Problem Set 1

Sum of Squares Seminar

September 6, 2017, Due Septmber 18, 2017

Problem 1 (10 points)

Recall that given equations $s_1(x_1, \cdots, x_n) = 0, s_2(x_1, \cdots, x_n) = 0$, etc., a degree d Positivstellensatz proof of infeasibility is an expression of the form $-1 = \sum_i f_i s_i + \sum_j g_j^2$ where

1. $\forall i, deg(f_i) + deg(s_i) \leq d$
2. $\forall j, deg(g_j) \leq \frac{d}{2}$

Give a degree 2 Positivstellensatz proof that a square has no triangle, i.e. that the following equations are unsatisfiable over $\mathbb{R}$:

1. $\forall i \in [1, 4], x_i^2 - x_i = 0$
2. $x_1 x_3 = x_2 x_4 = 0$
3. $\left(\sum_{i=1}^{4} x_i\right) - 3 = 0$

Choose one of the following two problems:

Problem 2: Linear Programming for Vertex Cover (20 points)

The vertex cover problem is as follows. Given an input graph G, what is the smallest size of a subset V of vertices of G such that every edge of G is incident with at least one vertex of V?

(a) 10 points: Give a linear programming relaxation of the vertex cover problem.

(b) 10 points: How much smaller can the value of your linear programming relaxation be than the true size of the minimal vertex cover? In other words, letting $opt(G)$ be the true size of the minimial vertex cover of G and letting $LP(G)$ be the value the linear program gives, what is the largest α such that for all input graphs G, $\alpha \cdot opt(G) \leq LP(G) \leq opt(G)$? Prove that your answer is correct by giving examples showing that $\frac{LP(G)}{opt(G)}$ can be arbitrarily close to α and proving that for all graphs G, $opt(G) \leq \frac{1}{\alpha} LP(G)$. (Hint is available)

Problem 3: Linear Programming for Maximum Matching (20 points)

Recall the maximum matching problem: Given an input graph G, what is the maximum k such that there are k edges which are all disjoint (i.e. no two of the are incident with the same vertex). Further recall the linear programming relaxation for this problem: We have a variable x_{ij} for each edge $(i,j) \in E(G)$ (where edges are always ordered so that $i < j$) and we maximize $\sum_{i,j:i<j,(i,j)\in E(G)} x_{ij}$ subject to

1. $\forall i < j : (i,j) \in E(G), 0 \leq x_{ij} \leq 1$
2. $\forall i, \sum_{j<i:(j,i)\in E(G)} x_{ji} + \sum_{j>i:(i,j)\in E(G)} x_{ij} \leq 1$

(a) 10 points: Prove that if the input graph G is bipartite, this linear programming relaxation gives the correct value of k (hint is available).

(b) 10 points: For general graphs, how much larger can the value for this linear programming relaxation be than the true size of the maximum matching? In other words, letting $LP(G)$ be the value of the linear progam on G and letting $opt(G)$ be the true size of the maximum matching in G, what is the smallest α such that for all input graphs G, $opt(G) \leq LP(G) \leq \alpha \cdot opt(G)$? Give an example where $LP(G) = \alpha \cdot opt(G)$.

Challenge question: Prove that for all input graphs G, $LP(G) \leq \alpha \cdot opt(G)$.

Problem 4: Conditions of Von Neumann's Minimax Theorem (15 points)

Recall Von Neumann's Minimax theorem

Theorem 0.1. *If X and Y are convex, compact subsets of $\mathbb{R}^n$ and $f : X \times Y \to \mathbb{R}$ is a continuous function which is convex in x and concave in y then*

$$\min_{x\in X}\max_{y\in Y} f(x,y) = \max_{y\in Y}\min_{x\in X} f(x,y)$$

(a) 5 points: Give an example showing that the theorem may fail if X and Y are closed and convex but not bounded.

(b) 5 points: Give an example showing that the theorem may fail if X and Y are compact but not convex.

(c) 5 points: Give an example showing that the theorem may fail if f is not convex in x.

(d) Challenge question: What happens if X and Y are bounded and convex but not closed?

Hints are available

Choose one of the following two problems:

Problem 5: Duality for max flow and min cut (25 points)

Recall the linear program for the max flow problem. Given non-negative capacities $\{c_{ij}\}$ (where we take $c_{n1} = \infty$), maximize x_{n1} subject to

1. $\forall i, j, 0 \leq x_{ij} \leq c_{ij}$ (no capacity is exceeded, no negative flow)
2. $\forall i, \sum_{j \neq i} x_{ji} = \sum_{j \neq i} x_{ij}$ (flow in = flow out)

(a) 10 points: What is the dual linear program?

(b) 5 points: Show that any cut separating x_1 and x_n gives a feasible solution to the dual program.

(c) 10 points: Prove the max flow/min cut theorem (assuming strong duality) by showing that for any feasible solution of the dual program, there is a cut which has equal or smaller value (hint is available).

Problem 6: Linear Programming Strong Duality (25 points)

This problem gives a proof that strong duality holds for linear programming. Recall the slack primal and dual forms for a linear program.
Primal: Maximize $c^T x$ subject to

1. $Ax = b$
2. $x \geq 0$

where these inequalities are taken coordinatewise.
Dual: Minimize $y^T b$ subject to

1. $A^T y \geq c$

(a) 10 points: Assuming that the primal is feasible (i.e. there is an $x \geq 0$ such that $Ax = b$), prove that strong duality holds using Farka's lemma, which says the following:

Lemma 0.2 (Farka's Lemma). *If $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^n$ then exactly one of the following holds:*

(a) $\exists x \in \mathbb{R}^n : Ax = b, x \geq 0$

(b) $\exists y \in \mathbb{R}^m : A^T y \geq 0, b^T y < 0$

(hint is available)

Challenge question: What happens if the primal is infeasible?

(b) 10 points: Prove Farka's lemma using the following version of the separating hyperplane theorem:

Theorem 0.3. *If X, Y are convex sets such that*

(a) $X \cap Y = \emptyset$

(b) X *is compact.*

(c) Y *is closed.*

then there is a hyperplane H such that X is entirely on one side of H, Y is entirely on the other side of H, and neither X nor Y intersect H.

(hint is available)

(c) 5 points: Show that if X and Y are convex sets and $(x, y) \in X \times Y$ has minimal distance (i.e. there is no $(x', y') \in X \times Y$ with $d(x', y') < d(x, y)$ then taking H to be hyperplane perpendicular to $y - x$ which passes through the midpoint between x and y, X is entirely on one side of H, Y is entirely on the other side of H, and neither X nor Y intersect H.

If you are familiar with compactness, complete the proof of the separating hyperplane theorem by proving that if X is compact and Y is closed then there must be a pair of points $(x, y) \in X \times Y$ of minimal distance.

Problem 7: Finite two person one round simultaneous zero-sum games (30 points)

Any finite two person zero-sum game which is played in one simultaneous round (i.e. both players choose a strategy and then simultaneously reveal them) can be represented in the following way. There is a payoff matrix M whose rows are labeled by strategies for player 1 and whose columns are labeled by strategies for player 2. M_{ij} is the amount that player 2 takes from player 1 if player 1 playes strategy i and player 2 plays strategy j. Thus, the row player is trying to minimize the resulting entry of M and the column player is trying to maximize the resulting entry of M.

For example, consider the game where both players simulataneously call out 1 or 2, player 1 wins if the sum of the numbers is odd, and player 2 wins if the sum is even. This game can be represented with the following payoff matrix: $M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

(a) 5 points: Give a payoff matrix for rock-paper-scissors.

(b) 10 points: Now consider the general game with an arbitrary payoff matrix and assume that the rules are changed so that the row player must go first but can choose a probability distribution over his/her strategies. Give a linear program which finds an optimal strategy for the row player in this setting. What is the dual linear program? How does this relate to strong duality and Nash equilibria?

(c) 15 points: Implement these linear programs to find a Nash equilibrium for the following payoff matrix:

$$M = \begin{bmatrix} 2 & 5 & 11 & 9 & 12 & 7 \\ 20 & 3 & 9 & 5 & 7 & 8 \\ 8 & 10 & 6 & 13 & 7 & 12 \\ 18 & 12 & 5 & 8 & 6 & 3 \\ 16 & 17 & 1 & 13 & 4 & 6 \\ 4 & 8 & 15 & 4 & 14 & 6 \end{bmatrix}$$

Hints

2b. For the examples, note that the linear programming relaxation always has a value of at most $\frac{n}{2}$ as it can always assign value $\frac{1}{2}$ to every vertex. For the proof, what happens if we take every vertex which has value at least $\frac{1}{2}$ in the output of the linear program?

3a. Use König's Theorem: For a bipartite graph G, the size of the maximum matching is equal to the size of the minimal vertex cover.

4a. There is an example where $X = Y = \mathbb{R}$.

4b. There is an example where $X = Y = \{-1, 1\}$.

4c. Consider the function $f(x, y) = xy$. If the y player plays first then $f(x) \leq 0$ so long x or y is 0 or x is chosen to have the opposite sign as y. On the other hand, if the x player plays first, he/she had better play $x = 0$. However, if the x player is penalized for playing $x = 0$...

5c. Show that any feasible solution for the dual can be written as a positive linear combination of a cut and another solution. For the cut, your dual linear program should involve weight variables w_{ij} for the edges of the graph where the sum of the weight variables along any path from $s = x_1$ to $t = x_n$ is 1. Take the cut $(S, \bar{S})$ where S is the set of vertices reachable from $s = x_1$ using edges of 0 or negative weight.

6a. Let z be a potential value for $c^T x$ which is larger than value for the given feasible point. Add the constraint $c^T x = z$ to the program by adding c as an extra row of A and z as an extra entry in b. Now use Farka's lemma to show that if the primal is not feasible with value z then the dual is feasible with some value less than z.

6b: Apply the separating hyperplane theorem to $X = \{0\}$ (the origin) and $Y = \{(b - Ax) : x \geq 0\}$