Local positivity of line bundles on smooth toric varieties and Cayley polytopes

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Introduction
For a smooth projective variety $X$ and a line bundle $L$ on $X$ there are various notions for measuring the local positivity of $L$ at a point $x \in X$. Two such measures are the dimension of the $k$-osculating space at $x$ and the Seshadri constant at $x$. The precise nature of the interplay between these two notions is in general an open question. Our main result gives a partial answer to this question for smooth toric varieties, by showing that these notions characterize Cayley sums.

Osculating Spaces
Let $L$ be a line bundle on a smooth variety $X$ and $x \in X$ a point with maximal ideal $m_x \subseteq O_x$. Consider the natural map
$$j_k^*: H^0(X, L) \to H^0(X, L \otimes (O_x/m_x^{k+1})).$$
The space $T_k^X(X, L) = \text{P}(\text{im}(j_k^*))$ is called the osculating space of order $k$ at $x \in X$. If $j_k^*$ is onto we say that $L$ is $k$-jet spanned at $x \in X$. The largest $k$ such that $X$ is $k$-jet spanned at $x \in X$ is denoted by $s(L, x)$.

Seshadri Constants
Let $L$ be a nef line bundle on a smooth projective variety $X$. The Seshadri constant of $L$ at a point $x \in X$ is the number
$$\varepsilon(L, x) := \inf_{C \subseteq X} \frac{L \cdot C}{m_x(C)}$$
where the infimum is taken over all irreducible curves $C$ passing through $x$ and $m_x(C)$ is the multiplicity of $C$ at $x$.

Cayley Polytopes
Let $P_0, \ldots, P_r \subseteq \mathbb{R}^k$ be polytopes. We define the Cayley sum
$$[P_0 \times \cdots \times P_r]^s := \text{Conv}\{(P_0 \times 0) \cup (P_1 \times \{0\}) \cup \cdots \cup (P_r \times \{0\})\} \subseteq \mathbb{R}^k \times \mathbb{R}^l$$
where $e_1, \ldots, e_l$ is the standard basis for $\mathbb{R}^l$. A polytope $P \subseteq \mathbb{R}^k$ is called a Cayley polytope of order $s$ and length $r + 1$ if there exist some lower dimensional polytopes $P_0, \ldots, P_r$ such that $P \cong [P_0 \times \cdots \times P_r]^s$. When $P_0, \ldots, P_r$ are normally equivalent then the associated toric variety is a projective fiber bundle $P(L_0 \oplus \cdots \oplus L_r)$.

Examples of Cayley Polytopes

Figure: Three Cayley polytopes in $\mathbb{R}^3$.

Existing Characterizations
Seshadri constants and osculating spaces have been shown to characterize certain polarized smooth toric varieties given by Cayley sums.

- In [3] D. Perkinson showed the following: Let $X$ be a smooth toric variety of dimension $\leq 3$ polarized by a complete linear series $|L|$. Then $s(L, x) = k$ at every point $x \in X$, for any fixed $k \in \mathbb{N}$, if and only if $X$ is a projective fiber bundle, i.e. if the associated polytope is a Cayley polytope.

- In [1] A. Ito proved that $\varepsilon(X, L, x) = 1$ at a very general point if and only if $L \cong [P_0 \times P_1]^1$.

Example
The following example shows that a direct generalization of Ito’s result in [1] is not true.

Figure: $(X, L) = (\text{Bl}_0(P^3), X)$

Here $\varepsilon(X, L, x) = 2$ at the general point and $\varepsilon(X, L, x) = 1$ at any point $x$ in the complement of the torus. But the polytope is not a Cayley sum.

Our Results
Our main results, in [2], generalize the characterizations of Perkinson and Ito.

Theorem 1
Let $(X, L)$ be a smooth polarized toric variety and let $P_L$ be the polytope associated to the complete linear series $|L|$. Then $L$ is $k$-jet spanned, but not $(k+1)$-jet spanned, at every point if and only if $P_L \cong [P_0 \times P_1]^k$ for some lower dimensional polytopes $P_0$ and $P_1$ and every edge of $P$ contains at least $k+1$ lattice points.

Theorem 2
Let $(X, L)$ be a smooth polarized toric variety and let $P_L$ be the polytope associated to the complete linear series $|L|$. Then $\varepsilon(X, L, x) = k$ at the fixpoints and at the general point if and only if $P_L \cong [P_0 \times P_1]^k$ for some lower dimensional polytopes $P_0$ and $P_1$ and every edge of $P$ contains at least $k+1$ lattice points.

As a corollary of our results we establish the equality between the integers $\varepsilon(X, L, x)$ and $s(L, x)$ under our assumptions. The exact relationship between these two quantities is in general an open and interesting question.

References