# Problem Set 4 Solutions

Sum of Squares Seminar

October 3, 2017, Due at the end of the seminar

## Problem 1: Sparsest Cut (50 points)

In this problem, we will run our algorithms for the sparsest cut problem on the following graphs (to be added)

(a) 25 points: Run the Leighton-Rao linear programming relaxation and the Goemans-Linial semidefinite programming relaxation (analyzed by Arora-Rao-Vazirani) and give the values obtained for each graph.

Solution: The linear and semidefinite programs give the following values:

(a) For the communities graph, the linear programming relaxation gives a value of $.3323$ and the semidefinite programming relaxation gives a value of $.3393$, which is the true answer.

(b) For the Cayley graph, the linear programming relaxation gives a value of $.2667$ and the semidefinite programming relaxation gives a value of $.3000$. The true answer is $.32$

(c) For the Peterson graph, both the linear programming relaxation and the semidefinite programming relaxation give a value of $.2$, which is the true answer.

(d) For the random sparse graph, both the linear programming relaxation and the semidefinite programming relaxation give a value of $0.0346$, which is the true answer.

(b) 25 points: Run the rounding algorithm for the linear programming relaxation and give the corresponding cuts.

## Problem 2: Minimizing a Homogeneous Polynomial over the Sphere (50 points)

Consider the following problem. Minimize the value of a homogeneous degree $k$ polynomial $h$ subject to the constraint that $\sum_{i=1}^{n} x_i^2 = 1$.

(a) 10 points: What are the constraints for the primal and dual semidefinite programs for degree $d$ SOS (giving the pseudo-expectation values and Positivstellensatz proof, respectively)? Write these constraints as linear constraints on the entries of the appropriate matrices and matrix PSDness constraints (i.e. the form you would put them in for implementing the semidefinite program).

Solution:

**Definition 0.1.** *Given a matrix $Q$ whose rows and columns are indexed by monomials $p, q$, define $f(Q) = \sum_{p,q} Q_{pq}pq$*

For the primal, we have the following semidefinite program: Choose a matrix $Q$ such that $f(Q) = h$ and minimize $Q \bullet M$ subject to the following constraints on $M$:

(a) $M_{11} = 1$ (as $\tilde{E}[1] = 1$)

(b) Whenever $p, q, p', q'$ are monomimals of degree at most $\frac{d}{2}$ such that $pq = p'q'$, $M_{p'q'} = M_{pq}$ (SOS symmetry)

(c) We can enforce the constraint $\sum_{i=1}^{n} x_i^2 = 1$ as follows: Whenever $p, q$ are monomials of degree at most $\frac{d}{2} - 1$, $\sum_i M_{(px_i)(qx_i)} = M_{pq}$

(d) $M \succeq 0$

For the dual, we have the following semidefinite program

**Definition 0.2.** *Given monomials $p, q$ of degree at most $\frac{d}{2} - 1$, define the matrix $A(p, q)$ so that $A(p, q)_{(px_i)(qx_i)} = 1$ for each $i \in [1, n]$, $A_{pq} = -1$, and all other entries of $A$ are equal to* $0$.

We now maximize $c$ such that there exist coefficients $a(p, q)$ and a matrix $Q$ which satisfy the following constraints:

(a) $h = f(Q + \sum_{p,q:deg(p) \leq \frac{d}{2}-1, deg(q) \leq \frac{d}{2}-1} a(p, q)A(p, q)) + c$

(b) $Q \succeq 0$

(b) 15 points: Show that in the special case where $d = k$ is even, it is sufficient to consider indices of degree exactly $\frac{d}{2}$. In other words, show the following:

(a) If we have pseudo-expectation values $\tilde{E}$ for all degree $d$ polynomials and the corresponding submatrix of the moment matrix is PSD, then these can be extended to valid pseudo-expectation values $\tilde{E}$ for all polynomials of degree $\leq d$.

(b) If there is a Positivstellensatz proof that $h \geq c$ with the the constraint that $\sum_{i=1}^{n} x_i^2 = 1$ then there is a Positivstellensatz proof that $h \geq c(\sum_{i=1}^{n} x_i^2)^{\frac{d}{2}}$ with no constraints.

What happens if $d, k$ are even and $d > k$?

Solution: Assume that we have pseudo-expectation values $\tilde{E}$ for all degree $d$ polynomials and the corresponding submatrix of the moment matrix is PSD. We can extend $\tilde{E}$ to all polynomials of degree at most $d$ as follows. Given a polynomial $f$, write $f = \sum_{k=0}^{d} f_k$ where $f_k$ is the part of $f$ with degree $k$. Now take

$$\tilde{E}[f] = \sum_{j=0}^{\frac{d}{2}} \tilde{E}\left[f_{2j}\left(\sum_{i=1}^{n} x_i^2\right)^{d-j}\right]$$

We need to check the following:

(a) For any monomial $p$ of degree at most $d-2$, $\tilde{E}\left[p(\sum_{i=1}^{n} x_i^2 - 1)\right] = 0$

(b) For any polynomial $g$ of degree at most $\frac{d}{2}$, $\tilde{E}[g^2] \geq 0$

For the first statement, observe that if $p$ has odd degree then $\tilde{E}\left[p(\sum_{i=1}^{n} x_i^2 - 1)\right] = 0$ and if $p$ has degree $2j$ where $j \in [0, \frac{d}{2} - 1]$ then

$$\tilde{E}\left[p(\sum_{i=1}^{n} x_i^2 - 1)\right] = \tilde{E}\left[p(\sum_{i=1}^{n} x_i^2)(\sum_{i=1}^{n} x_i^2)^{\frac{d}{2}-j-1} - p(\sum_{i=1}^{n} x_i^2)^{\frac{d}{2}-j}\right] = 0$$

For the second statement, given a polynomial $g$ of degree at most $\frac{d}{2}$, write $g = \sum_{k=0}^{\frac{d}{2}} g_k$ where $g_k$ is the part of $g$ which has degree $k$. Now observe that

$$\begin{aligned} g = &\sum_{j=0}^{\lfloor \frac{d}{4} \rfloor}\left(g_{\frac{d}{2}-2j}(\sum_{i=1}^{n} x_i^2)^j - g_{\frac{d}{2}-2j}\left(\sum_{i=1}^{n} x_i^2 - 1\right)\left(\sum_{l=0}^{j}\left(\sum_{i=1}^{n} x_i^2\right)^l\right)\right) \\ &+ \sum_{j=0}^{\lfloor \frac{d-2}{4} \rfloor}\left(g_{\frac{d}{2}-1-2j}(\sum_{i=1}^{n} x_i^2)^j - g_{\frac{d}{2}-1-2j}\left(\sum_{i=1}^{n} x_i^2 - 1\right)\left(\sum_{l=0}^{j}\left(\sum_{i=1}^{n} x_i^2\right)^l\right)\right)\end{aligned}$$

This implies that if we take $g' = g'_{\frac{d}{2}} + g'_{\frac{d}{2}-1}$ where

(a) $g'_{\frac{d}{2}} = \sum_{j=0}^{\lfloor \frac{d}{4} \rfloor}\left(g_{\frac{d}{2}-2j}(\sum_{i=1}^{n} x_i^2)^j\right)$

(b) $g'_{\frac{d}{2}-1} = \sum_{j=0}^{\lfloor \frac{d-2}{4} \rfloor}\left(g_{\frac{d}{2}-1-2j}(\sum_{i=1}^{n} x_i^2)^j\right)$

then

$$\begin{aligned}\tilde{E}[g^2] = \tilde{E}[g'^2] &= \tilde{E}[{g'_{\frac{d}{2}}}^2] + 2\tilde{E}[g'_{\frac{d}{2}} g'_{\frac{d}{2}-1}] + \tilde{E}[{g'_{\frac{d}{2}-1}}^2] = \tilde{E}[{g'_{\frac{d}{2}}}^2] + \tilde{E}[{g'_{\frac{d}{2}-1}}^2] \\ &= \tilde{E}[{g'_{\frac{d}{2}}}^2] + \tilde{E}[{g'_{\frac{d}{2}-1}}^2(\sum_{i=1}^{n} x_i^2)] - \tilde{E}[{g'_{\frac{d}{2}-1}}^2(\sum_{i=1}^{n} x_i^2 - 1)] \\ &= \tilde{E}[{g'_{\frac{d}{2}}}^2] + \sum_{i=1}^{n} \tilde{E}[(x_i g'_{\frac{d}{2}-1})^2] \geq 0\end{aligned}$$

For the Positivstellensatz proofs, assume that we have a proof that $h \geq c$ with the constraint that $\sum_{i=1}^{n} x_i^2 = 1$. This gives an equation

$$h = c + (\sum_{i=1}^{n} x_i^2 - 1)f + \sum_{j} g_j^2$$

For each $g_j$, first decompose $g_j$ as $g_{je} + g_{jo}$ where $g_{je}$ consists of the monomials of $g_j$ which have even degree and $g_{jo}$ consists of the monomials of $g_j$ which have odd degree. Also, decompose $f$ as $f_e + f_e$ where $f_e$ consists of the monomials of $f$ which have even degree and $f_o$ consists of the monomials of $f$ which have odd degree. Now we must have that

$$(\sum_{i=1}^{n} x_i^2 - 1)f_o + \sum_{j} 2g_{je}g_{jo} = 0$$

because $h$ is homogeneous of even degree and all other monomials in $c + (\sum_{i=1}^{n} x_i^2 - 1)f + \sum_j g_j^2$ have even degree. Thus, we may replace $f$ by $f_e$ and replace each $g_j^2$ with $g_{je}^2 + g_{jo}^2$.

Now we instead decompose $g_j$ as $g_j = \sum_{l=0}^{\frac{d}{2}} g_{j,l}$ where $g_{j,l}$ is the part of $g_j$ of degree $l$. We then replace $c$ with $c(\sum_{i=1}^{n} x_i^2)^{\frac{d}{2}}$ and replace $g_{je}^2 + g_{jo}^2$ with

$${g'}_{j,\frac{d}{2}}^2 + \sum_{i=1}^{n} x_i^2 (g'_{j,\frac{d}{2}-1})^2$$

where

(a) $g'_{j,\frac{d}{2}} = \sum_{l=0}^{\lfloor \frac{d}{4} \rfloor} \left( g_{j,\frac{d}{2}-2l} (\sum_{i=1}^{n} x_i^2)^l \right)$

(b) $g'_{j,\frac{d}{2}-1} = \sum_{l=0}^{\lfloor \frac{d-2}{4} \rfloor} \left( g_{j,\frac{d}{2}-1-2l} (\sum_{i=1}^{n} x_i^2)^l \right)$

Observe that both $c(\sum_{i=1}^{n} x_i^2)^{\frac{d}{2}} - c$ and ${g'}_{j,\frac{d}{2}}^2 + \sum_{i=1}^{n} x_i^2 (g'_{j,\frac{d}{2}-1})^2 - (g_{je}^2 + g_{jo}^2)$ are divisible by $\sum_{i=1}^{n} x_i^2 - 1$, so we have that

$$h = c(\sum_{i=1}^{n} x_i^2)^{\frac{d}{2}} + (\sum_{i=1}^{n} x_i^2 - 1)f' + \sum_{j} \left( {g'}_{j,\frac{d}{2}}^2 + \sum_{i=1}^{n} x_i^2 (g'_{j,\frac{d}{2}-1})^2 \right)$$

for some $f'$ of degree at most $d-2$. Now observe that $h$, $c(\sum_{i=1}^{n} x_i^2)^{\frac{d}{2}}$, and $\sum_j \left( {g'}_{j,\frac{d}{2}}^2 + \sum_{i=1}^{n} x_i^2 (g'_{j,\frac{d}{2}-1})^2 \right)$ are all homogeneous of degree $d$ so in fact we must have that $f' = 0$, which completes the proof.

(c) 25 points: Run both the primal and dual program on the polynomial $h = x_4^4 + x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 - 4x_1 x_2 x_3 x_4$. Letting $c$ be the value you obtain, give both the pseudo-expectation values $\tilde{E}$ such that $\tilde{E}[h] = c$ (giving the values for degree 4 polynomials is sufficient) and the Positivstellensatz proof that $h \geq c(\sum_{i=1}^{n} x_i^2)^2$