Local positivity of line bundles on smooth toric varieties and Cayley polytopes

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Introduction

For a smooth projective variety X and a line bundle $\mathscr{L}$ on X there are various notions for measuring the local positivity of $\mathscr{L}$ at a point $x \in X$. Two such measures are the dimension of the k-*osculating space* at x and the *Seshadri constat* at x. The precise nature of the interplay between these two notions is in general an open question. Our main result gives a partial answer to this question for smooth toric varieties, by showing that these notions characterize *Cayley sums*.

Osculating Spaces

Let $\mathscr{L}$ be a line bundle on a smooth variety X and $x \in X$ a point with maximal ideal $\mathfrak{m}_x \subseteq \mathscr{O}_X$. Consider the natural map

$$j_x^k : H^0(X, \mathscr{L}) \to H^0(X, \mathscr{L} \otimes (\mathscr{O}_X/\mathfrak{m}_x^{k+1})).$$

The space $\mathbb{T}_x^k(X, \mathscr{L}) := \mathbb{P}(\mathrm{im}(j_x^k))$ is called the *osculating space of order* k at $x \in X$. If j_x^k is onto we say that $\mathscr{L}$ is k-*jet spanned* at $x \in X$. The largest k such that X is k-jet spanned at $x \in X$ is denoted by $s(\mathscr{L}, x)$.

Seshadri Constants

Let $\mathscr{L}$ be a nef line bundle on a smooth projective variety X. The Seshadri constant of $\mathscr{L}$ at a point $x \in X$ is the number

$$\epsilon(X, \mathscr{L}; x) := \inf_{C \subseteq X} \frac{\mathscr{L} \cdot C}{m_x(C)},$$

where the infimum is taken over all irreducible curves C passing through x and $m_x(C)$ is the multiplicity of C at x.

Cayley Polytopes

Let $P_0, \dots, P_r \subset \mathbb{R}^k$ be polytopes. We define the Cayley sum

$$[P_0 * \cdots * P_r]^s := \mathrm{Conv}\{(P_0 \times 0) \cup (P_1 \times se_1) \cup \cdots \cup (P_r \times se_r)\} \subset \mathbb{R}^k \times \mathbb{R}^r$$

where $e_1, \dots, e_r$ is the standard basis for $\mathbb{R}^r$. A polytope $P \subseteq \mathbb{R}^n$ is called a *Cayley polytope of order* s *and length* $r+1$ if there exist some lower dimensional polytopes $P_0, \dots, P_r$ such that $P \cong [P_0 * \cdots * P_r]^s$. When $P_0, \dots, P_r$ are normally equivalent then the associated toric variety is a projective fiber bundle $\mathbb{P}(L_0 \oplus \cdots \oplus L_r)$.

Examples of Cayley Polytopes

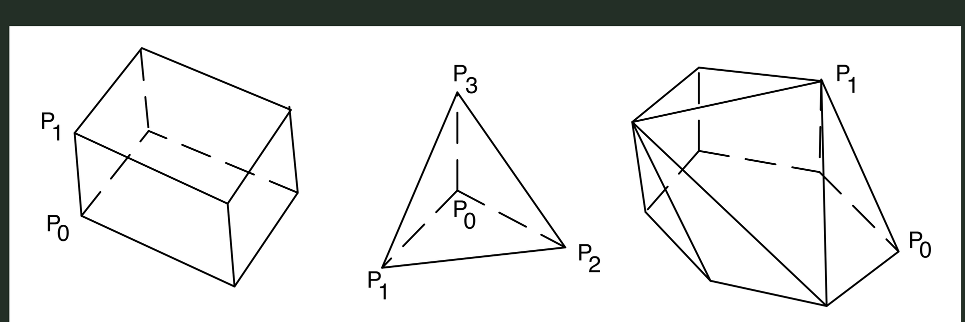

Figure: Three Cayley polytopes in $\mathbb{R}^3$.

Existing Characterizations

Seshadri constants and osculating spaces have been shown to characterize certain polarized smooth toric varieties given by Cayley sums.

- In [3] D. Perkinson showed the following: Let X be a smooth toric variety of dimension ≤ 3 polarized by a complete linear series $|\mathscr{L}|$. Then $s(\mathscr{L}, x) = k$ at every point $x \in X$, for any fixed $k \in \mathbb{N}$, if and only if X is a projective fiber bundle, i.e. if the associated polytope is a Cayley polytope.
- In [1] A. Ito proved that $\epsilon(X, \mathscr{L}; x) = 1$ at a very general point if and only if $P_{\mathscr{L}} \cong [P_0 * P_1]^1$.

Example

The following example shows that a direct generalization of Ito's result in [1] is not true.

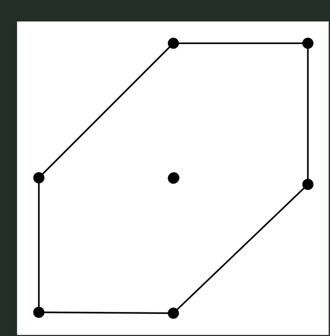

Figure: $(X, \mathscr{L}) = (\mathrm{Bl}_3(\mathbb{P}^2), K_X)$

Here $\epsilon(X, \mathscr{L}; x) = 2$ at the general point and $\epsilon(X, \mathscr{L}, x) = 1$ at any point x in the complement of the torus. But the polytope is not a Cayley sum.

Our Results

Our main results, in [2], generalize the characterizations of Perkinson and Ito.

Theorem 1
Let $(X, \mathscr{L})$ be a smooth polarized toric variety and let $P_{\mathscr{L}}$ be the polytope associated to the complete linear series $|\mathscr{L}|$. Then $\mathscr{L}$ is k-jet spanned, but not $(k+1)$-jet spanned at every point if and only if $P \cong [P_0 * P_1]^k$ for some lower dimensional polytopes P_0 and P_1 and every edge of P contains at least $k+1$ lattice points.

Theorem 2
Let $(X, \mathscr{L})$ be a smooth polarized toric variety and let $P_{\mathscr{L}}$ be the polytope associated to the complete linear series $|\mathscr{L}|$. Then $\epsilon(X, \mathscr{L}; x) = k$ at the fixpoints and at the general point if and only if $P \cong [P_0 * P_1]^k$ for some lower dimensional polytopes P_0 and P_1 and every edge of P contains at least $k+1$ lattice points.

As a corollary of our results we establish the equality between the integers $\epsilon(X, \mathscr{L}; x)$ and $s(\mathscr{L}, x)$ under our assumptions. The exact relationship between these two quantities is in general an open and interesting question.

References

[1] Atsushi Ito. Algebro-geometric characterization of cayley polytopes. *ArXiv e-prints*, pages 1–15, 2010.

[2] Anders Lundman. Local positivity of line bundles on smooth toric varieties and Cayley polytopes. *Preprint.*

[3] David Perkinson. Inflections of Toric Varieties. *Michigan Mathematical Journal*, 48(1):483–515, 2000.

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