Problem Set 2

Sum of Squares Seminar

September 14, 2017, Due Septmber 25, 2017

Adjacency matrix

Several problems involve the adjacency matrix of a graph, so we recall the definition here. The adjacency matrix A of a graph G is defined to be the matrix whose entries are as follows:

1. $\forall i, A_{ii} = 0$

2. $\forall i < j, A_{ij} = A_{ji} = 1$ if $(i, j) \in E(G)$ and $A_{ij} = A_{ji} = 0$ if $(i, j) \notin E(G)$

Problem 1: Vectors for PSD Matrices (20 points)

Recall the following very useful characterization of PSD matrices. A matrix M is PSD if and only if there are vectors $\{v_i\}$ such that $\forall i, j, M_{ij} = v_i \cdot v_j$. Further recall the Cholesky-Banachiewicz or Cholesky-Crout algorithm for finding the Cholesky decomposition (which gives us such a set of vectors).

1. Let c_{ia} be the ath coordinate of v_i. Set $c_{11} = \sqrt{M_{11}}$ and set $c_{ia} = 0$ whenever $a > i$.

2. For all $i < k$, take $c_{ki} = \frac{M_{ik} - \sum_{a=1}^{i-1} c_{ka}c_{ia}}{c_{ii}}$ (take $c_{ki} = 0$ if $M_{ik} - \sum_{a=1}^{i-1} c_{ka}c_{ia} = c_{ii} = 0$)

3. For all k, take $c_{kk} = \sqrt{M_{kk} - \sum_{a=1}^{k-1} c_{ka}^2}$

(a) 5 points: Let $M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$.

Find vectors v_1, v_2, v_3, v_4 such that $\forall i, j \in [1, 4], M_{ij} = v_i \cdot v_j$. Can you see the pattern?

(b) 15 points: Show that if this algorithm fails on a matrix M then M is not PSD. (hint is available)

Problem 2: Goemans-Williamson Dual (15 points)

(a) 10 points: What is the dual of the Geomans-Williamson semidefinite program?

(b) 5 points: Deduce that the value of the Goemans-Williamson semidefinite program is always at most $\frac{|E(G)|}{2} + \frac{n\lambda_{max}(-A)}{4}$ where $\lambda_{max}(-A)$ is the maximum eigenvalue of $-A$.

Problem 3: Goemans-Williamson on the Cycle (15 points)

Let G be the cycle graph on n vertices, i.e. $E(G) = \{(i, i+1) : i \in [1, n]\} \cup \{(1, n)\}$

(a) 10 points: What are the eigenvalues and eigenvectors of A? (hint is available)

(b) 5 points: What will the Goemans-Williamson program output when gven G? In particular, what vector v_i will be associated to each vertex i and what will the final value be?

Problem 4: Eigenvalues of the Hypercube Graph (15 points)

Let G be the graph with one vertex for each point of the hypercube $\{-1, 1\}^n$ and edges $E(G) = \{(x, y) : x, y \text{ differ in } k \text{ coordinates}\}$ where k is even.

(a) 10 points: What are the eigenvectors and eigenvalues of the adjacency matrix A? (hint is available)

Note: the expression for the eigenvalues will be somewhat messy. Challenge question: It is intuitively clear that when k is significantly greater than $\frac{n}{2}$, $\lambda_{max}(-A)$ is given by the coordinate cuts. Can you prove it?

(b) 5 points: Assuming that $\lambda_{max}(-A)$ is given by the coordinate cuts, deduce that Goemans-Williamson gives the correct value on G. Letting $\Theta = cos^{-1}(\frac{n-2k}{n})$, what happens when we round the feasible solution (to the SDP) where each vertex x is mapped to $\frac{x}{\sqrt{n}}$ (viewing x as a vector)?

Problem 5: Optimization Versus Feasibility Testing (10 points)

Let's say that we want to find the minimum value of some function h subject to a set of polynomial constraints. We have seen two alternative ways to use SOS to lower bound this. The two ways are as follows:

1. The first way (which was discussed briefly in Lecture 1) is to add $h \leq c$ (equivalently, $h = c - z^2$ for a new variable z) as a problem constraint and use SOS as a feasibility test to determine if this is feasible. We can then use binary search to find the minimal value of c where SOS thinks the equations are feasible and output this value.

2. The second way (which we saw in Lecture 3) is find the minimal possible value of $\tilde{E}[h]$ over any pseudo-expectation values $\tilde{E}$ which respect the problem constraints and output this value.

Does one of these alternatives give a better bound than the other? If so, why? If not, why not?

Problem 6: Applying Goemans-Williamson (25 points)

Apply the Goemans-Williamson semidefinite program and rounding algorithm to the following graphs (the adjacency matrices for the graphs are provided as .txt files on the course website):

(a) A random $G(n, \frac{1}{2})$ graph on $n = 30$ vertices.

(b) Half of the hypercube graph described in problem 4 with $n = 6$ and $k = 4$.

(c) A graph formed by taking two communities of size 15, adding each edge within a community with probability .3, and adding each edge between communities with probability .7.

(d) The Peterson graph.

For your answers, please give the value of the Goemans-Williamson progam and a cut obtained by the rounding algorithm. Optionally, you may also give the matrices outputted by the program and the radom vector used for the rounding, as this allows each step of the algorithm to be checked.

Hints

1b. To show that M is not PSD, find a vector w such that $w^T M w < 0$. If the algorithm fails because $M_{kk} < \sum_{a=1}^{k-1} c_{ka}^2$, we can instead take $c_{kk} = i\sqrt{\sum_{a=1}^{k-1} c_{ka}^2 - M_{kk}}$. Now write ie_k as a linear combination of the vectors $v_1, \cdots, v_k$. If the algorithm fails because $c_{ii} = 0$ and $M_{ik} \neq \sum_{a=1}^{i-1} c_{ka} c_{ia}$, show that there is a vector w on the first i coordinates such that

1. w is an eigenvector of the submatrix of M consisting of the first i rows and columns which has eigenvalue 0.

2. The inner product of w with the kth row/column of M is nonzero.

Now use this to find a vector w' such that ${w'}^T M w' < 0$

3a. This is one of the few times in this seminar where it is very useful to use complex numbers! Show that $\forall k \in [0, n-1]$, the vector v_k with jth coordinate $v_{kj} = e^{\frac{2\pi ijk}{n}}$ is an eigenvector of the adjacency matrix. To show this and find its eigenvalue, think geometrically! In other words, view these complex numbers as vectors in the complex plane.

4a. Use discrete Fourier analysis over the hypercube! In particular, by symmetry, all vectors of the following form are eigenvectors: Let $A \subseteq [1, n]$ and set $v_x = (-1)^{|A \cap L_x|}$ where $L_x = \{i \in [1, n] : x_i = -1\}$.