Stability of Systems Under Interference Feedback

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***Abstract*— Distributed power control problems in wireless communication can be modeled as a highly nonlinear feedback system. The nonlinear coupling appear when a large number of mobile stations interact through interference and it is well known that this may lead to instabilities. In this paper a number of results on existence and uniqueness of solution and boundedness and convergence of the solution are derived for systems with higher order control loops.**

I. Introduction

In control of wireless networks it is often desired to use decentralized control loops. This implies that only local information is used in a single local control loop. This is an area where a lot of research has been done, and there are several proposals of local control algorithms that are proven stable under certain conditions. A famous example is the decentralized power control (DPC) algorithm proposed by Foschini and Miljanic (1993) [3]. It uses the following local power update algorithm

$$\bar{p}_i(t+1) = \frac{\bar{\gamma}_i^T(t)}{\bar{\gamma}_i(t)}\bar{p}_i(t), \tag{1}$$

where $\bar{\gamma}_i(t)$ is the signal to interference ratio (SIR), $\bar{\gamma}_i^T(t)$ is the target SIR and $\bar{p}_i(t)$ is the power of user i. The bars of the variables indicate linear scale, while a variable without bar denotes logarithmic scale. The SIR can be modeled as

$$\bar{\gamma}_i(t) = \frac{\bar{\delta}_i \bar{g}_{ii}\bar{p}_i}{\sum_{j\neq i}\bar{g}_{ij}\bar{p}_j + (1-\bar{\delta}_i)\bar{g}_{ii}\bar{p}_i + \bar{\sigma}_i^2}$$

where $\bar{\delta}$ is a constant modeling auto interference due to imperfections in the receiver and phenomena such as scattering, $\bar{g}_{ij}$ is the channel gain from user j to user i, and $\bar{\sigma}_i^2$ is noise. If the interference function is defined as

$$\bar{I}_i(t) = \sum_{j\neq i}\bar{g}_{ij}\bar{p}_j + (1-\bar{\delta}_i)\bar{g}_{ii}\bar{p}_i + \bar{\sigma}_i^2, \tag{2}$$

then the SIR can be written as

$$\bar{\gamma}_i(t) = \frac{\bar{\delta}_i \bar{g}_{ii}\bar{p}_i}{\bar{I}_i(t)}.$$

Convergence of the basic algorithm by Foschini and Miljanic can be established by exploring monotonicity properties of the basic interference function, see e.g. [9], [6].

The real system is often subject to unmodeled dynamics such as propagation delay and this motivates the introduction of higher order control action [5], [4]. The resulting model is then of high order and fundamental questions such as boundedness and convergence of the solution are challenging to address.

In this paper we use two approaches to analyze the system. In the first approach we modify the framework introduced by Yates in [9] to prove global convergence of a higher dimensional generalization of the basic power control loop in (1). In the second approach we apply input-output theory to derive conditions for existence and uniqueness of a solution in a prescribed set of bounded power levels. The convergence to the desired equilibrium can also be established using this method. We show in the example section that the results successfully predict stability of the delay compensator suggested in [5], [4].

II. Model Class

In order to define a generalization of the Foschini and Miljanic model we introduce the gain matrix

$$M = \begin{bmatrix} (1-\bar{\delta}_1)\bar{g}_{11} & \bar{g}_{12} & \cdots & \bar{g}_{1n} \\ \bar{g}_{21} & (1-\bar{\delta}_2)\bar{g}_{22} & \cdots & \bar{g}_{2n} \\ \vdots & & \ddots & \vdots \\ \bar{g}_{n1} & & \cdots & (1-\bar{\delta}_n)\bar{g}_{nn} \end{bmatrix}, \tag{3}$$

where all entries are assumed to be nonnegative. The i^{th} row of M is denoted M^i in the following.

We consider the model

$$\prod_{k=0}^{m}\bar{p}_i(t+m-k)^{\alpha_k} = \prod_{k=0}^{m-1}\left(\frac{\bar{\gamma}_i^T(t+m-1-k)}{\bar{\gamma}_i(t+m-1-k)}\right)^{\beta_k}, \tag{4}$$

for $i = 1,\ldots,n$, where

$$\bar{\gamma}_i(t) = \frac{\bar{\gamma}_{i,G}(t)}{\bar{\gamma}_{i,I}(t)}$$

and

$$\prod_{k=0}^{l}\bar{\gamma}_{i,I}(t+l-k)^{\rho_k} = \prod_{k=0}^{l}(\bar{\sigma}_i^2 + M^i\bar{p}(t+l-k))^{\eta_k}$$

$$\prod_{k=0}^{r}\bar{\gamma}_{i,G}(t+r-k)^{\delta_k} = \prod_{k=0}^{r}(\bar{\delta}_i\bar{g}_{ii}\bar{p}_i(t+r-k))^{\epsilon_k}.$$

We can without loss of generality assume that $\alpha_0 = \rho_0 = \delta_0 = 1$. An important aspect is that all components in the power vector $\bar{p} = \begin{bmatrix}\bar{p}_1 & \ldots & \bar{p}_n\end{bmatrix}^T$ must be positive in order for the solution to be feasible.

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For the further analysis we define the following discrete time systems

$$R(q) = \frac{\beta_0 q^{m-1} + \beta_1 q^{m-2} + \cdots + \beta_{m-1}}{q^m + \alpha_1 q^{m-1} + \alpha_2 q^{m-2} + \cdots + \alpha_m}$$
$$F_I(q) = \frac{\eta_0 q^l + \eta_1 q^{l-1} + \cdots + \eta_l}{q^l + \rho_1 q^{l-1} + \rho_2 q^{l-2} + \cdots + \rho_l}$$
$$F_G(q) = \frac{\epsilon_0 q^r + \epsilon_1 q^{r-1} + \cdots + \epsilon_r}{q^r + \delta_1 q^{r-1} + \delta_2 q^{r-2} + \cdots + \delta_r},$$

where q is the time-shift operator.

In a receiver the measurements of the SIR are due to measurement noise, which motivates the use of filters. The signal and interference part of the signal can in practice be filtered separately, and these filters are here denoted by F_G and F_I, in accordance with [4].

We notice that the model (4) reduces to the Foschini and Miljanic algorithm in the special case when $F_I(q) = 1, F_G(q) = 1$, and $R(q) = \frac{1}{q-1}$.

III. Steady state analysis

In this section we show that under certain conditions there exists a unique equilibrium point to the system defined in (4).

In steady state we consider the system with all variables constant, e.g. $\bar{\gamma}_i(t+k) = \bar{\gamma}_i$, $\bar{\gamma}_i^T(t+k) = \bar{\gamma}_i^T$, $\bar{p}_i(t+k) = \bar{p}_i, \forall k$. Let the steady state solution be denoted by $\bar{p}^0 = \begin{bmatrix} \bar{p}_1^0 & \ldots \bar{p}_n^0 \end{bmatrix}^T$. If we plug this into (4), we get

$$(\bar{p}_i^0)^{\frac{1}{R(1)}} = \frac{\bar{\gamma}_i^T (\bar{\sigma}_i^2 + M^i \bar{p}^0)^{F_I(1)}}{(\bar{\delta}_i \bar{g}_{ii} \bar{p}_i^0)^{F_G(1)}}.$$

In this paper the special case when $R(1) = \infty$ and $F_I(1) = F_G(1) = 1$ is of particular interest. The equilibrium condition then reduces to

$$\bar{\gamma}_i^T = \frac{(\bar{\delta}_i \bar{g}_{ii} \bar{p}_i^0)}{(\bar{\sigma}_i^2 + M^i \bar{p}^0)} = \frac{(\bar{\delta}_i \bar{g}_{ii} \bar{p}_i^0)}{\bar{I}_i(\bar{p}^0)}, \quad i = 1, \ldots, n,$$

which implies that the target SIR is achieved provided that these equations are well defined in the sense that there exists a positive solution $\bar{p}_i^0 \geq 0$, $i = 1, \ldots, n$. The equation can be vectorized to

$$\bar{p}^0 = \bar{\Delta}^{-1} \bar{\Gamma}_T \left(M \bar{p}^0 + \bar{\sigma}^2 \right),$$

where

$$\bar{\Gamma}_T = \begin{bmatrix} \bar{\gamma}_1^T & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \bar{\gamma}_n^T \end{bmatrix}, \quad \bar{\Delta} = \begin{bmatrix} \bar{\delta}_1 \bar{g}_{11} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \bar{\delta}_n \bar{g}_{nn} \end{bmatrix}$$

and $\bar{\sigma}^2 = \begin{bmatrix} \bar{\sigma}_1^2 & \ldots & \bar{\sigma}_n^2 \end{bmatrix}^T$.

If the spectral radius condition $\rho(\bar{\Delta}^{-1} \bar{\Gamma}_T M) < 1$ holds, then it can be shown using the Perron Frobenius theorem that there exists a unique positive steady state power allocation

$$\bar{p}^0 = (I - \bar{\Delta}^{-1} \bar{\Gamma}_T M)^{-1} \bar{\Delta}^{-1} \bar{\Gamma}_T \bar{\sigma}^2, \tag{5}$$

see, for example [10], [6] for an early account of such results. The above condition on the spectral radius is assumed to hold throughout the paper.

IV. Analysis Using Interference Functions

In this section we use the framework developed by Yates in [9] to provide conditions under which the dynamics in (4) converges. We consider the special case when $\bar{\gamma}_i^T$ are constant for all i, $F_I(q) = 1$ and $F_G(q) = 1$. In this case the model (4) can be reformulated as

$$\bar{p}_i(t+1) = \Psi_i(\bar{p}(t), \ldots, \bar{p}(t+1-m)), \ i = 1, \ldots, n, \tag{6}$$

where

$$\Psi_i(\bar{p}(t), \ldots, \bar{p}(t+1-m)) = \prod_{k=0}^{m-1} \left(\frac{\bar{\gamma}_i^T \bar{I}_i(t-k)}{\bar{\delta}_i \bar{g}_{ii}} \right)^{\beta_k} \bar{p}_i(t-k)^{-\alpha_{k+1} - \beta_k} \tag{7}$$

where $\bar{I}_i$ is defined as in (2). We assume that the system (6) is initialized with $\bar{p}_i(0) = \bar{p}_i(-1) = \ldots = \bar{p}_i(1-m) = \bar{p}_{i,0} > 0$, for $i = 1, \ldots, n$.

For the next result on convergence of the model in (6)-(7) we assume that the feasibility condition $\rho(\bar{\Delta}^{-1} \bar{\Gamma}_T M) < 1$ holds.

Proposition 1: Suppose

$$\begin{gathered} 0 \leq \beta_k \leq -\alpha_{k+1}, \ k = 0, \ldots, m-1 \\ \sum_{k=0}^{m-1} \beta_k > 0, \ \text{and} \ \sum_{k=1}^{m} \alpha_k = -1 \end{gathered} \tag{8}$$

Then the system in (6)-(7) converges to the equilibrium $\bar{p}^0$ in (5).

Remark 1: We may also consider different dynamics for different users, i.e.

$$\Psi_i(\bar{p}(t), \ldots, \bar{p}(t+1-m_i)) = \prod_{k=0}^{m_i-1} \left(\frac{\bar{\gamma}_i^T \bar{I}_i(t-k)}{\bar{\delta}_i \bar{g}_{ii}} \right)^{\beta_{i,k}} \bar{p}_i(t-k)^{-\alpha_{i,k+1} - \beta_{i,k}}.$$

The conditions for convergence then becomes

$$\begin{gathered} 0 \leq \beta_{i,k} \leq -\alpha_{i,k+1}, \ k = 0, \ldots, m_i - 1 \\ \sum_{k=0}^{m_i-1} \beta_{i,k} > 0, \ \text{and} \ \sum_{k=1}^{m_i} \alpha_{i,k} = -1. \end{gathered} \tag{9}$$

Proof: The proof can be found in the appendix. ■

V. Equivalent Input-Output Model

By taking the logarithm of both sides of equation (4) and introducing the notation

$$p_i = \ln(\bar{p}_i), \quad \delta_i = \ln(\bar{\delta}_i), \quad g_{ii} = \ln(\bar{g}_{ii})$$
$$\delta g(t) = \begin{bmatrix} \delta_1(t) + g_{11}(t) & \ldots & \delta_n(t) + g_{nn}(t) \end{bmatrix}^T$$

we get the equivalent system in Figure 1 and Figure 2. Here the $\exp(\cdot)$ and $\ln(\cdot)$ operators are acting component-wise on the elements in $\bar{p}$ and $\bar{I}(\bar{p})$, respectively. For compactness of notation we write

$$\ln(\bar{p}) \triangleq \begin{bmatrix} \ln(\bar{p}_1) & \ln(\bar{p}_2) & \ldots & \ln(\bar{p}_n) \end{bmatrix}^T \tag{10}$$
$$e^p \triangleq \exp(p) = \begin{bmatrix} e^{p_1} & e^{p_2} & \ldots & e^{p_n} \end{bmatrix}^T \tag{11}$$

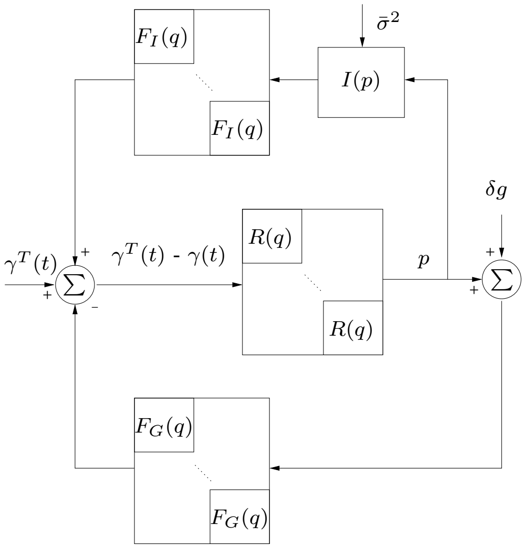

Fig. 1. Global control loop. The interference function $I(p)$ is described in Figure 2.

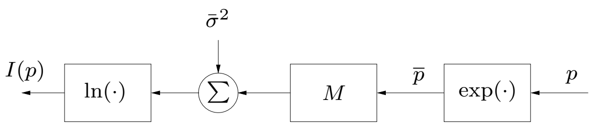

Fig. 2. Interference function

for any n-dimensional vectors $\bar{p}$ and p, respectively.

The same system model was proven locally stable in [4] using linearization of the interference function. Here the nonlinear system will be studied. We therefore transform the system further to the standard form in Figure 3, where $\Delta p = p - p^0$ and $\Delta r = r - r^0$ are deviations around the equilibrium point

$$(p^0, r^0) = \left(\ln(\bar{p}^0), \ln(Me^{p^0} + \bar{\sigma}^2) + \frac{1}{F_I}\gamma^T + \frac{F_G}{F_I}\delta g\right)$$

where we have assumed that F_I has a stable inverse. The linear and nonlinear subsystem in Figure 3 are defined by the block diagonal system

$$H(q) = \frac{R(q)F_I(q)}{1 + R(q)F_G(q)}\mathbf{I} \tag{12}$$

and

$$\Phi(\Delta p) = \begin{bmatrix} \ln\left(\frac{\bar{\sigma}_1^2 + M^1 e^{p^0+\Delta p}}{\bar{\sigma}_1^2 + M^1 e^{p^0}}\right) \\ \vdots \\ \ln\left(\frac{\bar{\sigma}_n^2 + M^n e^{p^0+\Delta p}}{\bar{\sigma}_n^2 + M^n e^{p^0}}\right) \end{bmatrix}, \tag{13}$$

where $e^{p^0+\Delta p}$ is defined as in (11) and $\mathbf{I}$ is the identity. The nonlinearity in (13) will be called *interference nonlinearity* in this paper. We note that $\Phi(0) = 0$ and therefore $\Delta p = 0$ is a unique equilibrium point of the system.

The input-output analysis in the next section provides conditions under which the power distribution remains in a prescribed neighborhood of the desired equilibrium when the system is subjected to disturbances. The disturbances are modeled as the perturbation Δr and could, for example, correspond to noise in the interence measurements or changes in the target SIR or gain matrix.

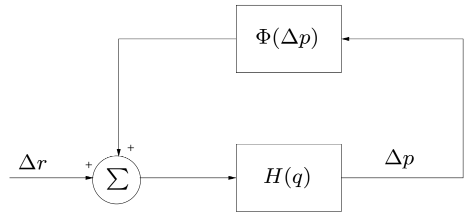

Fig. 3. The system on input-output form.

VI. Input-Output Analysis

The system in Figure 3 can equivalently be represented as the operator equation

$$\Delta p = H(\Delta r + \Phi(\Delta p)) \tag{14}$$

where the H in (12) and Φ in (13) are interpreted as operators on a Banach space X. We will address the questions of existence and uniqueness of solution to (14) as well as the boundedness of convergence of this solution [1] [2]. The results obtained are critically dependent on the choice of underlying space. The following notation will be used for norms and function spaces:

Finite dimensional vector spaces (spatial dimension)

- $R_2^n \triangleq (R^n, |\cdot|_2)$, where $|x|_2 = (\sum_{k=1}^{n} x_k^2)^{1/2}$
- $R_\infty^n \triangleq (R^n, |\cdot|_\infty)$, where $|x|_\infty = \max_{1\le k\le n} |x_k|$

Matrix norms

- $M : R_2^n \to R_2^n, \quad |M|_{R_2^n \to R_2^n} = \bar{\sigma}(M)$
- $M : R_\infty^n \to R_\infty^n, \ |M|_{R_\infty^n \to R_\infty^n} = |M|_1$

where $\bar{\sigma}(M)$ denotes the largest singular value of M and $|M|_1 = \max_{1\le i\le n} \sum_{j=1}^{n} |M_{ij}|$.

Signal space (function space)

- $l_2^n = \{z : R \to R_2^n : \|z\|_2 < \infty\}$
- $l_\infty^n = \{z : R \to R_\infty^n : \|z\|_\infty < \infty\}$,
- $l_{2,\infty}^n = \{z : R \to R_\infty^n : \|z\|_{2,\infty} < \infty\}$,

where the norms are defined as $\|z\|_2 = (\sum_{k=0}^{\infty} |z[k]|_2^2)^{1/2}$, $\|z\|_\infty = \sup_k |z[k]|_\infty$, and $\|z\|_{2,\infty} = (\sum_{k=0}^{\infty} |z[k]|_\infty^2)^{1/2}$. In the following the spatial dimension will often be suppressed.

[1] It is usually necessary to define the system on an extended Banach space in order to address these questions, see e.g. [2]. We will show that the operators involved in (14) are Lipschitz continuous. It is thus sufficient to apply the Banach fixed point theorem to obtain our main results.

[2] The operator Φ is memoryless and G is causal on all spaces considered. The assumptions of our main result in Theorem 2 imply that the closed loop system also is causal.

In the next few definitions we consider a nonlinear operator $F : X \to X$ such that $F(0) = 0$. The *gain* of F is defined as

$$\|F\|_{X\to X} = \sup_{z\in X; z\neq 0} \frac{\|F(z)\|_X}{\|z\|_X}$$

where $\|\cdot\|_X$ denotes the norm on X. A stronger assumption is Lipschitz continuity. The *global Lipschitz constant* of the operator F is defined as

$$L[F;X] = \sup_{z_1,z_2\in X, z_1\neq z_2} \frac{\|F(z_1)-F(z_2)\|_X}{\|z_1-z_2\|_X}$$

Notice that $\|F\|_{X\to X} \leq L[F;X]$. It will also be useful to consider the Lipschitz constant defined over the closed ball $B(X,\gamma) = \{z \in X : \|z\|_X \leq \gamma\}$. We define

$$L[F;B(X,\gamma)] = \sup_{z_1,z_2\in B(X,\gamma), z_1\neq z_2} \frac{\|F(z_1)-F(z_2)\|_X}{\|z_1-z_2\|_X}.$$

For linear operators the gain and Lipschitz constants coincide. The following gain characterization for the linear system H in (12) will be used in the sequel. We assume that H has the following time domain representation in terms of the impulse response

$$(Hu)(k) = \sum_{l=0}^{k} h(k-l)u(l), \quad k \geq 0$$

and define the l_1-norm as

$$\|H\|_1 = \sum_{k=0}^{\infty} |h(k)|.$$

Proposition 2: $\|H\|_{l^n_{2,\infty}\to l^n_{2,\infty}} \leq \|H\|_{l^n_\infty\to l^n_\infty} = \|H\|_1$.
Proof: It is well known that $\|H\|_{l^n_\infty\to l^n_\infty} = \|H\|_1$, see e.g. [1]. A proof that $\|H\|_{l^n_{2,\infty}\to l^n_{2,\infty}} \leq \|H\|_1$ is given in the appendix for completeness. ■

A. Analysis in l^n_2

It is often advantageous to analyze the system in a Hilbert space such as l^n_2. Then the inner product structure and Fourier domain tools may be used to capture phase information and frequency domain interpretations. Note also that the gain $\|H\|_{l^n_2\to l^n_2} = \sup_{\omega\in R} |H(j\omega)|$ is less than or equal to $\|H\|_1$. Despite these potential advantages it turns out that the interference nonlinearity in (13) has a structure that appears to be unsuitable for l_2-analysis. Our first negative result shows that its gain and Lipschitz constant grows with the number of users. The proof follows along the lines of Theorem 1 below.

Proposition 3: $\|\Phi\|_{l_2\to l_2} = L[\Phi; l_2] = \sqrt{n}$.
Our second negative observation shows that the interference nonlinearity violates the definition of incremental positivity in [8]. This implies that powerful characterizations of memoryless nonlinearities from the input-output theory cannot be used, see e.g. [8], [7] and the references therein. The proof of our claim follows because if

$$z_1 = \begin{bmatrix} z_{11} \\ z_{12} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad z_2 = \begin{bmatrix} z_{21} \\ z_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and $\bar{\delta}_i = 1$, $\bar{\sigma}_i^2 > 0$, $\quad i = 1, 2$, then one can show that

$$(z_1 - z_2)^T(\Phi(z_1) - \Phi(z_2)) = \ln\left(\frac{\bar{\sigma}_1^2 + \bar{g}_{12}}{\bar{\sigma}_1^2 + \bar{g}_{12}e}\right) - \ln\left(\frac{\bar{\sigma}_2^2 + g_{21}e}{\bar{\sigma}_2^2 + \bar{g}_{21}}\right) < 0,$$

which implies that the interference nonlinearity cannot be incrementally positive.

B. Main Results

Theorem 1: The interference nonlinearity in (13) has gain and Lipschitz constants

$$\|\Phi\|_{l_\infty\to l_\infty} = L[\Phi, l_\infty] = 1$$
$$\|\Phi\|_{l_{2,\infty}\to l_{2,\infty}} = L[\Phi, l_{2,\infty}] = 1.$$

Proof: The proof can be found in the appendix. ■
The next corollary gives the corresponding gain and Lipschitz constants over a closed ball.

Corollary 1: In the bounded sets $B(l_\infty, \gamma)$ the interference nonlinearity in (13) has gain and Lipschitz constants

$$\sup_{z\in B(l_\infty,\gamma),\ z\neq 0} \frac{\|\Phi(z)\|_{l_\infty}}{\|z\|_{l_\infty}} \leq L[\Phi, B(l_\infty,\gamma)] = K < 1$$
$$\sup_{z\in B(l_{2,\infty},\gamma),\ z\neq 0} \frac{\|\Phi(z)\|_{l_{2,\infty}}}{\|z\|_{l_{2,\infty}}} \leq L[\Phi, B(l_{2,\infty},\gamma)] = K < 1$$

where

$$K = \max_i \left(\frac{\eta_i}{\bar{\sigma}_i^2 + \eta_i}\right),$$

where

$$\eta_i = \max_{|z|_\infty\leq\gamma} \left(M^i e^{p^0+z}\right) = M^i e^{p^0} e^{\gamma}$$

Proof: The proof can be found in the appendix. Note that the maximization in the definition of η_i always is achieved by $z^* = \gamma\mathbf{1}$ since M^i and e^{p^0} has positive coefficients. Here $\mathbf{1}$ is the all one vector. ■
Next follows our main result on existence and uniqueness of a bounded and convergent solution.

Theorem 2: Suppose $\bar{\sigma}_i > 0$ for all i. Suppose furthermore that we have a desired bound on the power deviation (in logarithmic scale)

$$\|\Delta p\|_\infty \leq \gamma < \infty.$$

Let

$$\eta_i = \max_{|z|_\infty\leq\gamma} \left(M^i e^{p^0+z}\right) = M^i e^{p^0} e^{\gamma}$$

and let

$$K = \max_i \left(\frac{\eta_i}{\bar{\sigma}_i^2 + \eta_i}\right) < 1,$$

since $\eta_i < \infty$. Then if $\|H\|_1 < \frac{1}{K}$, there exists a unique power distribution with $\|\Delta p\|_\infty \leq \gamma$ for all

$$\|\Delta r\|_\infty \leq \frac{\gamma(1 - \|H\|_1 K)}{\|H\|_1}. \qquad (15)$$

Moreover if in addition $\|\Delta r\|_{2,\infty} < \infty$, it follows that $\Delta p_k \to 0$ as $k \to \infty$.

Proof: Define the saturation $sat_{[-\gamma\mathbf{1},\gamma\mathbf{1}]} : R^n \to R^n$ whose k^{th} component is

$$[sat_{[-\gamma\mathbf{1},\gamma\mathbf{1}]}(x)]_k = \begin{cases} \gamma & \text{if } x_k > \gamma \\ x_k & \text{if } -\gamma \le x_k \le \gamma \\ -\gamma & \text{if } x_k < -\gamma \end{cases}$$

and let

$$\Phi_\gamma(x) = \Phi(sat_{[-\gamma\mathbf{1},\gamma\mathbf{1}]}(x)).$$

Define $F(x) = H(\Delta r + \Phi_\gamma(x))$, then

$$\begin{aligned} \|F(x_1) - F(x_2)\|_\infty &= \|H(\Phi_\gamma(x_1) - \Phi_\gamma(x_2))\|_\infty \\ &\le \underbrace{\|H\|_1 K}_{L<1} \|x_1 - x_2\|_\infty, \end{aligned}$$

where we used Corollary 1. Hence F is a contraction on l_∞ and according to the Banach fixed point theorem there exists a unique solution Δp^0 to the fixed point equation $\Delta p^0 = F(\Delta p^0)$. Assume now that that the bound in (15) holds. Then the fixed point Δp^0 satisfies

$$\begin{aligned} \|\Delta p^0\|_\infty &= \|F(\Delta p^0)\|_\infty \\ &\le \|H\|_1 \left(\|\Delta r\|_\infty + K\|\Delta p^0\|_\infty \right) \\ &\le \|H\|_1 \frac{\gamma(1 - K\|H\|_1)}{\|H\|_1} + \|H\|_1 K \|\Delta p^0\|_\infty \end{aligned}$$

which is equivalent to $\|\Delta p^0\|_\infty \le \gamma$. This implies that there also exists a unique power distribution with $\|\Delta p\|_\infty \le \gamma$ to the real system because the saturation in the definition of Φ_γ is inactive.

The last statement in the theorem follows from the bound

$$\|\Delta p^0\|_{2,\infty} \le \frac{\|H\|_1}{1 - K\|H\|_1} \|\Delta r\|_{2,\infty}$$

which is derived in the same fashion as the previous bound. ∎

VII. Simulation Example

Consider a network with three users. Let the gains be

$$G = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0.05 & 0.01 \\ 0.07 & 1 & 0.08 \\ 0.04 & 0.06 & 1 \end{bmatrix}$$

$\bar{\sigma}_i^2 = 0.05$, $\bar{\gamma}_i = 9.025$ and $\bar{\delta}_i = 1, \quad \forall i$. The spectral radius satisfies $\rho(\overline{\Gamma}_T \overline{\Delta}^{-1} M) = 0.93 < 1$, and hence the problem is feasible and the optimal power vector is obtained from

$$\overline{p}^0 = (I - \overline{\Gamma}_T \overline{\Delta}^{-1} M)^{-1} \overline{\Gamma}_T \overline{\Delta}^{-1} \overline{\sigma}^2,$$

which in this case is (note the logarithmic scale)

$$p^0 = \begin{bmatrix} 1.57 \\ 2.12 \\ 1.90 \end{bmatrix}. \qquad (16)$$

For the following three cases, the initial power states were set to 0 and all filters were set to identity.

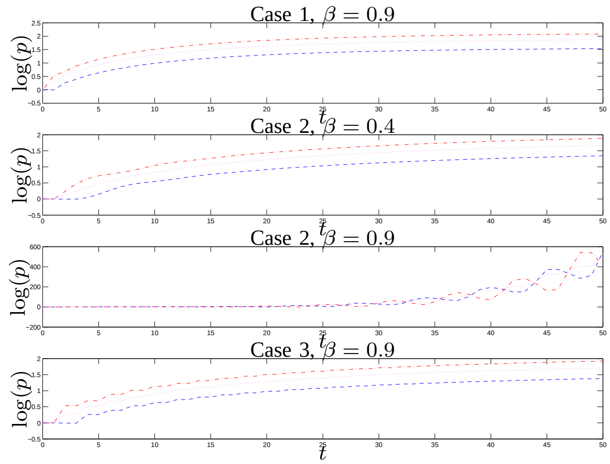

Fig. 4. Simulations of Case 1-3.

A. Case 1

Let all users have the control law

$$R(q) = R_0(q) \triangleq \frac{\beta}{q-1}.$$

Note that $\beta = 1$ corresponds to the power control algorithm proposed in (1). The convergence criterion in Proposition 1 is clearly satisfied for any $\beta > 0$. A simulation of Case 1 can be seen in Figure 4 and it confirms our theoretical conclusions.

To apply Theorem 2 we notice that

$$H(q) = \frac{\beta}{q - 1 + \beta},$$

and

$$\|H\|_1 = \sum_{k=0}^{\infty} |h(k)| = \sum_{k=0}^{\infty} \beta(1-\beta)^k = \frac{\beta}{1-(1-\beta)} = 1.$$

Theorem 2 thus ensures that power configuration remains bounded and converges when the system is subject to disturbances in $l_{2,\infty}$.

Note that if the system was to be analyzed in l_2, no conclusion on stability could be made using the small gain theorem. This is because $\|H\|_{l_2^n \to l_2^n} = \sup_{\omega \in R} |H(j\omega)| = 1$, and a linear approximation of the nonlinear interference function $\|\nabla\Phi(p^0)\|$ has the norm $\|\nabla\Phi(p^0)\|_{l_2^n \to l_2^n} \approx 1.06$, which gives a round trip gain which is bigger than one. This shows the inadequacy of the l_2-analysis with the small gain theorem and motivates the use of the $l_{2,\infty}$-signal space.

B. Case 2

Now consider the same system with a single delay. Then

$$R(q) = q^{-1} R_0(q) = \frac{q^{-1}\beta}{q-1} = \frac{\beta}{q^2 - q},$$

and

$$H(q) = \frac{\beta}{q^2 - q + \beta}.$$

The stability criterion in Proposition 1 is not satisfied since it would require that $\beta_1 = \beta > 0$ and $\alpha_2 = 0 < -\beta$, which is impossible. To use Theorem 2, $\|H\|_1$ can be computed as above, but the result depends on the value of β. For $\beta \in (0, 1)$ the sum is convergent, but $\|H\|_1 > 1$ for all β. For example, $\beta = 0.4$ gives $\|H\|_1 = 4.1773$. Even if we let $\Delta p = 0$, i.e. starting at the equilibrium point, we get the constraint $\|H\|_1 < 1.057$ to ensure stability. Since this is not the case, we can not make any conclusions on stability for this case. However, a simulation of the system shows that for $\beta = 0.4$ the system is stable and converges, see Figure 4. This shows that the stability criterion is conservative. However, for bigger β, for example $\beta = 0.9$, the system is unstable, see Figure 4. The fact that stable systems due to delay may go unstable motivates the use of higher order control action.

C. Case 3

In order to stabilize systems with delays, delay compensation using a Smith predictor was introduced in [4]. Let

$$R(q) = \frac{q^{-1}R_0(q)}{1 + R_0(q)(1 - q^{-1})} = \frac{\beta}{q^2 - q + \beta q - \beta},$$

We can easily concude that the criterion in Proposition 1 is satisfied when $\beta \in (0, 1]$, which is consistent with the simulation of Case 3 in Figure 4.

To apply Theorem 2 we notice that

$$H(q) = \frac{\beta}{q(q - (1 - \beta))}.$$

The derivation of $\|H\|_1$ is almost identical to that of Case 1, and gives the same value, $\|H\|_1 = 1$. Hence by the same argument as in Case 1, boundedness and convergence under disturbance of Case 3 can be ensured.

VIII. Concluding Remarks

Two approaches for analysis of a class of higher order power control loops in wireless communication networks have been considered. In the first approach Yates framework in [9] was generalized to fit the considered system model. This approach has the advantage that the case of heterogeneous control dynamics among the users can be considered, e.g. stability can be guaranteed in situations where different users have different, but known, delays.

In the second approach we used input-output theory to prove boundedness of the solution when the system is subject to disturbances. This approach also allow convergence to be established. It has the advantage that it allows generalization to the case with time-varying gain matrices and additional filters can be included in the analysis. However, the case with heterogeneous user dynamics appears to be harder since it adds conservatism to our criterion.

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Appendix: Proofs

1) Proof of Proposition 1: The assumptions on the β_k and α_k imply that $R(1) = \infty$. Hence it follows from the previous section that $\bar{p}^0$ in (5) is an equilibrium point. To prove that the equilibrium is unique and that the system converges to this equilibrium we use the framework in [9]. First to obtain a compact notation we let

$$P(t) = \begin{bmatrix} \bar{p}(t) \\ \vdots \\ \bar{p}(t+1-m) \end{bmatrix}, \qquad \mathcal{I}(P(t)) = \begin{bmatrix} \Psi(P(t)) \\ \bar{p}(t) \\ \vdots \\ \bar{p}(t+2-m) \end{bmatrix}$$

where

$$\Psi(P(t)) = \begin{bmatrix} \Psi_1(\bar{p}(t), \dots, \bar{p}(t+1-m)) \\ \vdots \\ \Psi_n(\bar{p}(t), \dots, \bar{p}(t+1-m)) \end{bmatrix}$$

The dynamics in (6)-(7) can now be formulated as

$$P(t+1) = \mathcal{I}(P(t)), \quad P(0) = \begin{bmatrix} \bar{p}_0 & \dots & \bar{p}_0 \end{bmatrix}^T. \qquad (17)$$

The extended interference function $\mathcal{I}$ satisfies the following properties

(i) $\mathcal{I}(P) \geq 0$, $\forall P \geq 0$ and $\mathcal{I}(P) > 0$, $\forall P > 0$

(ii) If $P' \geq P$ then $\mathcal{I}(P') \geq \mathcal{I}(P)$

(iii) For all $\theta > 1$, $\theta\mathcal{I}(P) \geq \mathcal{I}(\theta P)$

where all inequalities should be interpreted componentwise and where in (ii) and (iii) $P, P' \geq 0$. These properties follow since

$$\begin{aligned} \widehat{\bar{I}}_i(\bar{p}) &\triangleq \frac{\bar{\gamma}_i^T \bar{I}_i(t-k)}{\bar{\delta}_i \bar{g}_{ii}} \\ &= \frac{\bar{\gamma}_i^T}{\bar{\delta}_i \bar{g}_{ii}} \left(\sum_{i \neq j} \bar{g}_{ij}\bar{p}_j + (1 - \bar{\delta}_i)\bar{g}_{ii}\bar{p}_i + \bar{\sigma}_i^2 \right) \end{aligned}$$

in the definition of Ψ_i in (7) satisfies $(i)-(iii)$ and since the β_k and α_k satisfies (8). We will here also use the stronger properties that

(i') $\widehat{\bar{I}}_i(\bar{p}) > 0$, $\forall \bar{p} \geq 0$,

(ii') If $\bar{p}' \geq \bar{p}$ then $\mathcal{I}(\bar{p}') \geq \mathcal{I}(\bar{p})$

(iii') For all $\theta > 1$, $\theta \widehat{\bar{I}}_i(\bar{p}) > \widehat{\bar{I}}_i(\theta\bar{p})$,

which follows since $\bar{\sigma}_i > 0$. We also use the property that if $\bar{p} \geq \bar{p}^0$ then

$$\bar{p}_i \geq \widehat{\bar{I}}_i(\bar{p}), \quad i = 1, \ldots, n \tag{18}$$

which follows since $\rho(\bar{\Delta}^{-1}\bar{\Gamma}_T M) < 1$ and since we have equality in (18) when $\bar{p} = \bar{p}^0$ due to (5).

Based on these properties we use a similar proof technique as in [9] to show that 1) $\bar{p}^0$ is the unique strictly positive equilibrium of the system, 2) any strictly positive initial condition of the system converges to the equilibrium $\bar{p}^0$.

Uniqueness of the equilibrium: Suppose that there exists another nonzero equilibrium $\bar{p}^1$ of the system in (6)-(7). Since both equilibria are strictly positive one can establish that there either exists $\theta > 1$ such that $\theta\bar{p}^1 \geq \bar{p}^0$ and for some i, $\theta\bar{p}_i^1 = \bar{p}_i^0$ or that the analogous identity holds in the case when the roles of $\bar{p}^1$ and $\bar{p}^0$ are interchanged. We assume without loss of generality that the first alternative holds. Then

$$\begin{aligned}\bar{p}_i^0 &= \Psi_i(\bar{p}^0, \ldots, \bar{p}^0) \leq \Psi_i(\theta\bar{p}^1, \ldots, \theta\bar{p}^1)\\ &< \theta\Psi_i(\bar{p}^1, \ldots, \bar{p}^1) = \theta\bar{p}_i^1\end{aligned}$$

which is a contradiction. The first and second inequalities follows from (ii') and (iii'), respectively. For example, the second inequality follows since

$$\begin{aligned}\Psi_i(\theta\bar{p}^1, \ldots, \theta\bar{p}^1) &= \widehat{\bar{I}}(\theta\bar{p}^1)^{\sum_{k=0}^{m-1}\beta_k}(\theta\bar{p}_i^1)^{-\sum_{k=0}^{m-1}(\alpha_{k+1}+\beta_k)}\\ &< (\theta\widehat{\bar{I}}(\bar{p}^1))^{\sum_{k=0}^{m-1}\beta_k}(\theta\bar{p}_i^1)^{-\sum_{k=0}^{m-1}(\alpha_{k+1}+\beta_k)}\\ &= \theta\widehat{\bar{I}}(\bar{p}^1)^{\sum_{k=0}^{m-1}\beta_k}(\bar{p}_i^1)^{-\sum_{k=0}^{m-1}(\alpha_{k+1}+\beta_k)}\\ &= \theta\Psi_i(\bar{p}^1, \ldots, \bar{p}^1)\end{aligned}$$

where we used (8).

Convergence to the equilibrium: The idea of the proof is to use the monotonicity condition in (ii) to sandwich the solution (17) between an increasing strictly positive lower bound and a decreasing upper bound. Since $P^0 = \begin{bmatrix}\bar{p}^0 & \ldots \bar{p}^0\end{bmatrix}^T$ is the unique equilibrium it follows that $P(t) \to P^0$.

Let us consider the system in (17) with initial conditions $\bar{P}(0) = \theta\mathbf{1}$ and $\underline{P}(0) = \frac{1}{\theta}\mathbf{1}$, respectively. If θ is sufficiently large then it follows that $\bar{P}(0) \geq \mathcal{I}(\bar{P}(0))$ because each of the first n components will satisfy such an inequality due to (18) and the definition of Ψ_i in (7) while we will have equality in the remaining components. Let $\bar{P}(t) = \mathcal{I}^t(\bar{P}(0))$. It follows by induction that $\bar{P}(t+1) = \mathcal{I}^{t+1}(\bar{P}(0)) = \mathcal{I}(\bar{P}(t)) \leq \bar{P}(t)$ for all $t \geq 0$.

Similarly, if θ is sufficiently large then $\mathcal{I}(\underline{P}(0)) \geq \underline{P}(0)$ because each of the first n components will satisfy such an inequality when $\theta\bar{\sigma}_i^2 >> 1$ while the remaining components are equal. It follows by induction that $\underline{P}(t+1) = \mathcal{I}^{t+1}(\underline{P}(0)) = \mathcal{I}(\underline{P}(t)) \geq \underline{P}(t)$, for all $t \geq 0$.

It follows from the monotonicity condition in (ii) that $P(t) = \mathcal{I}^t(P(0))$ satisfies

$$\underline{P}(t) \leq P(t) \leq \bar{P}(t), \quad \forall t \geq 0.$$

Since $\bar{P}(t)$ is monotonically decreasing and $\underline{P}(t)$ is monotonically increasing it follows that all three sequences converges to the unique equilibrium P^0. This concludes the proof.

Proof of Proposition 2

The first bound can be proven using the following arguments. Using that H is diagonal, it follows that at any time instance k

$$\begin{aligned}|\sum_{l=0}^{k} h(k-l)u(l)|_\infty &\leq \sum_{l=0}^{k} |h(k-l)||u(l)|_\infty\\ &\leq \Big(\sum_{l=0}^{k} |h(k-l)||u(l)|_\infty^2\Big)^{\frac{1}{2}}\Big(\sum_{l=0}^{k}|h(k-l)|\Big)^{\frac{1}{2}}\end{aligned}$$

Taking the l_2-norm on both sides gives

$$\begin{aligned}\|Hu\|_{2,\infty}^2 &\leq \sum_{k=0}^{\infty}\sum_{l=0}^{k}|h(k-l)||u(l)|_\infty^2\Big(\sum_{l=0}^{k}|h(k-l)|\Big)\\ &\leq \sum_{k=0}^{\infty}\sum_{l=0}^{k}|h(k-l)||u(l)|_\infty^2\|H\|_1\\ &= \sum_{l=0}^{\infty}\underbrace{\Big[\sum_{k=l}^{\infty}|h(k-l)|\Big]}_{=\|H\|_1}|u(l)|_\infty^2\|H\|_1 = \|H\|_1^2\|u\|_{2,\infty}^2,\end{aligned}$$

which shows that $\|H\|_{l_{2,\infty}^n \to l_{2,\infty}^n} \leq \|H\|_1$.

Proof of Theorem 1

Let us first consider the interference nonlinearity as a multivariable function $\Phi : R_\infty^n \to R_\infty^n$. It follows that

$$\begin{aligned}|\Phi(x) - \Phi(y)|_\infty &= |\int_0^1 \nabla\Phi(y + \theta(x-y))(x-y)d\theta|_\infty\\ &\leq \int_0^1 |\nabla\Phi(y+\theta(x-y))|_1 d\theta|x-y|_\infty\\ &\leq \sup_{z\in R^n}|\nabla\Phi(z)|_1|x-y|_\infty\end{aligned}$$

This gives the Lipschitz bound

$$\begin{aligned}L[\Phi, R_\infty^n] &\leq K \triangleq \sup_z |\nabla\Phi(z)|_1\\ &= \sup_z \left|\begin{bmatrix}\frac{(1-\delta_1)g_{11}e^{p_1^0+z_1}}{\bar{\sigma}_1^2+M^1e^{p^0+z}} & \ldots & \frac{g_{1n}e^{p_n^0+z_n}}{\bar{\sigma}_1^2+M^1e^{p^0+z}}\\ \vdots & & \vdots\\ \frac{g_{n1}e^{p_1^0+z_1}}{\bar{\sigma}_n^2+M^ne^{p^0+z}} & \ldots & \frac{(1-\delta_n)g_{nn}e^{p_n^0+z_n}}{\bar{\sigma}_n^2+M^ne^{p^0+z}}\end{bmatrix}\right|_1\\ &= \sup_z \max_i\left(\frac{M^ie^{p^0+z}}{\bar{\sigma}_i^2+M^ie^{p^0+z}}\right) = 1.\end{aligned}$$

where M^i is the i:th row in (3). We then have

$$\begin{aligned}\big\|\Phi(z_1)-\Phi(z_2)\big\|_{2,\infty} &= \sqrt{\sum_{k=0}^{\infty}|\Phi(z_1[k])-\Phi(z_2[k])|_\infty^2} \\ &\le K\sqrt{\sum_{k=0}^{\infty}|z_1[k]-z_2[k]|_\infty^2} \\ &= K\big\|z_1-z_2\big\|_{2,\infty}.\end{aligned}$$

which shows that $L[\Phi,l_{2,\infty}]\le K=1$. We will next see that the bound can be achieved asymptotically by considering the $l_{2,\infty}$-signal,

$$z=\begin{cases}\mathbf{1}, & k=0\\ 0, & \text{otherwise.}\end{cases}$$

Let $e^{p^0+\alpha\mathbf{1}}$ be defined as in (11). We have

$$\begin{aligned}\frac{1}{\alpha}\|\Phi(\alpha z)-\Phi(0)\|_{2,\infty} &= \left(\sum_{k=0}^{\infty}\frac{1}{\alpha}|\Phi(\alpha z_k)|_\infty^2\right)^{1/2}\\ &= \left|\frac{1}{\alpha}\begin{bmatrix}\ln\left(\frac{\bar\sigma_1^2+M^1e^{p^0+\alpha}}{\bar\sigma_1^2+M^1e^{p^0}}\right)\\ \vdots\\ \ln\left(\frac{\bar\sigma_n^2+M^ne^{p^0+\alpha}}{\bar\sigma_n^2+M^ne^{p^0}}\right)\end{bmatrix}\right|_\infty\\ &= \left|\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}+\begin{bmatrix}\frac{1}{\alpha}\ln\left(\frac{\bar\sigma_1^2e^{-\alpha}+M^1e^{p^0}}{\bar\sigma_1^2+M^1e^{p^0}}\right)\\ \vdots\\ \frac{1}{\alpha}\ln\left(\frac{\bar\sigma_n^2e^{-\alpha}+M^ne^{p^0}}{\bar\sigma_n^2+M^ne^{p^0}}\right)\end{bmatrix}\right|_\infty \to 1\end{aligned}$$

as $\alpha\to\infty$. It follows that $L[\Phi,l_{2,\infty}]=1$ as well as $\|\Phi\|_{l_{2,\infty}\to l_{2,\infty}}=1$. The case with l_∞-space follows using essentially the same arguments.

Proof of Corollary 1

The proof that $L[\Phi,B(l_{2,\infty},\gamma)]\le K$ is analogous to the first half of the proof of Theorem 1. To see that equality can be achieved we assume

$$K=\sup_z|\nabla\Phi(z)|_1=\frac{\eta_{i^*}}{\bar\sigma_{i^*}^2+\eta_{i^*}},\quad \eta_{i^*}=M^{i^*}e^{p^0}e^{\gamma},$$

i.e. that the maximum defining K is achieved at index i^* with $z^*=\gamma\mathbf{1}$, the argument that achieves the maximum defining η_{i^*} (this follows since M is a positive matrix). Further, let δz be a unit length vector in R_∞^n such that $|\nabla\Phi(z^*)\delta z|_\infty=|\nabla\Phi(z^*)|_1=K$. We note that δz must have positive components since $\nabla\Phi(z^*)$ is a positive matrix.

Let $y=z^*$ and $x=z^*-\epsilon\delta z$. We get

$$\begin{aligned}&\epsilon^{-1}|\Phi(x)-\Phi(y)|_\infty=|\int_0^1\nabla\Phi(z^*-\epsilon\theta\delta z)\delta z d\theta|_\infty\\ &\quad\ge|\nabla\Phi(z^*)|_1-|\int_0^1(\nabla\Phi(z^*)-\nabla\Phi(z^*-\epsilon\theta\delta z)\delta z d\theta|_\infty\\ &\quad\ge|\nabla\Phi(z^*)|_1-\epsilon L[\nabla\Phi;R_1^{n\times n}]\end{aligned}$$

where $L[\nabla\Phi;R_1^{n\times n}]$ denotes the Lipschitz bound of the Jacobian $\nabla\Phi:R_\infty^n\to R_1^{n\times n}$ and $R_1^{n\times n}$ is the vector space of real valued $n\times n$ matrices equipped with the matrix $|\cdot|_1$-norm.

Hence, if we define the $l_{2,\infty}$-signals,

$$z_1=\begin{cases}z^*-\epsilon\delta z, & k=0\\ 0, & \text{otherwise}\end{cases}$$

$$z_2=\begin{cases}z^*, & k=0\\ 0, & \text{otherwise}\end{cases}$$

we get

$$\epsilon^{-1}\|\Phi(z_1)-\Phi(z_2)\|_{l_{2,\infty}}\ge|\nabla\Phi(z^*)|_1-\epsilon L[\nabla\Phi;R_1^{n\times n}].$$

Since ϵ is arbitrary it follows that $L[\Phi,B(l_{2,\infty}^n,\gamma)]\ge K$. We conclude that $L[\Phi,B(l_{2,\infty}^n,\gamma)]=K$. The proof that $L[\Phi,B(l_\infty^n,\gamma)]=K$ follows analogously and the two gains are obviously less than the Lipschitz constants.