

## OPTIMALITY CONDITIONS

- **FoNoc** = First-order Necessary optimality conditions
- **SoNoc** = Second-order Necessary optimality conditions
- **SoSoc** = Second-order Sufficient optimality conditions
- Assume  $f, g$  and  $h \in \mathcal{C}^2$ . Also assume that  $x^*$  is a regular point.

### Optimality conditions for unconstrained problems

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x)$$

- **FoNoc**:  $\nabla f(x^*) = 0$ .
- **SoNoc**: FoNoc and  $\nabla^2 f(x^*) \succeq 0$ .
- **SoSoc**: SoNoc and  $\nabla^2 f(x^*) \succ 0$ .

### Optimality conditions for linear equality constrained problems

$$\begin{aligned} &\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) \\ &\text{subject to} \quad Ax = b \end{aligned}$$

- **FoNoc**:  $Ax^* = b$  and  $Z^T \nabla f(x^*) = 0$ .
- **SoNoc**: FoNoc and  $Z^T \nabla^2 f(x^*) Z \succeq 0$ .
- **SoSoc**: SoNoc and  $Z^T \nabla^2 f(x^*) Z \succ 0$ .

where  $Z$  denotes a matrix (not unique) whose columns form a base of  $\mathcal{N}(A)$ .  
e.g. let  $A = (B \ N)$ , then a

$$Z = \begin{pmatrix} -B^{-1}N \\ I \end{pmatrix}.$$

- An alternative of **FoNoc** is:  $Ax^* = b$  and  $\nabla f(x^*) = A^T \lambda^*$ ,  
where  $\lambda^*$  is a Lagrangian multiplier vector.

### Optimality conditions for linear inequality constrained problems

$$\begin{aligned} &\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) \\ &\text{subject to} \quad Ax \geq b \end{aligned}$$

- **FoNoc**:  $Ax^* \geq b$  and  $\nabla f(x^*) = A_A^T \lambda_A^*$ ,  $\lambda_A^* \geq 0$ .
- **SoNoc**: FoNoc and  $Z_A^T \nabla^2 f(x^*) Z_A \succeq 0$ .
- **SoSoc**: SoNoc and  $Z_+^T \nabla^2 f(x^*) Z_+ \succ 0$ .

where  $A_A$  correspond to the active constraints in  $Ax^* \geq b$  (i.e.  $A_A x^* = b_A$ ).  
where  $Z_A$  is a matrix whose columns form a base of  $\mathcal{N}(A_A)$ .  
where  $Z_+$  is a matrix whose columns form a base of  $\mathcal{N}(A_+)$ .  
where  $A_+$  is a matrix of the rows of  $A_A$  for which  $\lambda_A^* > 0$ .

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**Optimality conditions for equality constrained problems**

$$\begin{aligned} & \underset{x \in \mathbb{R}^n}{\text{minimize}} && f(x) \\ & \text{subject to} && g(x) = 0 \end{aligned}$$

- **FoNoc:**  $g(x^*) = 0$  and  $\nabla f(x^*) = A(x^*)^T \lambda^*$ .
- **SoNoc:** FoNoc and  $Z(x^*)^T \nabla_{xx}^2 \mathcal{L}(x^*, \lambda^*) Z(x^*) \succeq 0$ .
- **SoSoc:** SoNoc and  $Z(x^*)^T \nabla_{xx}^2 \mathcal{L}(x^*, \lambda^*) Z(x^*) \succ 0$ .

where

$$A(x) = \begin{pmatrix} \nabla g_1(x)^T \\ \vdots \\ \nabla g_m(x)^T \end{pmatrix}$$

where  $\mathcal{L}(x, \lambda) = f(x) - \sum_{i=1}^m \lambda_i g_i(x)$

where  $Z(x^*)$  is a matrix whose columns form a base of  $\mathcal{N}(A(x^*))$ .

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**Optimality conditions for inequality constrained problems**

$$\begin{aligned} & \underset{x \in \mathbb{R}^n}{\text{minimize}} && f(x) \\ & \text{subject to} && g(x) \geq 0 \end{aligned}$$

- **FoNoc:**  $g(x^*) \geq 0$  and  $\nabla f(x^*) = A_A(x^*)^T \lambda_A^*$ ,  $\lambda_A^* \geq 0$ .
- **SoNoc:** FoNoc and  $Z_A(x^*)^T \nabla_{xx}^2 \mathcal{L}(x^*, \lambda^*) Z_A(x^*) \succeq 0$ .
- **SoSoc:** SoNoc and  $Z_+(x^*)^T \nabla_{xx}^2 \mathcal{L}(x^*, \lambda^*) Z_+(x^*) \succ 0$ .

where  $A_A$  correspond to the active constraints in  $g(x^*) \geq 0$  (i.e.  $j : g_j(x^*) = 0$ ).

where  $Z_A(x^*)$  is a matrix whose columns form a base of  $\mathcal{N}(A_A(x^*))$ .

where  $Z_+$  is a matrix whose columns form a base of  $\mathcal{N}(A_+)$ .

where  $A_+(x^*)$  is a matrix of the rows of  $A_A(x^*)$  for which  $\lambda_A^* > 0$ .

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**Optimality conditions for inequality and equality constrained problems**

$$\begin{aligned} & \underset{x \in \mathbb{R}^n}{\text{minimize}} && f(x) \\ & \text{subject to} && g(x) \geq 0 \\ & && h(x) = 0 \end{aligned}$$

- **FoNoc:**  $g(x^*) \geq 0$ ,  $h(x^*) = 0$  and  $\nabla f(x^*) = A(x^*)^T \lambda^*$ ,  $\lambda_i^* \geq 0 \forall i \in \mathcal{I}$ ,  $\lambda_i^* g_i(x^*) = 0 \forall i \in \mathcal{I}$ , (where  $\mathcal{I}$  is the inequalities).

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Please contact Anders M (amolle@kth.se) if you find an error.