Maximal symmetry groups of hyperbolic 3-manifolds

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Abstract

Every finite group acts as the full isometry group of some compact hyperbolic 3-manifold. In this paper we study those finite groups which act *maximally*, that is when the ratio

$$|\mathrm{Isom}^+(M)|/\mathrm{vol}(M)$$

is maximal among all such manifolds. In two dimensions maximal symmetry groups are called Hurwitz groups, and arise as quotients of the (2,3,7)–triangle group. Here we study quotients of the minimal co-volume lattice Γ of hyperbolic isometries in three dimensions, and its orientation-preserving subgroup Γ^+, and we establish results analogous to those obtained for Hurwitz groups. In particular, we show that for every prime p there is some $q = p^k$ such that either $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$ is quotient of Γ^+, and that for all but finitely many n, the alternating group A_n and the symmetric group S_n are quotients of Γ and also quotients of Γ^+, by torsion-free normal subgroups. We also describe all torsion-free subgroups of index up to 120 in Γ^+ (and index up to 240 in Γ), and explain how other infinite families of quotients of Γ and Γ^+ can be constructed.

1 Introduction

A well known theorem of Greenberg [13] states that every finite group acts on a Riemann surface as the full group of conformal automorphisms, and a famous theorem of Hurwitz [15] states that if S is a compact Riemann surface of genus $g > 1$, then the number of conformal automorphisms of S is bounded above by $|\mathrm{Aut}^+(S)| \leq 84(g-1)$, with equality attained if and only if the conformal automorphism group $\mathrm{Aut}^+(S)$ is a homomorphic image of the $(2, 3, 7)$ triangle group.

A reason for Hurwitz's result (although not his original argument) is that S can be identified with a quotient space $\mathbb{U}/\Gamma$, where $\mathbb{U}$ is the hyperbolic upper half-plane and Γ is a Fuchsian group. The conformal automorphism group of S is isomorphic to the factor group Δ/Γ where Δ is the normalizer of Γ in $\mathrm{PSL}(2, \mathbb{R})$. The Riemann-Hurwitz formula gives

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$|\mathrm{Aut}^+(S)| = |\Delta/\Gamma| = \mu(\Gamma)/\mu(\Delta) = 4\pi(g-1)/\mu(\Delta)$, where $\mu(\cdot)$ is the area of a fundamental region. A theorem of Siegel [24] states that among all Fuchsian groups, the $(2, 3, 7)$ triangle group has a fundamental region of uniquely smallest hyperbolic area, viz. $\mu(\Delta) = \pi/21$, so the Riemann-Hurwitz formula gives $|\mathrm{Aut}^+(S)| = 2\pi(2g-2)/\mu(\Delta) = 84(g-1)$.

Studies of automorphisms of surfaces have since focussed on determining genera for which this bound is sharp, finding sharp bounds for other genera, and finding the minimum genus of a surface on which a given group acts faithfully, among other things; see [3, 4, 5] and some of the references therein, for example. The smallest Hurwitz group is the simple group $\mathrm{PSL}(2, 7)$ of order 168, and Macbeath [20] proved that $\mathrm{PSL}(2, q)$ is Hurwitz precisely when $q = 7$, or $q = p$ for some prime $p \equiv \pm 1$ modulo 7, or $q = p^3$ for some prime $p \equiv \pm 2$ or ± 3 modulo 7. Conder [2] proved that the alternating group A_n is a Hurwitz group for all $n \geq 168$ (and for all but 64 smaller values of $n > 1$), and later [3] determined all Hurwitz groups of order less than 1,000,000. More recently, it has been shown that many more of the classical simple groups of Lie type are Hurwitz (see [19] and other references listed there), as are exactly 12 of the 26 sporadic simple groups, including the Monster (see [30]). Other families of examples are described in two survey articles [4, 5].

In this paper we investigate the 3-dimensional analogues of some of these questions. Kojima [17] has shown that every finite group acts as the full isometry group of a closed hyperbolic 3-manifold; this does not appear to be known for higher dimensions. We may, however, follow the above argument for all $n \geq 2$. An orientable n-dimensional hyperbolic manifold is the quotient space $M = \mathbb{H}^n/\Gamma$, where Γ is some torsion-free discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^n)$, the group of orientation-preserving isometries of n-dimensional hyperbolic space $\mathbb{H}^n$. The orientation-preserving isometry group $\mathrm{Isom}^+(M)$ of M is then isomorphic to Δ/Γ where Δ is the normalizer of Γ in $\mathrm{Isom}^+(\mathbb{H}^n)$. Letting O be the orientable n-dimensional orbifold $\mathbb{H}^n/\Delta$, we have $\mathrm{vol}(O) = \mathrm{vol}(M)/|\mathrm{Isom}^+(M)|$ and

$$O = \mathbb{H}^n/\Delta \cong (\mathbb{H}^n/\Gamma)/(\Delta/\Gamma) \cong M/\mathrm{Isom}^+(M),$$

and so the ratio $\frac{|\mathrm{Isom}^+(M)|}{\mathrm{vol}(M)}$ is largest precisely when $O = \mathbb{H}^n/\Delta$ is of minimum possible volume. Note that O depends only on the normaliser $\Delta = N_{\mathrm{Isom}^+(\mathbb{H}^n)}(\Gamma)$ of Γ in $\mathrm{Isom}^+(\mathbb{H}^n)$ and not on the subgroup Γ itself.

Now, in each dimension it is known that there are orientable n-dimensional orbifolds of minimum volume; for $n = 3$ this is due to Jørgensen, see [12], and for $n \geq 4$ this is due to Wang, [27]. Only in two and three dimensions, however, have these minimal volume orbifolds been identified and uniqueness established. The discrete subgroup Γ of $\mathrm{Isom}(\mathbb{H}^3)$ of uniquely smallest co-volume $\mathrm{vol}(\mathbb{H}^n/\Gamma)$ was found by Gehring and Martin [7, 8, 9, 10] and Marshall and Martin [22] to be the normaliser of the $[3, 5, 3]$-Coxeter group (which will be described in detail in the next section). Its orientation-preserving subgroup Γ^+ is a split extension by C_2 of the orientation-preserving subgroup of the $[3, 5, 3]$-Coxeter group, of index 2 in Γ.

Jones and Mednykh [16] proved that the smallest quotient of Γ^+ by a torsion-free normal subgroup is $\mathrm{PGL}(2, 9)$, of order 720, while that of the group Γ itself is $\mathrm{PGL}(2, 11) \times C_2$, of order 2640, and investigated the 3-manifolds associated with these and a number of other such quotients of small order.

In this paper we will investigate further properties of Γ and Γ^+, describing all torsion-free (but not normal) subgroups of small index, and proving the following:

Theorem A. *For every prime p there is some $q = p^k$ (with $k \le 8$) such that either* $\mathrm{PSL}(2, q)$ *or* $\mathrm{PGL}(2, q)$ *is a quotient of Γ^+ by some torsion-free normal subgroup.*

Theorem B. *For all but finitely many n, both the alternating group A_n and the symmetric group S_n are quotients of Γ, and also quotients of Γ^+, by torsion-free normal subgroups.*

The paper is structured as follows. In section 2 we describe the groups Γ and Γ^+, determine all conjugacy classes of their torsion elements of prime order, and describe all conjugacy classes of torsion-free subgroups of index up to 120 in Γ^+ (index up to 240 in Γ). We give a quick proof of Theorem A in section 3; a more detailed proof was given in the third author's doctoral thesis [26]. In section 4 we prove Theorem B, and in the final section we explain how other infinite families of quotients of Γ and Γ^+ can be constructed.

2 The extended [3,5,3] Coxeter group Γ

Let C be the $[3, 5, 3]$ Coxeter group generated by a, b, c and d with defining relations

$$a^2 = b^2 = c^2 = d^2 = (ab)^3 = (bc)^5 = (cd)^3 = (ac)^2 = (ad)^2 = (bd)^2 = 1.$$

This group can be realised as a group of hyperbolic isometries, generated by the reflections in the faces of a hyperbolic tetrahedron, two of which intersect at an angle $\frac{\pi}{5}$, with each intersecting another face at an angle $\frac{\pi}{3}$, and all other angles being $\frac{\pi}{2}$.

This tetrahedron T has a rotational symmetry that is naturally exhibited as a reflection of the Dynkin diagram of the Coxeter group (and called the graph automorphism of C), corresponding to an orientation-preserving hyperbolic isometry t of order 2 that interchanges the faces reflected by a and d and also the faces reflected by b and c.

The group Γ is a split extension (semi-direct product) of the group C by the cyclic group of order 2 generated by t, and has the finite presentation

$$\Gamma = \langle\, a, b, c, d, t \mid a^2, b^2, c^2, d^2, t^2, atdt, btct, (ab)^3, (ac)^2, (ad)^2, (bc)^5, (bd)^2, (cd)^3 \,\rangle.$$

This is the group now known to be the unique discrete subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ of smallest co-volume. Its torsion-free normal subgroups act fixed-point-freely on hyperbolic 3-space $\mathbb{H}^3$ and give rise to hyperbolic 3-manifolds with maximal symmetry group.

The orientation-preserving subgroup Γ^+ is the index 2 subgroup generated by ab, bc, cd and t, and is a split extension of $C^+ = \langle ab, bc, cd\rangle$ by $\langle t\rangle$. In fact by Reidemeister-Schreier theory, it is not difficult to show that Γ^+ is generated by the three elements $x = ba$, $y = ac$ and $z = t$, subject to defining relations $x^3 = y^2 = z^2 = (xyz)^2 = (xzyz)^2 = (xy)^5 = 1$.

We need to determine the torsion elements of Γ. To do this, note that any finite group in $\mathrm{Isom}(\mathbb{H}^3)$ fixes a point of $\mathbb{H}^3$ equivalent (under the group action) to a vertex or point on some edge (in the case of cyclic groups) of the fundamental domain obtained by bisecting the tetrahedron appropriately. Identifying the vertex stabilisers gives

Proposition 1 *Every torsion element of Γ is conjugate to an element of one of the following six subgroups of Γ stabilizing a point of the hyperbolic tetrahedron T:*
(1) $\langle a, b, c\rangle$, *isomorphic to* $A_5 \times C_2$, *of order* 120,
(2) $\langle a, b, d\rangle$, *isomorphic to* $D_3 \times C_2$, *of order* 12,
(3) $\langle a, c, d\rangle$, *isomorphic to* $D_3 \times C_2$, *of order* 12,
(4) $\langle b, c, d\rangle$, *isomorphic to* $A_5 \times C_2$, *of order* 120,
(5) $\langle a, d, t\rangle$, *isomorphic to* D_4, *of order* 8,
(6) $\langle b, c, t\rangle$, *isomorphic to* $D_5 \times C_2$, *of order* 20.

This leads to the following:

Proposition 2 *A subgroup of Γ^+ is torsion-free if and only if none of its elements is conjugate in Γ to any of ab, ac, bc or t. A subgroup Δ of Γ is torsion-free if and only if none of its elements is conjugate to any of $a, ab, ac, bc, (abc)^5, t, adt$ or $bcbcbt$.*

Proof. We need only find from among the elements of the six subgroups listed in Proposition 1 a list of representatives of conjugacy classes of torsion elements of (prime) orders 2, 3 and 5, up to inversion. This is not difficult:

(1) In the subgroup $\langle a, b, c\rangle \cong A_5 \times C_2$, every element of order 2 is conjugate to either a or ac or the central involution $(abc)^5$, while every element of order 3 is conjugate to ab, and every element of order 5 is conjugate to bc or its inverse.

(2) In the subgroup $\langle a, b, d\rangle \cong D_3 \times C_2$, every element of order 2 is conjugate to a or d or ad, while every element of order 3 is conjugate to ab, and there are no elements of order 5.

(3) In the subgroup $\langle a, c, d\rangle \cong C_2 \times D_3$, every element of order 2 is conjugate to a or c or ac, while every element of order 3 is conjugate to cd, and there are no elements of order 5.

(4) In the subgroup $\langle b, c, d\rangle \cong A_5 \times C_2$, every element of order 2 is conjugate to b or bc or the central involution $(bcd)^5$, while every element of order 3 is conjugate to cd, and every element of order 5 is conjugate to bc or its inverse.

(5) In the subgroup $\langle a, d, t\rangle \cong D_4$, every element of order 2 is conjugate to a or ad or t, and there are no elements of order 3 or 5.

(6) In the subgroup $\langle b, c, t\rangle \cong D_5 \times C_2$, every element of order 2 is conjugate to b or t or the central involution $bcbcbt$, while every element of order 5 is conjugate to bc, and there are no elements of order 3.

Finally note that each of b, c and d is conjugate to a in at least one of these six subgroups, that cd is conjugate in Γ to $(cd)^t = ba = (ab)^{-1}$, that ad is conjugate in $\langle a, c, d\rangle$ to ac, and that $(bcd)^5$ is conjugate in Γ to $(cba)^5 = (abc)^5$. The result follows. □

In order to work purely within Γ^+, without any reference to Γ, the following corollary of the above proposition can be more useful:

Corollary 3 *A subgroup of* Γ^+ *is torsion-free if and only if none of its elements is conjugate to any of* ab, ac, bc, t *or* adt*.*

Proof. Every torsion element of prime order in Γ^+ lies in the Γ-conjugacy class of one of ab, ac, bc or t, and each Γ-conjugacy class g^Γ is the union of one or two conjugacy classes in Γ^+, namely g^{Γ^+} and $(g^u)^{\Gamma^+}$ for any $u \in \Gamma \setminus \Gamma^+$. Since $(ab)^a = ba = (ab)^{-1}$, and $(ac)^a = ca = ac$, and $(bc)^b = cb = (bc)^{-1}$, while $t^a = ata = adt$, the result follows. □

Proposition 2 also feeds into a useful test for torsion in subgroups of finite index in Γ, derivable from the standard permutation representation of Γ on the right cosets of the subgroup by right multiplication.

Proposition 4 *A subgroup* Σ *of finite index in* Γ *is torsion-free if and only if none of the elements* $a, ab, ac, bc, (abc)^5, t, adt$ *or* $bcbcbt$ *has a fixed point in the natural action of* Γ *on the right coset space* $(\Gamma : \Sigma) = \{\Sigma g : g \in \Gamma\}$ *by right multiplication. Similarly, a subgroup* Σ *of finite index in* Γ^+ *is torsion-free if and only if none of the elements* ab, ac, bc, t *or* adt *has a fixed point in the natural action of* Γ^+ *on the coset space* $(\Gamma^+ : \Sigma)$*.*

Proof. If g is any of the given torsion elements and g has a fixed point Σu, then ugu^{-1} is a torsion element of finite order in Σ. Conversely, by Proposition 2 any torsion element in Σ must be of the form ugu^{-1}, where g is one of the given elements, and then g fixes the coset Σu. □

Corollary 5 *Any torsion-free subgroup of finite index in* Γ^+ *must have index in* Γ^+ *divisible by 60, and index in* Γ *divisible by 120.*

Proof. Any torsion-free subgroup Σ of finite index in Γ must intersect trivially the torsion subgroup $\langle a, b, c\rangle$ of order 120, and hence in the natural action of Γ on the coset space $(\Gamma : \Sigma)$, all orbits of $\langle a, b, c\rangle$ must have length 120, and in the corresponding action of Γ^+ on $(\Gamma^+ : \Sigma)$, all orbits of $\langle ab, bc\rangle$ must have length 60. □

Using these facts one can seek all torsion-free subgroups of small index in Γ, with the help of the `LowIndexSubgroups` command in the computational algebra system MAGMA [1].

There are exactly two Γ-conjugacy classes of torsion-free subgroups of index 60 in Γ^+ (and index 120 in Γ). A representative of one such class is the subgroup Σ_{60a} generated by $x = abt$ and $y = bcbacdbcbcdb$. The `Rewrite` command in MAGMA (using Reidemeister-Schreier theory) gives $\langle\, x, y \mid x^3yx^3yx^3y^{-1}xy^{-1}, x^4y^{-1}xy^{-2}xy^{-1}xy^{-2}xy^{-1} \,\rangle$ as a defining presentation for this subgroup, and forcing the generators to commute reveals that the abelianisation (first homology group) $\Sigma_{60a}/[\Sigma_{60a}, \Sigma_{60a}]$ is isomorphic to $\mathbb{Z}_{70}$, the cyclic group of order 70. The quotient $\Gamma^+/\mathrm{core}_{\Gamma^+}(\Sigma_{60a})$ by the core of Σ_{60a} in Γ^+ (the kernel of the action of Γ^+ on the coset space $(\Gamma^+ : \Sigma)$) is isomorphic to $\mathrm{PSL}(2, 29)$, of order 12180, and the quotient $\Gamma/\mathrm{core}_\Gamma(\Sigma_{60a})$ is isomorphic to a split extension of $\mathrm{PSL}(2,29)\times\mathrm{PSL}(2,29)$ by C_2.

We can give a description of this manifold as follows. It is obtained from the cusped manifold m017 of volume 2.828122... and homology $\mathbb{Z}_7+\mathbb{Z}$ of Weeks' census [28] by performing $(-4,3)$ Dehn surgery [25] on the cusp. These and subsequent data can be checked by comparing the fundamental groups and/or identifying and comparing the arithmetic data using Goodman's Snap program and arithmetic census data [11].

The covolume of Σ_{60a} is 2.3430... and its symmetry group is $\mathbb{Z}_2$ with Chern-Simons invariant 29/120. A representation in $\mathrm{PSL}(2,\mathbb{C})$ is given by two matrices A and B with $\mathrm{tr}(A)=i\sqrt{(1+\sqrt{5})/2}$, $\mathrm{tr}(B)=z$, where $z^4-2z^3+z-1=0$ is complex, and

$$\mathrm{tr}[A,B]=-4.5450849718747\ldots+i1.58825139255011\ldots$$

in the (trace) field of degree 8, $\mathbb{Q}(z,\sqrt{5})$.

Also we have the following data on finite-sheeted covers, with CS denoting the Chern-Simons invariant:

- 2 sheets: 1 cyclic cover, homology $\mathbb{Z}_{35}$, CS$=-1/60$;
- 5 sheets: 1 cyclic cover, homology $\mathbb{Z}_{14}$, CS=5/24;
- 7 sheets: 2 covers, homology $\mathbb{Z}_2+\mathbb{Z}_{20}+\mathbb{Z}$ and $\mathbb{Z}_{29}+\mathbb{Z}_{290}$ (cyclic cover), CS =23/120;
- 8 sheets: 3 covers, homology $(\mathbb{Z}_2)^3+\mathbb{Z}_{210}$, $(\mathbb{Z}_2)^3+\mathbb{Z}_{70}$ and $\mathbb{Z}_{1260}$, CS $=-1/15$;
- 9 sheets: 1 cover, homology $\mathbb{Z}_{210}$, CS=7/40;
- 10 sheets: 5 covers, homology $\mathbb{Z}_{112}+\mathbb{Z}+\mathbb{Z}$, $\mathbb{Z}_{42}+\mathbb{Z}$ (two), $\mathbb{Z}_5+\mathbb{Z}_{560}$, $\mathbb{Z}_{11}+\mathbb{Z}_{77}$ (cyclic cover), CS$=-1/12$.

Note that one of the two covers of 7 sheets and three of the five covers of 10 sheets have infinite homology. There are no other covers of fewer than 11 sheets.

A representative of the other class is the subgroup Σ_{60b} generated by $x=abt$ and $y=bacbacbcdbcb$, with presentation $\langle\, x,y \mid x^3yx^3yx^{-1}y^2x^{-1}y, x^4y^{-2}xy^{-3}xy^{-2}\,\rangle$, and abelianisation $\mathbb{Z}_{58}$. The quotient $\Gamma^+/\mathrm{core}_{\Gamma^+}(\Sigma_{60b})$ is isomorphic to $\mathrm{PSL}(2,59)$, of order 102660, and the quotient $\Gamma/\mathrm{core}_\Gamma(\Sigma_{60b})$ is a split extension of $\mathrm{PSL}(2,59)\times\mathrm{PSL}(2,59)$ by C_2.

This manifold is obtained from the cusped manifold m016 of volume 2.828122... and homology $\mathbb{Z}$ of Weeks' census [28] by performing $(-4,3)$ Dehn surgery [25] on the cusp. Its volume is again 2.3430... (of course) and its symmetry group is again $\mathbb{Z}_2$ but this time the Chern-Simons invariant is $-11/120$. A representation in $\mathrm{PSL}(2,\mathbb{C})$ is given by matrices A and B with $\mathrm{tr}(A)=i\sqrt{(\sqrt{5}-1)/2}$, $\mathrm{tr}(B)=0.77168095921341\ldots+i0.63600982475703\ldots$, and

$$\mathrm{tr}[A,B]=-3.73606797749979\ldots+i2.56984473582543\ldots,$$

again in the field $\mathbb{Q}(z,\sqrt{5})$. Also we have the following data on finite-sheeted covers, none of which has infinite homology:

- 2 sheets: 1 cyclic cover, homology $\mathbb{Z}_{29}$, CS$=-11/60$;
- 8 sheets: 1 cover, homology $\mathbb{Z}_2+\mathbb{Z}_{870}$, CS $=-7/30$;

- 9 sheets: 1 cover, homology $\mathbb{Z}_{522}$, CS=7/40;
- 10 sheets: 2 covers, homology $\mathbb{Z}_2+\mathbb{Z}_2+\mathbb{Z}_{464}$, $\mathbb{Z}_{1392}$, CS=$-1/12$.

Only the two-sheeted cover is cyclic, and there are no other covers of fewer than 11 sheets.

Next, there are exactly seven Γ-conjugacy classes of torsion-free subgroups of index 120 in Γ^+ (index 240 in Γ). Details of representatives of these classes are summarised below.

Subgroup Σ_{120a}

Generators:	$x=abcd,\ y=acbacbdcab$
Defining relations:	$xy^{-1}x^2yx^2yx^2y^{-1}x^2yx^2yx^2y^{-1}xy^{-2}$ $=yx^2yx^2y^{-1}xy^{-1}x^2yx^2y^{-1}xy^{-1}x^2yx^2yx=1$
Abelianisation:	$\mathbb{Z}_{29}$
Quotient by core in Γ^+:	$\mathrm{PGL}(2,59)$, of order 205320
Quotient by core in Γ:	a C_2 extension of a subdirect product of two copies of $\mathrm{PGL}(2,59)$, of order 42156302400
Manifold:	$(-3,2)$ Dehn surgery on s897
Chern-Simons:	11/60
Covers of ≤ 10 sheets:	8 of 6 sheets, 2 of 7 sheets, 1 of 8 sheets, 1 of 9 sheets, 8 of 10 sheets (two with infinite homology: $\mathbb{Z}_2+\mathbb{Z}_{58}+\mathbb{Z}$).

Subgroup Σ_{120b}

Generators:	$x=abcd,\ y=acbacdcbcb$
Defining relations:	$x^2yxyx^2yx^{-1}yx^2yx^{-1}y^2x^{-1}yx^2yx^{-1}y$ $=xy^{-1}x^{-2}y^{-1}xy^{-1}x^{-2}y^{-1}xyx^{-1}yx^2yx^{-1}y^3x^{-1}yx^2yx^{-1}y=1$
Abelianisation:	$\mathbb{Z}_{35}$
Quotient by core in Γ^+:	$\mathrm{PSL}(2,29)\times C_2$, of order 24360
Quotient by core in Γ:	$(C_2\times \mathrm{PSL}(2,29)\times \mathrm{PSL}(2,29)){:}C_2$, of order 593409600
Manifold:	$(-3,2)$ Dehn surgery on s900
Chern-Simons:	1/60
Covers of ≤ 10 sheets:	5 of 5 sheets (four with infinite homology: $\mathbb{Z}_{42}+\mathbb{Z}$), 8 of 6 sheets (two with infinite homology: $\mathbb{Z}_{140}+\mathbb{Z}$), 10 of 7 sheets: (seven with infinite homology: $\mathbb{Z}_{65}+\mathbb{Z}+\mathbb{Z}$ in four cases, $\mathbb{Z}_2+\mathbb{Z}_{10}+\mathbb{Z}+\mathbb{Z}$ in two cases, $\mathbb{Z}_{120}+\mathbb{Z}$ in one), 15 of 8 sheets (none with infinite homology), 5 of 9 sheets (none with infinite homology), 24 of 10 sheets (nineteen with infinite homology: $\mathbb{Z}_4+\mathbb{Z}_4+\mathbb{Z}_{168}+\mathbb{Z}+\mathbb{Z}$ in one case, $\mathbb{Z}_7+\mathbb{Z}_{224}+\mathbb{Z}$ in two cases, $\mathbb{Z}_{14}+\mathbb{Z}_{42}+\mathbb{Z}$ in four, $\mathbb{Z}_3+\mathbb{Z}_{84}+\mathbb{Z}$ in eight, and $\mathbb{Z}_2+\mathbb{Z}_{84}+\mathbb{Z}$ in four).

There is another interesting way to see this group. If we take the knot 5_2 in Rolfsen's tables and take the two-sheeted cyclic cover we get s900. Then $(-3,2)$ Dehn surgery will produce the hyperbolic manifold with fundamental group Σ_{120b}.

Subgroup Σ_{120c}

Generators:	$x = abcd,\ y = acbacbacdbcb,\ z = acbacdcbcdbcab$
Defining relations:	$xy^{-1}xz^{-1}y^2z^{-1} = x^2y^{-1}xy^{-1}x^2zy^{-1}z$ $= x^2yz^{-1}x^{-1}z^{-1}yx^2z = 1$
Abelianisation:	$\mathbb{Z}_{29}$
Quotient by core in Γ^+:	$\mathrm{PSL}(2,59) \times C_2$, of order 205320
Quotient by core in Γ:	$(C_2 \times \mathrm{PSL}(2,59) \times \mathrm{PSL}(2,59)){:}C_2$, of order 42156302400
Manifold:	$(3,2)$ Dehn surgery on v2051
Chern-Simons:	$-1/15$
Covers of ≤ 10 sheets:	4 of 6 sheets, 8 of 7 sheets, 12 of 8 sheets, 6 of 9 sheets, and 6 of 10 sheets (none with infinite homology).

Subgroup Σ_{120d}

Generators:	$x = abcd,\ y = acbacbcdcbcb,\ z = acbadcbacdbcab$
Defining relations:	$xy^{-1}zx^2zy^{-1} = x^2y^{-1}z^{-1}yx^{-1}zx^2z^{-1}yx^{-1}z$ $= x^2y^{-1}zy^{-1}x^3y^{-1}zy^{-1}x^2z = 1$
Abelianisation:	$\mathbb{Z}_9$
Quotient by core in Γ^+:	$(Z_3)^{19}{:}\mathrm{PGL}(2,19)$, of order 7949868434280
Quotient by core in Γ:	a C_2 extension of a subdirect product of two copies of $(Z_3)^{19}{:}\mathrm{PGL}(2,19)$, of order 63200408122361538679118400
Manifold:	$(3,2)$ Dehn surgery on s890
Chern-Simons:	$1/30$
Covers of ≤ 10 sheets:	1 of 3 sheets, 5 of 4 sheets, 9 of 6 sheets (two with infinite homology $\mathbb{Z}_3 + \mathbb{Z} + \mathbb{Z}$), 35 of 8 sheets (18 with infinite homology: $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_{36} + \mathbb{Z}$ in 12 cases, $\mathbb{Z}_{36} + \mathbb{Z}$ in six), 9 of 9 sheets (two with infinite homology $\mathbb{Z}_3 + \mathbb{Z}_9 + \mathbb{Z}$), and 2 of 10 sheets.

Subgroup Σ_{120e}

Generators:	$x = abcbt,\ y = adcbacbadct$
Defining relations:	$xyx^{-2}yx^{-2}yxyxyx^{-2}yx^{-2}yxy^2xy^2$ $= x^2yxyx^{-2}yx^{-2}yxyx^2y^{-1}x^2y^{-1}xy^{-1}x^2y^{-1} = 1$
Abelianisation:	$\mathbb{Z}_{80}$
Quotient by core in Γ^+:	a subdirect product of $\mathrm{PGL}(2,9)$ and $\mathrm{PGL}(2,11)$, of order 475200
Quotient by core in Γ:	a C_2 extension of a subdirect product of two copies of each of $\mathrm{PGL}(2,9)$ and $\mathrm{PGL}(2,11)$, of order 342144000
Manifold:	$(-3,2)$ Dehn surgery on v2413,
Chern-Simons:	$-1/5$
Covers of ≤ 10 sheets:	1 of 2 sheets, 1 of 3 sheets, 4 of 4 sheets, (one with infinite homology $\mathbb{Z}_2 + \mathbb{Z}_{20} + \mathbb{Z}$), 1 of 5 sheets, 7 of 6 sheets (one with infinite homology $\mathbb{Z}_4 + \mathbb{Z}_{40} + \mathbb{Z}$), 2 of 7 sheets, 25 of 8 sheets (15 with infinite homology: one $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_{40} + \mathbb{Z} + \mathbb{Z}$, two $\mathbb{Z}_2 + \mathbb{Z}_{20} + \mathbb{Z} + \mathbb{Z}$,

one $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_{10} + \mathbb{Z} + \mathbb{Z}$, one $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_{240} + \mathbb{Z}$, one $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_4 + \mathbb{Z}_{60} + \mathbb{Z}$, one $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_{30} + \mathbb{Z}$, six $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_{160} + \mathbb{Z}$, and two $\mathbb{Z}_4 + \mathbb{Z}_4 + \mathbb{Z}_{20} + \mathbb{Z}$), 3 of 9 sheets, 10 of 10 sheets, (four with infinite homology: one $\mathbb{Z}_{40} + \mathbb{Z}_{40} + \mathbb{Z}$, one $\mathbb{Z}_{12} + \mathbb{Z}_{24} + \mathbb{Z}$, two $\mathbb{Z}_{96} + \mathbb{Z}$).

Subgroup Σ_{120f}

Generators:	$x = acdbcb,\ y = cbdcabcd$
Defining relations:	$yxyxy^2xy^2x^{-1}yx^{-2}yx^{-1}y^2xy$
	$= xyx^2y^{-1}x^2y^{-1}xy^{-1}x^2y^{-1}x^2yxy^2 = 1$
Abelianisation:	$\mathbb{Z}_{11} \oplus \mathbb{Z}_{11}$
Quotient by core in Γ^+:	$\mathrm{PGL}(2, 11)$, of order 1320
Quotient by core in Γ:	$\mathrm{PGL}(2, 11) \times C_2$, of order 2640.

This group is particularly interesting. If we perform $(5, 0)$ orbifold Dehn surgery on the figure of eight knot complement — so the underling space is the 3-sphere branched along the figure eight knot — and then take the cyclic cover of 5 sheets, we obtain a manifold whose fundamental group is the group Σ_{120f}. This manifold, which does not appear on the census, has an exceptionally large symmetry group, being amphichiral of order 40 and having presentation

$$\langle\, a, b : a^{10}, b^4, (ba)^2, (ba^2ba^{-3})^2, (ba^{-1})^2, b^2a^3b^2a^{-3} \,\rangle.$$

In the census, only $v3215(1, 2)$ with volume $4.7494\ldots$ has a larger symmetry group (non-abelian of order 72). The order of the symmetry group could be at most 120 here. The Chern-Simons invariant of this manifold is 0.

There are also the following non-cyclic covers of up to 10 sheets:

- 6 sheets: 14 covers, with homology $\mathbb{Z}_{11} + \mathbb{Z}_{66}$ in ten cases and $\mathbb{Z}_{11} + \mathbb{Z}_{33}$ in four;
- 7 sheets: 10 covers, all with homology $\mathbb{Z}_{44} + \mathbb{Z}_{88}$;
- 8 sheets: 5 covers, all with infinite homology $\mathbb{Z}_{33} + \mathbb{Z}_{33} + \mathbb{Z}$;
- 9 sheets: 10 covers, all with infinite homology $\mathbb{Z}_{11} + \mathbb{Z}_{11} + \mathbb{Z} + \mathbb{Z}$;
- 10 sheets: 17 covers, with 15 having infinite homology ($\mathbb{Z}_{66} + \mathbb{Z}_{66} + \mathbb{Z}$ in ten cases and $\mathbb{Z}_{55} + \mathbb{Z}_{55} + \mathbb{Z}$ in five), and two with finite homology $\mathbb{Z}_{11} + \mathbb{Z}_{11} + \mathbb{Z}_{11} + \mathbb{Z}_{264}$.

In fact there is another way to see this manifold. If one performs $(2, 0)$ and $(5, 0)$ orbifold Dehn surgery on the two-bridge link 6^2_2 of Rolfsen's tables, one obtains the orbifold $O_{2,5}$ whose fundamental group is a lattice with presentation

$$\pi_{2,5} = \langle\, a, b : a^2 = b^5 = (abab^{-1}ab^{-1})^2ababab^{-1}ab \,\rangle.$$

Geometrically the underlying orbifold $O_{2,5}$ is the sphere with branching of degree 2 along one link and degree 5 along the other, and has homology $\mathbb{Z}_{10}$. The group $\pi_{2,5}$ has index 12 in Γ^+. The 10-fold cyclic cover (obtained by unwrapping both branching links) is the manifold corresponding to Σ_{120f}.

Subgroup Σ_{120g}

Generators:	$x = acdbcb,\ y = abacbdcabcdc,\ z = abacbcbabcbcda$
Defining relations:	$x^{-1}yxzyz^{-1}y = x^2zyxzx^{-1}zy^{-1}z = x^2yz^{-1}xy^{-1}zxzy^{-1}z = 1$
Abelianisation:	$\mathbb{Z}_2 \oplus \mathbb{Z}_{18}$
Quotient by core in Γ^+:	$(Z_3)^{19}:\mathrm{PGL}(2,19)$, of order 7949868434280
Quotient by core in Γ:	a C_2 extension of a subdirect product of two copies of $(Z_3)^{19}:\mathrm{PGL}(2,19)$, of order 63200408122361538679118400
Manifold:	$(-1,2)$ Dehn surgery on v3318
Chern-Simons:	$-1/30$
Covers of ≤ 10 sheets:	3 of 2 sheets, 1 of 3 sheets, 2 of 4 sheets, 11 of 6 sheets (one with infinite homology $\mathbb{Z}_3 + \mathbb{Z} + \mathbb{Z}$), 23 of 8 sheets (one with infinite homology $\mathbb{Z}_6 + \mathbb{Z}_{18} + \mathbb{Z} + \mathbb{Z}$), 12 of 9 sheets (two with infinite homology: $\mathbb{Z}_{18} + \mathbb{Z}_{54} + \mathbb{Z}$ and $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_6 + \mathbb{Z}_{18} + \mathbb{Z}$), and 14 of 10 sheets (one with infinite homology $\mathbb{Z}_{72} + \mathbb{Z} + \mathbb{Z}$).

Note that the core of the subgroup Σ_{120f} in Γ^+ is also the core of Σ_{120f} in Γ, and hence normal in Γ, while for each of the other eight representative subgroups above, the core of the subgroup in Γ is the intersection of two normal subgroups of Γ (one being the core in Γ^+ and the other its conjugate under an element of $\Gamma \setminus \Gamma^+$).

Corollary 5 tells us that any torsion-free subgroup of Γ^+ must have index a multiple of 60. The data on low index covers with infinite homology show that for $k \in \{7, 8, 10, 12, 18\}$, there are torsion-free subgroups of Γ^+ of index $60kn$ for every positive integer n. The complete spectrum of possible indices (of torsion-free subgroups of Γ^+) will be investigated in a subsequent paper.

3 Projective linear groups as quotients of Γ^+

The following is a partial generalisation of Macbeath's theorem [20]:

Theorem 6 *For any prime p there exists a positive integer k such that either* $\mathrm{PSL}(2, p^k)$ *or* $\mathrm{PGL}(2, p^k)$ *is a quotient of the orientation-preserving subgroup Γ^+ by a torsion-free normal subgroup.*

A complete proof of this theorem appears in the third author's doctoral thesis [26]. We give a sketch of the proof only.

First, as observed in the previous section, Γ^+ has an alternative defining presentation

$$\Gamma^+ = \langle\, x, y, z \mid x^3 = y^2 = z^2 = (xyz)^2 = (xzyz)^2 = (xy)^5 = 1 \,\rangle,$$

with $x = ba$, $y = ac$ and $z = t$.

When $p = 2$ we find $\mathrm{PSL}(2,16) \cong \mathrm{SL}(2,16)$ is a quotient of Γ^+, obtainable from the homomorphism $\theta : \Gamma^+ \to \mathrm{SL}(2,16)$ such that

$$\theta(x) = \begin{bmatrix} \lambda & \lambda^9 \\ \lambda & \lambda^4 \end{bmatrix}, \quad \theta(y) = \begin{bmatrix} 1 & \lambda^4 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \theta(z) = \begin{bmatrix} \lambda^6 & \lambda^7 \\ \lambda^4 & \lambda^6 \end{bmatrix},$$

where λ is a zero of the polynomial s^4+s+1 over GF(2). Similarly when $p = 3$, the group PGL(2,9) is a quotient of Γ^+, obtainable from the homomorphism $\theta : \Gamma^+ \to \mathrm{SL}(2,9)$ with

$$\theta(x) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \theta(y) = \begin{bmatrix} 1 & 0 \\ \lambda & -1 \end{bmatrix} \quad \text{and} \quad \theta(z) = \begin{bmatrix} 0 & 1 \\ \lambda & 0 \end{bmatrix},$$

where λ is a zero of the polynomial $s^2 + s + 2$ over GF(3). In both cases, the kernel of the homomorphism is torsion-free by Proposition 2, since the images of the elements $x^{-1}\,(= ab)$, $y\,(= ac)$, $xy\,(= bc)$ and $z\,(= t)$ are all nontrivial.

When $p \geq 5$, let λ be a zero of the polynomial $f(u) = u^{16}+12u^{12}-122u^8+972u^4+6561$ in a finite extension field K over GF(p), and define matrices X, Y and Z in GL(2, K) by

$$X = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 3/\lambda^2 \\ \lambda^2 & -1 \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} 0 & 1 \\ \lambda^2 & 0 \end{bmatrix}.$$

One can verify the map $(x, y, z) \mapsto (X, Y, Z)$ induces a homomorphism $\theta : \Gamma^+ \to \mathrm{PGL}(2,K)$, with torsion-free kernel. Dickson's classification of subgroups of the groups PSL(2, q) [14], and the fact that the images of x and y generate an insoluble subgroup isomorphic to A_5, then imply that the image of θ is PGL(2, p^k) for some k.

Note that in this construction, X, Y and Z are matrices over $F(\lambda^2)$ where $F = \mathrm{GF}(p)$, and hence k is at most 8. This bound is larger than necessary, however. In the case $p = 3$, for example, the polynomial $f(u) = u^{16} + 12u^{12} - 122u^8 + 972u^4 + 6561$ factorises over GF(3) as $s^4(s^2+s+2)(s^2+2s+2)$ where $s = u^2$, so choosing λ as a zero of the polynomial $s^2 + s + 2$ we obtain the homomorphism from Γ^+ onto PGL(2,9) given above.

A more precise version of Theorem 6 can be obtained using ideas from [18], [20] and [21], and will be given in a sequel to this paper, stating explicit values of q for which PSL(2, q) or PGL(2, q) is a quotient of Γ^+, and showing that $q = p, p^2$ or p^4 in all cases. The case $q \equiv 1$ mod 10 was also dealt with by Paoluzzi in [23]. We are grateful to Colin Maclachlan, Alan Reid and Bruno Zimmerman for pointing out these references.

4 Maximal alternating and symmetric groups

The main result of this paper is the following analogue of the main theorem from [2]:

Theorem 7 *For all sufficiently large n, the alternating group A_n and the symmetric group S_n are quotients of the extended [3,5,3] Coxeter group Γ, and also quotients of its orientation-preserving subgroup Γ^+, by torsion-free normal subgroups. In particular, this holds whenever n and $n-159$ are expressible in the form $52r + 365s$ for integers r and s satisfying $1 \leq r \leq s$, and hence for all $n \geq 152000$ (as well as many smaller values of n).*

We prove this theorem using permutation representations of the group Γ on 52, 365 and 159 points, and linking these together to construct transitive permutation representations of Γ of arbitrarily large degree. Specifically, we will link together r copies of the first representation with s copies of the second one, and show that the resulting transitive permutation representation on $n = 52r + 365s$ points gives a homomorphism from Γ onto A_n if n is even, and S_n if n is odd. Adjoining one extra copy of the third representation will provide a representation on $52r + 365s + 159$ points, giving a homomorphism from Γ onto S_n if n is even, and A_n if n is odd.

The representations we use as building blocks (on 52, 365 and 159 points) were found with the help of MAGMA [1], and can be defined as follows:

(A) *Representation A* is the transitive permutation representation of Γ of degree 52 corresponding to the action of Γ by (right) multiplication on right cosets of the subgroup Σ_A generated by a, b, *cbcbtabc*, *bctatatatcb*, *dcbacbcdbcbcabcd*, *bctabctabatcbatcb*, *bctabctacatcbatcb*, *dcbacbcdbcbcabcd* and *dcbacbacbtabcabcabcd*, of index 52 in Γ.

The permutations induced by each of a, b, c, d and t may be written as follows:

$$\begin{aligned}
a &\mapsto (4,9)(5,10)(7,15)(11,18)(12,19)(13,21)(16,24)(17,23)(20,27)(22,28)(25,30)(26,29)\\
&\quad (31,34)(32,37)(35,39)(40,43)(45,48)(46,50)(47,51)(49,52),\\
b &\mapsto (2,5)(4,7)(6,12)(8,16)(9,17)(11,20)(13,18)(14,22)(15,23)(21,27)(25,32)(26,33)\\
&\quad (30,35)(31,36)(37,39)(40,45)(41,46)(43,47)(44,49)(48,51),\\
c &\mapsto (1,2)(3,8)(5,11)(6,14)(7,13)(10,18)(12,19)(15,21)(16,25)(17,26)(23,29)(24,30)\\
&\quad (33,38)(35,40)(36,41)(39,43)(42,44)(46,50)(47,52)(49,51),\\
d &\mapsto (1,3)(2,6)(5,12)(8,14)(10,19)(11,18)(13,20)(16,22)(21,27)(24,28)(26,31)(29,34)\\
&\quad (33,36)(38,42)(40,45)(41,44)(43,48)(46,49)(47,51)(50,52),\\
t &\mapsto (1,4)(2,7)(3,9)(5,13)(6,15)(8,17)(10,20)(11,18)(12,21)(14,23)(16,26)(19,27)\\
&\quad (22,29)(24,31)(25,33)(28,34)(30,36)(32,38)(35,41)(37,42)(39,44)(40,46)(43,49)\\
&\quad (45,50)(47,51)(48,52).
\end{aligned}$$

For later use, we observe that the permutations induced by a, b, c and d all have cycle structure $1^{12}2^{20}$ (where k^{n_k} indicates n_k cycles of length k), while the one induced by t has cycle structure 2^{26}. In particular, all of these permutations are even. Also we note that the element *bcabat* induces the following permutation of cycle structure $2^2 6^8$:

$$\begin{aligned}
&(1,20,13,7,14,29)(2,12,34,30,47,33)(3,31,45,38,22,27)(4,10,19,21,11,5)(6,15,25,\\
&\quad 41,35,17)(8,44,46,28,23,18)(9,16)(24,42,48,43,39,36)(26,32)(37,52,50,40,51,49).
\end{aligned}$$

(B) *Representation B* is the transitive permutation representation of Γ of degree 365 corresponding to the action of Γ by (right) multiplication on right cosets of the subgroup Σ_B generated by a, b, *cbcbtabc*, *bctatatatcb*, *bctabctabatcbatcb*, *bctabctacatcbatcb*, *dcbacbacbadcbcdabcabcabcd*, *dcbacbacbacbcdcbatbcdbcabcabcabcd*, *dcbacbcbdcbacbdcbcabcdb cabcdbcbcabcd*, *dcbacbcbdcbacbcdcbcbtabcdcbcabcdbcbcabcd*, *dcbacbacbcdcbcbadcbacbacbcdcb cabcabcdabcbcdcbcabcabcd*, *dcbacbacbcdcbcbadcbcbadcbacbcabcdabcabcdabcbcdcbcabcabcd*, *dcbacbacbcdcbcbadcbacbcdcbacbcabcdcbcabcdabcbcdcbcabcabcd* and *dcbacbacbcdcbcbadcbcbdc bacbdcbacbctcabcdbcabcdbcbcdabcbcdcbcabcabcd*, of index 365 in Γ.

The permutations induced by a, b and t may be written as follows:

$a \mapsto$ (4, 9)(5, 10)(7, 15)(11, 18)(12, 19)(13, 21)(16, 24)(17, 23)(20, 27)(22, 28)(25, 30)(26, 29)(31, 34)
(32, 37)(35, 39)(40, 43)(45, 51)(46, 49)(47, 54)(48, 55)(50, 59)(52, 60)(53, 63)(56, 64)(57, 65)
(58, 70)(61, 72)(62, 71)(66, 79)(67, 76)(68, 81)(69, 78)(73, 86)(77, 93)(80, 92)(83, 96)(84, 99)
(87, 101)(91, 108)(94, 109)(98, 111)(103, 116)(104, 118)(106, 120)(113, 126)(117, 130)(121, 133)
(122, 132)(123, 135)(125, 138)(129, 142)(131, 134)(136, 143)(137, 146)(140, 148)(144, 152)
(145, 151)(149, 157)(150, 156)(153, 161)(155, 163)(158, 165)(159, 167)(164, 171)(166, 172)
(168, 173)(175, 181)(176, 179)(177, 184)(178, 185)(180, 189)(182, 190)(183, 193)(186, 194)
(187, 195)(188, 200)(191, 202)(192, 201)(196, 209)(197, 206)(198, 211)(199, 208)(203, 216)
(207, 223)(210, 222)(212, 228)(213, 227)(214, 231)(217, 234)(221, 239)(224, 242)(225, 241)
(226, 245)(229, 248)(230, 247)(232, 252)(236, 258)(237, 260)(240, 261)(243, 264)(244, 263)
(246, 269)(249, 270)(250, 275)(251, 277)(254, 266)(255, 278)(256, 282)(259, 285)(262, 286)
(265, 287)(267, 290)(268, 294)(272, 295)(273, 299)(276, 302)(279, 303)(280, 292)(281, 305)
(284, 307)(288, 309)(289, 311)(291, 312)(293, 314)(296, 315)(298, 317)(301, 319)(304, 313)
(310, 326)(320, 333)(321, 328)(322, 336)(323, 332)(325, 340)(327, 341)(329, 343)(331, 345)
(334, 347)(335, 342)(337, 350)(338, 346)(348, 354)(351, 356)(364, 365),

$b \mapsto$ (2, 5)(4, 7)(6, 12)(8, 16)(9, 17)(11, 20)(13, 18)(14, 22)(15, 23)(21, 27)(25, 32)(26, 33)(30, 35)
(31, 36)(37, 39)(40, 45)(41, 47)(43, 48)(44, 50)(46, 52)(49, 57)(51, 55)(58, 68)(60, 65)
(62, 73)(63, 74)(64, 75)(66, 77)(69, 82)(70, 83)(71, 84)(72, 85)(76, 90)(79, 94)(80, 95)
(81, 96)(86, 99)(88, 104)(91, 106)(93, 109)(98, 113)(101, 115)(102, 117)(107, 121)(108, 122)
(110, 123)(111, 125)(116, 128)(119, 131)(120, 132)(124, 136)(126, 138)(129, 140)(141, 149)
(142, 150)(145, 153)(146, 154)(147, 155)(148, 156)(151, 159)(152, 160)(158, 166)(161, 167)
(164, 172)(165, 171)(168, 175)(169, 177)(173, 178)(174, 180)(176, 182)(179, 187)(181, 185)
(188, 198)(190, 195)(192, 203)(193, 204)(194, 205)(196, 207)(199, 212)(200, 213)(201, 214)
(202, 215)(206, 220)(208, 224)(209, 225)(210, 226)(211, 227)(216, 231)(218, 237)(221, 238)
(222, 240)(223, 241)(228, 242)(230, 251)(232, 254)(233, 256)(234, 257)(235, 259)(243, 266)
(244, 268)(245, 261)(246, 271)(247, 273)(248, 274)(249, 276)(252, 264)(253, 279)(255, 280)
(258, 283)(263, 289)(265, 291)(267, 292)(270, 296)(272, 297)(275, 300)(277, 299)(278, 290)
(281, 293)(284, 301)(286, 308)(287, 310)(294, 311)(302, 315)(304, 320)(305, 321)(306, 322)
(307, 323)(309, 324)(312, 326)(313, 327)(314, 328)(316, 329)(317, 330)(318, 331)(319, 332)
(333, 341)(335, 349)(338, 352)(340, 353)(347, 357)(348, 358)(350, 359)(351, 360)(362, 364),

$t \mapsto$ (1, 4)(2, 7)(3, 9)(5, 13)(6, 15)(8, 17)(10, 20)(11, 18)(12, 21)(14, 23)(16, 26)(19, 27)(22, 29)(24, 31)
(25, 33)(28, 34)(30, 36)(32, 38)(35, 41)(37, 42)(39, 44)(40, 47)(43, 50)(45, 53)(46, 54)(48, 58)
(49, 59)(51, 61)(52, 63)(55, 66)(56, 68)(57, 70)(60, 72)(62, 74)(64, 77)(65, 79)(67, 81)(69, 83)
(71, 85)(73, 88)(75, 91)(76, 93)(78, 94)(80, 96)(82, 98)(84, 100)(86, 102)(87, 104)(89, 106)(90, 108)
(92, 109)(95, 111)(97, 113)(99, 114)(101, 117)(103, 118)(105, 120)(107, 122)(110, 125)(112, 126)
(115, 129)(116, 130)(119, 132)(121, 134)(123, 137)(124, 138)(127, 140)(128, 142)(135, 144)
(136, 146)(139, 148)(141, 150)(143, 152)(145, 154)(147, 156)(149, 158)(151, 160)(153, 162)
(155, 165)(157, 166)(159, 169)(161, 170)(163, 172)(164, 171)(167, 174)(168, 177)(173, 180)
(175, 183)(176, 184)(178, 188)(179, 189)(181, 191)(182, 193)(185, 196)(186, 198)(187, 200)
(190, 202)(192, 204)(194, 207)(195, 209)(197, 211)(199, 213)(201, 215)(203, 218)(205, 221)
(206, 223)(208, 225)(210, 227)(212, 230)(214, 233)(216, 235)(217, 237)(219, 238)(220, 239)
(222, 241)(224, 244)(226, 247)(228, 249)(229, 251)(231, 253)(232, 256)(234, 259)(236, 260)
(240, 263)(242, 265)(243, 268)(245, 270)(246, 273)(248, 276)(250, 277)(252, 279)(254, 281)

$(255,282)(257,284)(258,285)(261,287)(262,289)(264,291)(266,293)(267,294)(269,296)$
$(271,298)(272,299)(274,301)(275,302)(278,303)(280,305)(283,307)(286,310)(288,311)$
$(290,312)(292,314)(295,315)(297,317)(300,319)(304,321)(306,323)(308,325)(309,326)$
$(313,328)(316,330)(318,332)(320,335)(322,338)(324,340)(327,342)(329,344)(331,346)$
$(333,348)(334,349)(336,351)(337,352)(339,353)(341,354)(343,355)(345,356)(347,358)$
$(350,360)(357,362)(361,364)(363,365).$

The permutations induced by c and d can be obtained using the relations $atdt = 1$ and $btct = 1$. The permutations induced by a, b, c and d all have cycle structure $1^{77}2^{144}$, and are therefore even, while the one induced by t has cycle structure $1^3 2^{181}$ and is odd. Also we note that the element $bcabat$ induces a permutation with cycle structure $2^8 4^4 6^{14} 7^1 8^4 12^5 14^1 23^2 24^2 42^1$.

(C) *Representation C* is the transitive permutation representation of Γ of degree 159 corresponding to the action of Γ by (right) multiplication on right cosets of the subgroup Σ_C generated by a, b, $cbcbtabc$, $bctatatatcb$, $bctabctabatcbatcb$, $bctabctacatcbatcb$, $dcbacbacbdcbtdbcabcabcd$ and $dcbacbacbacbcdcbacbacbtabcabcdcbcabcabcabcd$, of index 159.

The permutations induced by a, b and t may be written as follows:

$$\begin{aligned}
a \mapsto\ & (4,9)(5,10)(7,15)(11,18)(12,19)(13,21)(16,24)(17,23)(20,27)(22,28)\\
& (25,30)(26,29)(31,34)(32,37)(35,39)(40,43)(45,51)(46,49)(47,54)(48,55)\\
& (50,59)(52,60)(53,63)(56,64)(57,65)(58,70)(61,72)(62,71)(66,79)(67,76)\\
& (68,81)(69,78)(77,89)(80,88)(87,95)(91,96)(93,98)(99,103)(100,102)(101,106)\\
& (104,109)(105,108)(107,112)(110,113)(111,118)(114,122)(115,120)(116,117)\\
& (119,127)(121,126)(123,131)(124,125)(128,137)(129,138)(130,136)(132,135)\\
& (133,144)(139,146)(140,149)(141,142)(143,148)(147,150)(151,152)(156,157),
\end{aligned}$$

$$\begin{aligned}
b \mapsto\ & (2,5)(4,7)(6,12)(8,16)(9,17)(11,20)(13,18)(14,22)(15,23)(21,27)(25,32)\\
& (26,33)(30,35)(31,36)(37,39)(40,45)(41,47)(43,48)(44,50)(46,52)(49,57)\\
& (51,55)(58,68)(60,65)(62,73)(63,74)(64,75)(66,77)(69,70)(72,82)(76,86)\\
& (78,81)(79,80)(84,91)(87,93)(88,89)(94,99)(95,100)(97,101)(98,102)\\
& (104,105)(107,114)(108,116)(109,117)(110,111)(112,119)(113,121)(115,123)\\
& (118,126)(120,129)(122,127)(125,134)(128,140)(130,142)(131,138)(135,145)\\
& (136,147)(137,139)(141,150)(143,153)(144,154)(146,149)(152,158)(156,159),
\end{aligned}$$

$$\begin{aligned}
t \mapsto\ & (1,4)(2,7)(3,9)(5,13)(6,15)(8,17)(10,20)(11,18)(12,21)(14,23)(16,26)\\
& (19,27)(22,29)(24,31)(25,33)(28,34)(30,36)(32,38)(35,41)(37,42)(39,44)\\
& (40,47)(43,50)(45,53)(46,54)(48,58)(49,59)(51,61)(52,63)(55,66)(56,68)\\
& (57,70)(60,72)(62,74)(64,77)(65,79)(67,81)(71,82)(73,84)(75,87)(76,89)\\
& (78,80)(83,91)(85,93)(86,95)(90,96)(92,98)(94,100)(97,102)(99,105)\\
& (101,108)(103,110)(106,113)(107,116)(109,111)(112,121)(114,124)(115,117)\\
& (119,130)(120,126)(122,132)(123,125)(127,139)(128,142)(129,136)(131,135)\\
& (133,134)(138,146)(140,151)(143,147)(144,145)(148,149)(150,152)(153,156)\\
& (154,155)(157,158).
\end{aligned}$$

Again the permutations induced by c and d can be obtained using the relations $atdt = 1$ and $btct = 1$. Those induced by a, b, c and d all have cycle structure $1^{31}2^{64}$, while the one induced by t has cycle structure $1^7 2^{76}$, and so all are even. Also we note that the element $bcabat$ induces a permutation with cycle structure $1^3 2^2 3^2 5^1 6^7 10^2 16^1 19^1 22^2$.

In each of these three permutation representations, it is helpful to consider the orbits of two subgroups, $S = \langle b, c\rangle$ and $T = \langle a, b, c\rangle$. From the relations $a^2 = b^2 = c^2 = (ab)^3 = (ac)^2 = (bc)^5 = 1$, we know that S is dihedral of order 10, while T is isomorphic to $A_5 \times C_2$ (the Coxeter group $[3, 5]$, of order 120). Every orbit of the subgroup T (in any permutation representation of Γ) can be decomposed into orbits of the subgroup S, which are linked together by 2-cycles of the permutation induced by a. Similarly, as $d = tat$ we know that Γ is generated by a, b, c and t, and it follows that every transitive permutation representation of Γ can be decomposed into orbits of the subgroup T, which are linked together by 2-cycles of the permutation induced by t to form a single orbit of Γ.

For example, the representation A above decomposes into four orbits of T, namely $\{1, 2, 4, 5, 7, 9, 10, 11, 13, 15, 17, 18, 20, 21, 23, 26, 27, 29, 33, 38\}$, $\{3, 8, 16, 24, 25, 30, 32, 35, 37, 39, 40, 42, 43, 44, 45, 47, 48, 49, 51, 52\}$, $\{6, 12, 14, 19, 22, 28\}$ and $\{31, 34, 36, 41, 46, 50\}$. This is illustrated in Figure 1.

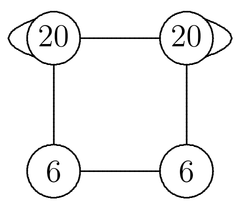

Figure 1: Representation A as a t-linkage of orbits of $T = \langle a, b, c\rangle$

Here each circle represents an orbit of T, with the number inside being its length, and each edge represents a linkage of a pair of orbits of S using the permutation induced by t. For example, the two orbits $\{6, 12, 14, 19, 22, 28\}$ and $\{31, 34, 36, 41, 46, 50\}$ of length 6 are linked by the transposition $(28, 34)$, while the first of these is also linked to the orbit of length 20 containing the point 1 by the transpositions $(6, 15)$, $(12, 21)$, $(14, 23)$, $(19, 27)$ and $(22, 29)$.

More detail is given in Figure 2. Each of the two orbits of length 20 decomposes further into four orbits of S of length 5, while each of the two orbits of length 6 contains an orbit of S of length 5 and a fixed point of S (the fixed points being 28 and 34), with the permutation induced by a interchanging the fixed point of S with one of the points of the orbit of length 5 (and also interchanging two of the other points of the latter orbit).

In Figure 2 the thin lines represent transpositions of the permutation induced by a while each thick line between two orbits of S indicate that they are linked by t. (For example, t interchanges the two orbits $\{28\}$ and $\{34\}$ of length 1, shown by the thick line between the bottom two small squares.) Relations can be verified by chasing points around the figure.

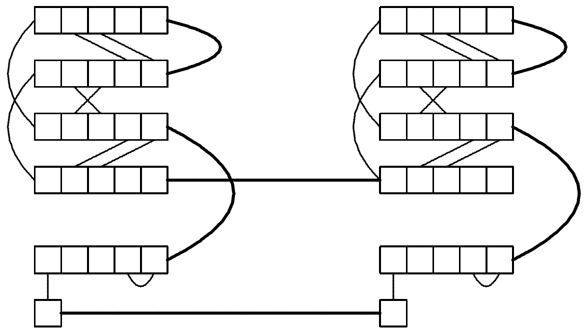

Figure 2: Representation A as a t-linkage of orbits of $S = \langle b, c\rangle$

The linkage here between two of the orbits of S of length 5 inside an orbit of T of length 20 is important, as we will later undo such linkages in order to join together copies of representations of A, B and C. We will call any orbit of T that has this property (namely an internal t-linkage between two of its sub-orbits of S) a *handle*. From Figure 2 we see that the representation A has two handles; these are the T-orbits containing the points 1 and 42.

The building block B on 365 points can viewed similarly, as in Figure 3.

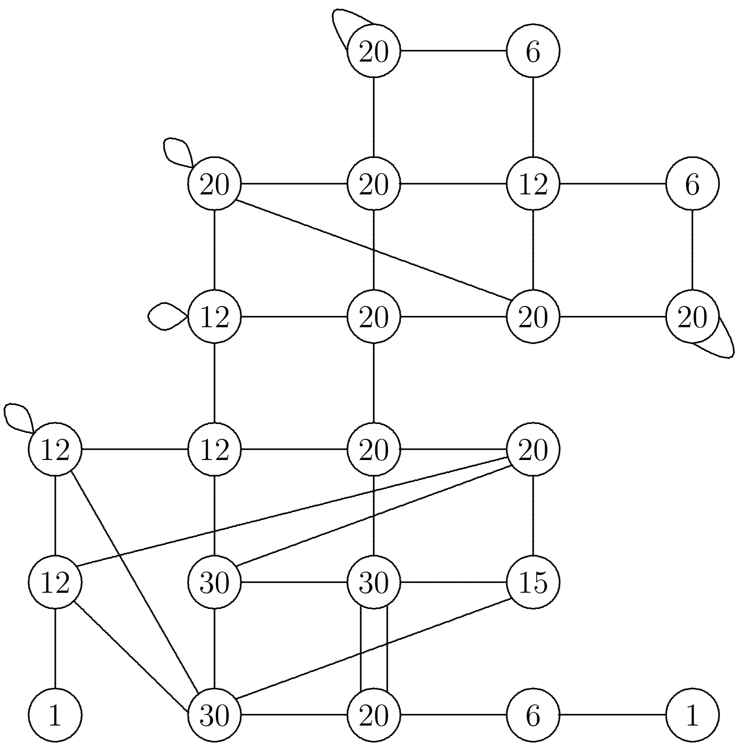

Figure 3: Representation B as a t-linkage of orbits of $T = \langle a, b, c\rangle$

Note that B has two handles: the T-orbits of length 20 containing the points 1 and 139, both at the top right of Figure 3. (Also B has another T-orbit of length 20 with a sub-orbit of S linked to itself, but we will not use this.)

Finally representation C can viewed as in Figure 4. In this case there is just one handle: the T-orbit containing the point 1, the left-most orbit of length 20 in Figure 4. This will be used to join a single copy of C to chains of copies of A and B, to alter the parity of the number of points (without affecting the parity of the permutations induced by any of the generators).

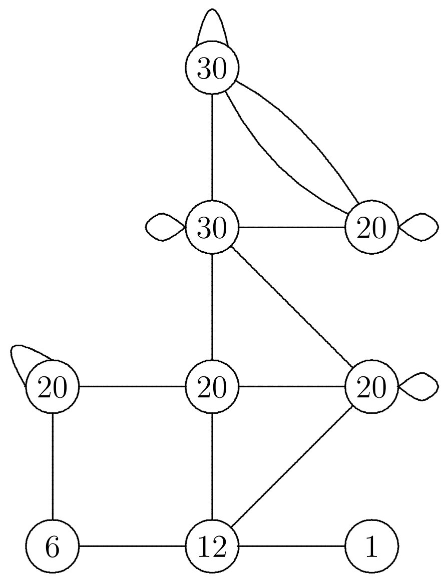

Figure 4: Representation C as a t-linkage of orbits of $T = \langle a, b, c \rangle$

We now consider the process of t-linkage of handles in more detail.

Suppose we have two permutation representations P and Q of Γ such that each contains a handle. Let $(p_1, p_2, p_3, p_4, p_5)$ and $(p_6, p_7, p_8, p_9, p_{10})$ be the cycles induced by bc on the two orbits of S which are linked by t in the given handle of P, and let $(q_1, q_2, q_3, q_4, q_5)$ and $(q_6, q_7, q_8, q_9, q_{10})$ be those induced by bc on the analogous orbits of S in the given handle of Q, such that before linkage the action of t on these points is given by

$$(p_1, p_{10})(p_2, p_9)(p_3, p_8)(p_4, p_7)(p_5, p_6) \quad \text{and} \quad (q_1, q_{10})(q_2, q_9)(q_3, q_8)(q_4, q_7)(q_5, q_6).$$

Note that this will always happen if p_4 and p_{10} are chosen to be points like the left-most points in the first two S-orbits of the 20-point representation of $T = \langle a, b, c \rangle$ at the top left of Figure 2, and likewise for q_4 and q_{10}.

The t-links between orbits of S in the same handles can be removed and replaced by new t-links from the given handle of P to the given handle of Q, by defining a new action of t on the set $X = \{p_1, ..., p_{10}, q_1, ..., q_{10}\}$ by

$$(p_1, q_{10})(p_2, q_9)(p_3, q_8)(p_4, q_7)(p_5, q_6) \quad \text{and} \quad (p_6, q_5)(p_7, q_4)(p_8, q_3)(p_9, q_2)(p_{10}, q_1).$$

or equivalently, by multiplying the permutation induced by t by the permutation which swaps each p_i with the corresponding q_i. In particular, we note that the parity of the permutation induced by t is unchanged.

In order to show that this still gives a permutation representation of Γ, we need only show that the relations $t^2 = btct = (at)^4 = 1$ are still satisfied, because d can be defined by $atdt = 1$, and all other relations involving d follow from these: $(ad)^2 = (atat)^2 = (at)^4$, $(bd)^2 = (btat)^2 = t(tbta)^2t = t(ca)^2t = 1$, and $(cd)^3 = (ctat)^3 = t(tcta)^3t = t(ba)^3t = 1$.

By defining the action of t in the way that we have, linking pairs of S-orbits of equal length, the relations $t^2 = 1$ and $c = tbt$ will always hold; in fact t normalises the dihedral subgroup $S = \langle b, c\rangle$ (by interchanging its generators). Hence only the relation $(at)^4$ has to be checked. Now it is clear that cycles of the element at containing none of the points of the set $X = \{p_1, ..., p_{10}, q_1, ..., q_{10}\}$ are unaffected by the linkage, and so we can ignore them. On the other hand, before linkage each cycle of at containing a point of X is either a 1-cycle fixing one of the points p_3, p_8, q_3 or q_8, or a 4-cycle of the form (p_i, p_j, x, y) or (q_i, q_j, u, v) where x and y are points of P not in X and u and v are points of Q not in X. In the above process of t-linkage, the fixed points of at on X are replaced by 2-cycles (p_3, q_3) and (p_8, q_8), while the 4-cycles are replaced by cycles of the form (p_i, q_j, u, v) and (q_i, p_j, x, y). With all its cycle lengths dividing 4, the permutation induced by at still has order 4, and thus all the relations of Γ are satisfied.

This same sort of analysis can be used to find the cycle structure of any element of the form wt where w lies in the subgroup $T = \langle a, b, c\rangle$. In particular, let us consider the permutation induced by the element $bcabat$. Recall that the cycle structures of the permutations induced by this element in each of the representations A, B and C are

$$2^2 6^8,\ \ 2^8 4^4 6^{14} 7^1 8^4 12^5 14^1 23^2 24^2 42^1 \quad \text{and} \quad 1^3 2^2 3^2 5^1 6^7 10^2 16^1 19^1 22^2$$

respectively. As in the argument above, we can see the effect of making new t-linkages on the permutation induced by $bcabat$ on the n points of our chain, by considering just the cycles that contain handle points. In each case, before t-linkage we have the following:

Representation A: The cycles induced by bc on the two orbits of S linked by t in the handle containing the point 1 are $(p_1, p_2, p_3, p_4, p_5) = (1, 2, 11, 20, 5)$ and $(p_6, p_7, p_8, p_9, p_{10}) = (13, 10, 18, 7, 4)$, while those for the two orbits in the handle containing the point 42 are $(p'_1, p'_2, p'_3, p'_4, p'_5) = (42, 44, 51, 48, 49)$ and $(p'_6, p'_7, p'_8, p'_9, p'_{10}) = (43, 52, 47, 39, 37)$, such that the element t interchanges p_i with p_{11-i} and p'_i with p'_{11-i} for $1 \le i \le 10$. There are six cycles of $bcabat$ that affect the relevant points from the two handles, all having length 6, and these can be written in the form

$(p_1, p_4, p_6, p_9, x_1, x_2)$, $(p_2, x_3, x_4, x_5, p'_8, x_6)$, $(p_3, p_5, p_{10}, p_7, x_7, x_8)$,
$(p'_1, p'_4, p'_6, p'_9, x'_1, x'_2)$, $(p'_2, x'_3, x'_4, x'_5, p_8, x'_6)$, and $(p'_3, p'_5, p'_{10}, p'_7, x'_7, x'_8)$.

Representation B: The cycles induced by bc on the two orbits of S linked by t in the handle containing the point 1 are $(q_1, q_2, q_3, q_4, q_5) = (1, 2, 11, 20, 5)$ and $(q_6, q_7, q_8, q_9, q_{10}) =$

$(13, 10, 18, 7, 4)$, while those for the orbits in the handle for the point 139 are $(q'_1, q'_2, q'_3, q'_4, q'_5)$ $= (139, 147, 164, 172, 155)$ and $(q'_6, q'_7, q'_8, q'_9, q'_{10}) = (165, 163, 171, 156, 148)$, again such that the element t interchanges q_i with q_{11-i} and q'_i with q'_{11-i} for $1 \leq i \leq 10$. There are six cycles of *bcabat* that affect the relevant points from the two handles — four of length 6 and two of length 23 — and these can be written in the form

$$(q_1, q_4, q_6, q_9, y_1, y_2),\ (q_3, q_5, q_{10}, q_7, y_{24}, y_{25}),$$
$$(q_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, q_8, y_{12}, y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18}, y_{19}, y_{20}, y_{21}, y_{22}, y_{23}),$$
$$(q'_1, q'_4, q'_6, q'_9, y'_1, y'_2),\ (q'_3, q'_5, q'_{10}, q'_7, y'_{24}, y'_{25}),\ \text{and}$$
$$(q'_2, y'_3, y'_4, y'_5, y'_6, y'_7, y'_8, y'_9, y'_{10}, y'_{11}, y'_{12}, y'_{13}, y'_{14}, y'_{15}, y'_{16}, q'_8, y'_{17}, y'_{18}, y'_{19}, y'_{20}, y'_{21}, y'_{22}, y'_{23}).$$

Representation C: The cycles induced by bc on the two orbits of S linked by t in the handle containing the point 1 are $(r_1, r_2, r_3, r_4, r_5) = (1, 2, 11, 20, 5)$ and $(r_6, r_7, r_8, r_9, r_{10}) =$ $(13, 10, 18, 7, 4)$, such that the element t interchanges r_i with r_{11-i} for $1 \leq i \leq 10$. There are three cycles of *bcabat* that affect the relevant points, of lengths 6, 19 and 6, and these can be written in the form

$$(r_1, r_4, r_6, r_9, z_1, z_2),\ (r_3, r_5, r_{10}, r_7, z_{20}, z_{21}),\ \text{and}$$
$$(r_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9, z_{10}, z_{11}, z_{12}, z_{13}, r_8, z_{14}, z_{15}, z_{16}, z_{17}, z_{18}, z_{19}).$$

Now suppose that a copy of A is linked to a copy of B by joining the handle of A containing the point $p'_1 = 42$ to the handle of B containing the point $q_1 = 1$. We will denote this by $A_{42} -_1 B$. Then in the process of t-linkage, the seven cycles of *bcabat* containing the points p'_i and q_j will be replaced by

$$(p_2, x_3, x_4, x_5, q_8, y_{12}, y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18}, y_{19}, y_{20}, y_{21}, y_{22}, y_{23},$$
$$p'_2, x'_3, x'_4, x'_5, p_8, x'_6, q_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, p'_8, x_6),$$
$$(p'_1, q_4, p'_6, q_9, y_1, y_2),\ (p'_3, q_5, p'_{10}, q_7, y_{24}, y_{25}),\ (p'_4, q_6, p'_9, x'_1, x'_2, q_1),$$
$$\text{and}\ (p'_5, q_{10}, p'_7, x'_7, x'_8, q_3).$$

In summary, six cycles of length 6 and one of length 23 are replaced by four of length 6 and one of length 35.

Similarly, if instead the handle of A containing the point $p_1 = 1$ is joined by t-linkage to the handle of B containing the point $q'_1 = 139$, giving a representation denoted by $B_{139} -_1 A$, then the seven cycles of *bcabat* containing the points p_i and q'_j will be replaced by

$$(p_1, q'_4, p_6, q'_9, y'_1, y'_2),\ (p_3, q'_5, p_{10}, q'_7, y'_{24}, y'_{25}),$$
$$(p_2, x_3, x_4, x_5, p'_8, x_6, q'_2, y'_3, y'_4, y'_5, y'_6, y'_7, y'_8, y'_9, y'_{10}, y'_{11}, y'_{12}, y'_{13}, y'_{14}, y'_{15}, y'_{16}, p_8, x'_6,$$
$$p'_2, x'_3, x'_4, x'_5, q'_8, y'_{17}, y'_{18}, y'_{19}, y'_{20}, y'_{21}, y'_{22}, y'_{23}),$$
$$(p_4, q'_6, p_9, x_1, x_2, q'_1)\ \text{and}\ (p_5, q'_{10}, p_7, x_7, x_8, q'_3);$$

so again six cycles of length 6 and one of length 23 are replaced by four of length 6 and one of length 35.

Next suppose two copies of B and one copy of A are linked together into a chain $B - A - B$ by joining the handle of A containing the point $p_1 = 1$ to the handle of one copy of B containing the point $q'_1 = 139$, and the handle of A containing the point $p'_1 = 42$ to the handle of the other copy of B containing the point $q_1 = 1$. This can be denoted by $B_{139} -_1 A_{42} -_1 B$. Then in the process of t-linkage, ten cycles of length 6 and two of length 23 are replaced by eight of length 6 and two of length 29 (with one of the latter containing the points $p_2 = 2$ and $p'_2 = 44$ and the other containing the points $p_8 = 18$ and $p'_8 = 47$), and no new cycles of length 35 are introduced.

If two copies of B are joined by t-linkage using the handle containing the point $q_1 = 1$ from one and the handle containing the point $q'_1 = 139$ from the other, then the four cycles of length 6 and the two cycles of length 23 for the element $bcabat$ containing the relevant points of the two handles will be replaced by another four cycles of length 6, plus one of length 18 and one of length 28 (with the last one being formed out of 15 points of one 23-cycle and 13 of the other).

Finally if a copy of A is linked to a copy of C by joining their handles containing the points $p_1 = 1$ and $r_1 = 1$, then the six cycles of length 6 and single cycle of length 19 for the element $bcabat$ containing the relevant points of the two handles will be replaced by

$$(p_1, r_4, p_6, r_9, z_1, z_2), \quad (p_3, r_5, p_{10}, r_7, z_7, z_8),$$
$$(p_2, x_3, x_4, x_5, p'_8, x_6, r_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9, z_{10}, z_{11}, z_{12}, z_{13}, p_8, x'_6, p'_2, x'_3, x'_4, x'_5, r_8, z_{14}, z_{15}, z_{16}, z_{17}, z_{18}, z_{19}),$$
$$(r_1, p_4, r_6, p_9, x_1, x_2), \text{ and } (r_3, p_5, r_{10}, p_7, x_{20}, x_{21});$$

that is, four cycles of length 6 and a single cycle of length 31 (containing the points p'_2 and p'_8 from the other handle of A, with $(bcabat)^{15}$ taking p'_2 to p'_8). It follows that if we extend this to a chain $C_1 -_1 A_{42} -_1 B$ by joining the other handle of A (containing the point $p'_1 = 42$) to the first handle of a copy of B (containing the point $q_1 = 1$), then the latter cycle of length 31 (from the $C - A$ join) and the cycle of length 23 from the copy of B are replaced by cycles of lengths 26 and 28, namely

$$(p_2, x_3, x_4, x_5, q_8, y_{12}, y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18}, y_{19}, y_{20}, y_{21}, y_{22}, y_{23}, p'_2, x'_3, x'_4, x'_5, r_8, z_{14}, z_{15}, z_{16}, z_{17}, z_{18}, z_{19}), \text{ and}$$
$$(p'_8, x_6, r_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9, z_{10}, z_{11}, z_{12}, z_{13}, p_8, x'_6, q_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}),$$

while all other cycles containing points affected by this join have length 6.

We are now (at last) in a position to prove the theorem.

Proof of Theorem 7. First observe that as 52 and 365 are relatively prime, every sufficiently large positive integer n can be written in the form $n = 52r + 365s$ for integers r and s satisfying $1 \le r \le s$. For any such n, we may construct a transitive permutation representation of the extended $[3, 5, 3]$ group Γ on n points by taking r copies of the representation A (on 52 points) and s copies of the representation B (on 365 points), and linking them together into a chain of the form

$$A - B - A - B - \cdots - A - B - A - B - B - B - \cdots - B - B$$

using the process of t-linkage described above. Note that this chain consists of r copies of the sub-chain $A-B$ followed by $s-r$ copies of B, and that such a construction is possible since each of A and B has two handles.

In fact there are many ways to form the chain, but we will do it in such a way that t-linkage occurs as follows: reading the chain from left to right, in each sub-chain $A - B$ the permutation t will interchange the point 42 in the copy of representation A with the point 1 in the copy of representation B, while in each sub-chain $B - A$ the permutation t will interchange the point 139 in the copy B with the point 1 in the copy of A, and in each subchain $B - B$ the permutation t will interchange the point 139 in the first copy of B with the point 1 in the second copy of B. The handle containing the point 1 of the first copy of A will be unaffected (and therefore left free for subsequent t-linkage with a copy of C), as will the handle containing the point 139 in the final copy of B. In summary, the chain will have the form

$${}_1A_{42} - {}_1B_{139} - {}_1A_{42} - {}_1B_{139} - \cdots - {}_1A_{42} - {}_1B_{139} - {}_1B_{139} - \cdots - {}_1B_{139}.$$

The group generated by the resulting permutations (induced by a, b, c, d and t) is a transitive subgroup of S_n, because the orbits of the subgroup $T = \langle a, b, c\rangle$ are linked together by the action of the (new) permutation induced by t. The elements a, b, c and d all induce even permutations (as they do on each of the building blocks A, B and C), while t induces an even permutation if and only if s is even, and hence if and only if $n = 52r + 365s$ is even, because each copy of B adds an odd number of 2-cycles to the permutation induced by t.

We will show that this group contains a single cycle of prime length 23 (fixing the other $n-23$ points), and then use this element to prove that the group is primitive. We can then apply Jordan's theorem ([29]), which says that any primitive permutation group of degree n containing a single p-cycle for some prime $p < n-2$ has to be A_n or S_n.

To do this, consider the cycle structure of the permutation π induced by the element *bcabat* on these n points.

Before the building blocks are linked together by altering the definition of t, the cycle structure of this permutation is $2^{2r+8s}4^{4s}6^{8r+14s}7^{s}8^{4s}12^{5s}14^{s}23^{2s}24^{2s}42^{s}$, as each copy of A provides $2^{2}6^{8}$ and each copy of B provides $2^{8}4^{4}6^{14}7^{1}8^{4}12^{5}14^{1}23^{2}24^{2}42^{1}$.

In the process of t-linkage, each sub-chain of the form $B_{139}{-}_1A_{42}{-}_1B$ reduces the number of 6-cycles by two and the number of 23-cycles by two, and introduces two new 29-cycles, while the first linkage $A_{42}-{}_1B$ reduces the number of 6-cycles by two and the number of 23-cycles by one, and introduces one new 35-cycle. Similarly each link of the form $B_{139}{-}_1B$ reduces the number of 23-cycles by two, and introduces one new 18-cycle and one new 28-cycle. As the number of sub-chains of the form $B_{139} - {}_1A_{42} - {}_1B$ is $r-1$ and the number of links of the form $B_{139} - {}_1B$ is $s - r$, the resulting cycle structure of the permutation π induced by *bcabat* is $2^{2r+8s}4^{4s}6^{6r+14s}7^s8^{4s}12^{5s}14^s18^{s-r}23^124^{2s}28^{s-r}29^{2r-2}35^142^s$.

In particular, the number of 23-cycles is $2s - 2(r-1) - 1 - 2(s-r) = 1$. Moreover, none of the other cycle-lengths of π is divisible by 23; in fact their least common multiple is $73080 = 8 \cdot 9 \cdot 5 \cdot 7 \cdot 29$, and so it follows that π^{73080} is a single cycle of length 23. The points of this 23-cycle come from the very last copy of B that is added to the chain (so this 23-cycle is not killed by any new t-linkage). In fact it is a cycle of the form

$$(q'_2, y'_3, y'_4, y'_5, y'_6, y'_7, y'_8, y'_9, y'_{10}, y'_{11}, y'_{12}, y'_{13}, y'_{14}, y'_{15}, y'_{16}, q'_8, y'_{17}, y'_{18}, y'_{19}, y'_{20}, y'_{21}, y'_{22}, y'_{23}),$$

where q'_2 is the point numbered 147 in the original description of the representation B.

This cycle contains eight fixed points of a (including the point $q'_2 = 147$ itself), three fixed points of $S = \langle b, c\rangle$ (namely the points $y'_4 = 118$, $y'_{15} = 103$ and $y'_{20} = 133$), and one fixed point of t (namely $y'_{20} = 133$).

We can now prove the group generated by our permutations is primitive. For assume the contrary. Then since 23 is prime, all the points of the 23-cycle π must lie in the same block of imprimitivity, say X, as otherwise the blocks moved by π would contain only one point each. This block X contains all those points of π which are fixed by a, $\langle b, c\rangle$ or t, and hence X is preserved by each of a, b, c and t. But these elements generate Γ, and so X is preserved by Γ, which implies it cannot be a proper block, contradiction.

Jordan's theorem [29] now applies, so we have either the alternating group A_n or the symmetric group S_n. In fact since the permutation induced by t is even if and only if r is even, we have A_n if n is even and S_n if n is odd.

To obtain A_n for all large odd n and S_n for all large even n, we suppose that $n = 159 + 52r + 365s$, and repeat the above construction and argument, but with a single copy of the representation C joined to the copy of A the end of the chain, thus:

$$C_1 - {}_1A_{42} - {}_1B_{139} - {}_1A_{42} - {}_1B_{139} - \cdots - {}_1A_{42} - {}_1B_{139} - {}_1B_{139} - \cdots - {}_1B_{139}.$$

This changes the parity of n, but not the parity of any of the permutations induced by the generators. The cycle structure of the permutation induced by *bcabat* is again easily determined: before linkage the cycle structure of this element in the representation C is $1^32^23^25^16^710^216^119^122^2$, and in the process of linkage, the cycle of length 19 from this representation and the single cycle of length 35 on the other $n - 159$ points are replaced by two cycles of length 26 and 28, while again all other cycle lengths are unaffected. Hence the resulting cycle structure of the permutation π induced by *bcabat* is $1^32^{2r+8s+2}3^24^{4s}5^16^{6r+14s+7}7^s8^{4s}10^212^{5s}14^s16^118^{s-r}22^223^124^{2s}26^128^{s-r+1}29^{2r-2}42^s$.

Again there is a single 23-cycle, involving the same points as previously, but other cycle lengths are introduced, so the 23-cycle we need is $\pi^{16\cdot9\cdot5\cdot7\cdot11\cdot13\cdot19\cdot29} = \pi^{20900880}$. This can be used in an application of Jordan's theorem to show that the permutations induced by a, b, c and t generate the alternating group or symmetric group on the n points of the chain, and as the permutation induced by t is even if and only if r is even, we have A_n if $n = 159 + 52r + 365s$ is odd, and S_n if n is even.

Hence for every such large n there exists a homomorphism ϕ from the extended $[3,5,3]$ Coxeter group Γ onto the alternating group A_n, and another homomorphism ψ from Γ onto the symmetric group S_n.

By our construction, both ϕ and ψ map the Coxeter group $C = \langle a, b, c, d\rangle$ of index 2 in Γ to a subgroup of A_n (since the generators all induce even permutations), and as A_n has no subgroup of index 2 (for $n \geq 5$), it follows that these both map C onto A_n. The very same argument now also shows that both map the subgroup $C^+ = \langle ab, ac, ad\rangle$ onto A_n, and therefore ϕ and ψ map $\Gamma^+ = \langle ab, ac, ad, t\rangle$ onto A_n and S_n respectively.

Finally we observe that the permutations induced by the elements ab, ac, bc, t and adt are all non-trivial, and so by Proposition 3 we find the kernels of the restrictions $\phi\lceil_{\Gamma^+}$ and $\psi\lceil_{\Gamma^+}$ are torsion-free normal subgroups of Γ^+, completing the proof. □

5 Other families of quotients

The two main theorems from the preceding sections give us three infinite families of quotients of Γ^+ by torsion-free normal subgroups: 2-dimensional projective linear groups (one in each prime characteristic p), alternating groups (of large degree), and symmetric groups (of large degree). We will now show how an infinite family of quotient groups of Γ^+ can be obtained using just one given quotient, under certain circumstances.

Let Σ be any torsion-free subgroup of index n in Γ^+, with core K (the intersection of all conjugates of Σ by elements of Γ^+) of index N in Γ^+. Then of course K itself is torsion-free, and the quotient Γ^+/K (which is isomorphic to the permutation group induced by Γ^+ on right cosets of Σ by right multiplication) has order N.

Now suppose that the abelianisation $\Sigma/\Sigma' = \Sigma/[\Sigma,\Sigma]$ of the subgroup Σ is infinite. Then Σ/Σ' must have at least one infinite cyclic factor, and therefore Σ contains a normal subgroup Λ such that $\Sigma/\Lambda \cong \mathbb{Z}$. It then follows that for every positive integer m the subgroup Σ has a normal subgroup Λ_m of index m such that $\Sigma/\Lambda_m \cong \mathbb{Z}_m$ (namely the pre-image of $m\mathbb{Z}$ under the resulting homomorphism from Σ to $\mathbb{Z}$). In particular, Λ_m is a torsion-free subgroup of index $|\Gamma^+\!:\!\Lambda_m| = |\Gamma^+\!:\!\Sigma||\Sigma\!:\!\Lambda_m| = nm$ in Γ^+.

The core K_m of Λ_m in Γ^+ will be a torsion-free normal subgroup of Γ^+, with quotient Γ^+/K_m isomorphic to the permutation group of degree nm induced by Γ^+ by multiplication on cosets of Λ_m. Moreover, K/K_m is a normal subgroup of Γ^+/K_m, with quotient $(\Gamma^+/K_m)/(K/K_m) \cong \Gamma^+/K$. Since Σ/Λ_m is abelian (indeed cyclic) and of order m, we know that Λ_m contains $\Sigma'\Sigma^m$ and therefore also contains $K'K^m$, which is a characteristic subgroup of K and is therefore normal in Γ. It follows that K_m contains $K'K^m$, and hence K/K_m is abelian and of exponent m.

Thus we obtain an infinite sequence of quotients of Γ^+ by torsion-free normal subgroups of Γ^+, all being extensions of an abelian group K/K_m of exponent m (for increasing m) by the initial quotient group Γ^+/K. In particular, the orders of the groups in this sequence exhibit polynomial growth. The corresponding 3-manifolds (with maximal symmetry group) will be covers of the 3-manifold associated with the initial subgroup Σ.

For small n it is easy to determine computationally whether a given torsion-free subgroup Σ of index n has infinite abelianisation, using the Reidemeister-Schreier process (or the `AQInvariants` command in Magma).

The torsion-free subgroups of index 60 in Γ^+ all have abelianisations that are cyclic of order 58 or 70 (see section 2). Similarly the abelianisations of the kernels of the homomorphisms from Γ^+ onto $\mathrm{PSL}(2,16)$ and $\mathrm{PGL}(2,9)$ are isomorphic to the direct products $(\mathbb{Z}_2)^9 \times (\mathbb{Z}_4)^{10}$ and $(\mathbb{Z}_3)^6$ respectively.

On the other hand, the kernels of homomorphisms from Γ^+ onto $\mathrm{PSL}(2,25)$ and $\mathrm{PGL}(2,11)$ have abelianisations isomorphic to $\mathbb{Z}^{26}$ and $\mathbb{Z}^{10}$ respectively. It follows that there are torsion-free normal subgroups of Γ^+ with quotients isomorphic to extensions of $\mathbb{Z}_m^{26}$ by $\mathrm{PSL}(2,25)$, of order $7800m^{26}$, and extensions of $\mathbb{Z}_m^{10}$ by $\mathrm{PGL}(2,11)$, of order $1320m^{10}$, for every positive integer m. There are numerous other such examples.

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