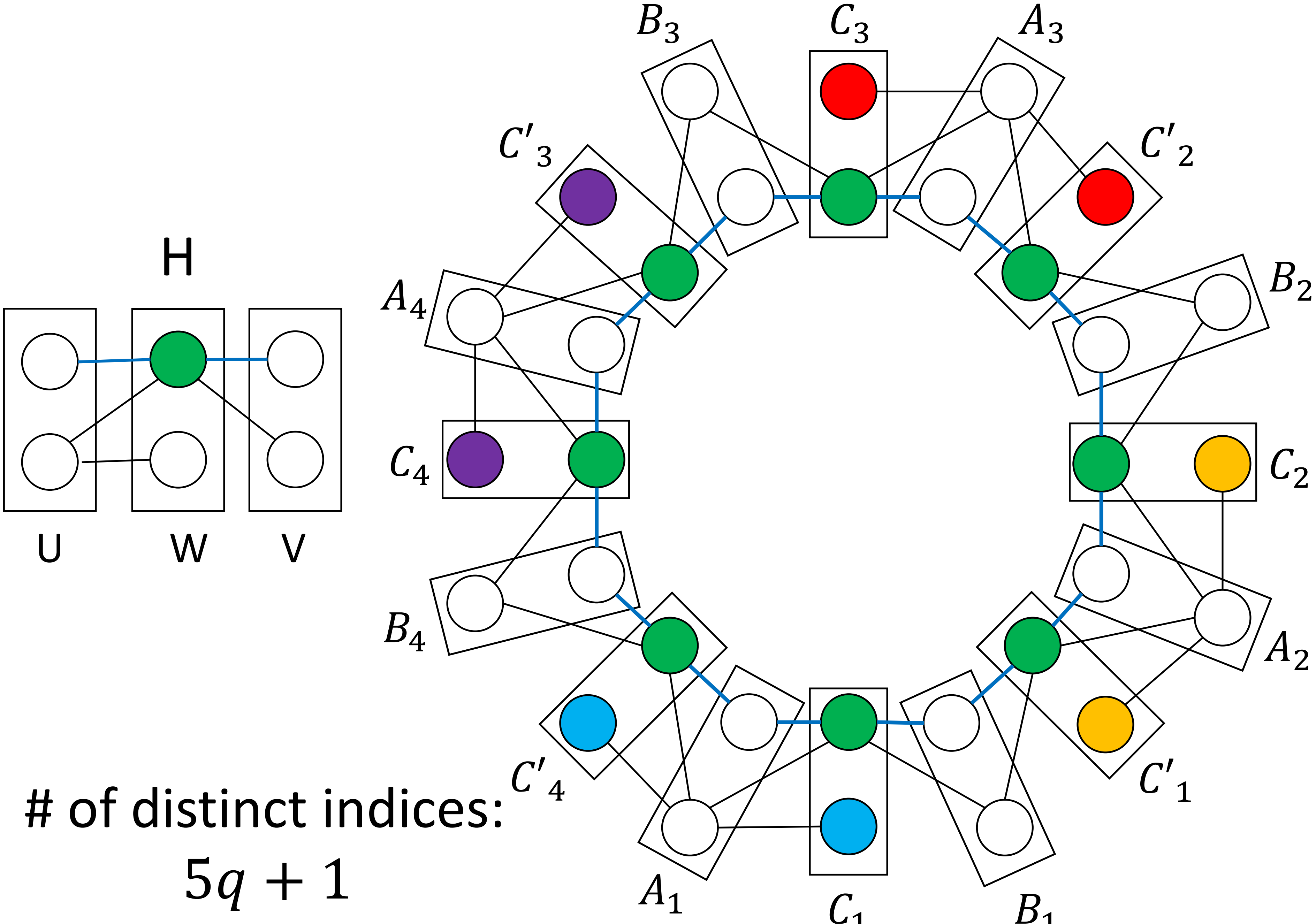
A_1
B_1
# of distinct indices:
$4q + 2$



# of distinct indices:
$5q + 1$

Part III: Open Problems

Open Problems

- With more careful analysis, can we tighten the norm bounds and remove the logarithmic factors?
- More ambitiously, can we determine the spectrum of these matrices?

References

- [MP16] D. Medarametla, A. Potechin. Bounds on the Norms of Uniform Low Degree Graph Matrices. RANDOM 2016. https://arxiv.org/abs/1604.03423
- [Wig55] E. Wigner. Characteristic Vectors of Bordered Matrices with Infinite Dimensions. Ann. of Math. 62, p. 548-564. 1955
- [Wig58] E. Wigner. On the Distribution of the Roots of Certain Symmetric Matrices. Ann. of Math. 67, p. 325-328, 1958.

Appendix: Definition of R_H with Correct Constant

Definition of R_H with Correct Constant

- Define

$$R_H(A,B) = \sum_{G':\exists\sigma:V(H)\to V(G):\sigma(H)=G'} \chi_{E(G')}$$

where G' is a graph on a subset of the vertices of G and we require that σ is injective, $\sigma(U) = A$, $\sigma(V) = B$, and σ respects the orderings on U, A, V, B.

- Remark: This definition avoids counting the same Fourier character multiple times for a given matrix entry $R_H(A,B)$.