Problem Set 0

Sum of Squares Seminar

September 1,2017

Problem 1

What are the eigenvalues and eigenvectors of the matrix $M = \begin{bmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{bmatrix}$?

Problem 2: Orthonormal eigenvector basis for symmetric matrices

This problem gives a short proof of the very useful fact that if M is a symmetric real matrix then there is an orthonormal basis of eigenvectors of M. Recall that a real matrix S is orthogonal if $S^{-1} = S^T$. Equivalently, S is orthogonal if if the rows and the columns of S are both orthonormal sets of vectors.

(a) Show that if M is a symmetric real matrix then there is an orthogonal matrix S such that $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is an eigenvector of $S^T MS$. What does this imply about the first row and column of $S^T MS$?

(b) Prove by induction that there is an orthogonal matrix S such that $S^T MS$ is a diagonal matrix.

(c) Deduce that there is an orthonormal basis of eigenvectors of M.

Problem 3

Prove that if $X \succeq 0$ and $Y \succeq 0$ are $n \times n$ PSD (positive semidefinite) matrices then $X \bullet Y \geq 0$ where $X \bullet Y = tr(XY^T) = \sum_{i=1}^{n} \sum_{j=1}^{n} X_{ij}Y_{ij}$ is the entrywise dot product of X and Y. When is $X \bullet Y = 0$?

Problem 4: Singular Value Decomposition and Induced Norm

Recall that:

1. A singular value decomposition of a rank r real matrix M decomposes M as $M = \sum_{i=1}^{r} \lambda_i u_i v_i^T$ where $\forall i, \lambda_i > 0$ and the $\{u_i\}$ and $\{v_i\}$ are orthonormal.
2. The induced norm of a matrix M is $||M|| = \max_{v:||v||=1} \{||Mv||\}$

(a) Show that if M has singular value decomposition $M = \sum_{i=1}^{r} \lambda_i u_i v_i^T$ then $||M|| = \max_{i \in [1,r]} \{\lambda_i\}$

(b) If M is a symmetric matrix and $\{v_i\}$ is an orthonormal basis of eigenvectors of M with eigenvalues $\{\lambda_i\}$, what is a singular value decomposition of M? What does this imply about $||M||$?

Problem 5

Let $A = \begin{bmatrix} 2 & -11 \\ 5 & 10 \end{bmatrix}$.

(a) What are the eigenvalues of A?

(b) Find a singular value decomposition of A (hint is available).

Hints

Problem 5b: Recall the following facts about singular value decompositions. If $M = \sum_{i=1}^{r} \lambda_i u_i v_i^T$ then:

1. The vectors $\{v_i\}$ are orthonormal eigenvectors of $M^T M$ with eigenvalues $(\lambda_i)^2$.
2. The vectors $\{u_i\}$ are orthonormal eigenvectors of $M M^T$ with eigenvalues $(\lambda_i)^2$.
3. $u_i = \frac{1}{\lambda_i} M v_i$ and $v_i = \frac{1}{\lambda_i} M^T u_i$