Higher order Gauss Maps

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Motivation

Higher order Gauss maps describe tangency properties of higher order relating both to higher fundamental forms in differential geometry and local positivity in algebraic geometry.

The classical Gauss map

The classical Gauss map is defined as follows:

Definition

Let $X \subset \mathbb{P}^N$ be an irreducible, non-degenerate projective n-dimensional variety over $\mathbb{C}$. The **Gauss map** *is the rational morphism*

$$X \overset{\gamma}{\dashrightarrow} \mathrm{Gr}(n, N)$$

that assigns to a smooth point $x \in X$ the projective tangent space $\mathbb{T}_{X,x}$ of X at the point x.

Two known facts about the classical Gauss map due to Zak (1993) are

1. If X is smooth then the Gauss map γ is finite and birational, unless X is $\mathbb{P}^n \hookrightarrow \mathbb{P}^n$.
2. The general fiber of γ is a linear space.

Osculating spaces

Let $\mathscr{L}$ be a line bundle on X, $k \in \mathbb{N}$ and $x \in X$ be a point corresponding to a maximal ideal $\mathfrak{m}_x$. Then consider the map

$$j_{k,x} : H^0(X, \mathscr{L}) \to H^0(X, \mathscr{L} \otimes \mathscr{O}_X/\mathfrak{m}_x^{k+1}),$$

which is defined by evaluating a global section s and its derivatives of order at most k at the point x.

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We call the projectivisation of the image of $j_{k,x}$ the k-**th osculating space**, $\mathbb{T}^k_{X,x}$ of X at x.

Osculating spaces II

More explicitly for a choice of coordinates $x_1, x_2, \ldots, x_n$ around x:

$$\begin{aligned} j_{k,x} : H^0(X, \mathscr{L}) \to & H^0(X, \mathscr{L} \otimes \mathscr{O}_X / \mathfrak{m}_x^{k+1}) \\ s \mapsto & \left(s(x), \frac{\partial s}{\partial x_1}(x), \ldots, , \frac{\partial^{t_1 + \ldots + t_n} s}{\partial x_1^{t_1} \partial x_2^{t_2} \cdots \partial x_n^{t_n}}(x), \ldots\right) \end{aligned}$$

for $t = (t_1, \ldots, t_n)$ and $0 \leq t_1 + \ldots + t_n \leq k$.

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The line bundle $\mathscr{L}$ is said to be ***k*-jet spanned at a point** x if the map $j_{k,x}$ is onto at the point $x \in X$. We say that $\mathscr{L}$ is ***k*-jet spanned** if it is k-jet spanned at all points.

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Around a point x, a basis for the global sections of $\mathscr{O}(d)$ on $\mathbb{P}^n$ is given by all monomials of degree at most d in n variables.

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$$\frac{\partial^{t_1+\ldots+t_n}}{\partial x_1^{t_1} \cdots \partial x_n^{t_n}} (x_1^{t_1} \cdots x_n^{t_n}) \Big|_{\mathbf{x}=0} = \prod_{i=1}^{n} t_i!$$

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Hence $j_{k,x}$ is onto if and only if $d \leq k$.

Higher order Gauss maps

The Gauss map of order k is defined as follows

Definition

Let $X \subset \mathbb{P}^N$ be an irreducible projective n-dimensional variety over $\mathbb{C}$, that is k-jet spanned at the general point. The **Gauss map of order** k *is the rational morphism*

$$X \overset{\gamma^k}{\dashrightarrow} \mathrm{Gr}\left(\binom{n+k}{k} - 1, N\right)$$

that assigns to a general point $x \in X$ the k-th osculating space $\mathbb{T}^k_{X,x}$ of X at the point x.

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If X is smooth then the Gauss map is finite and birational, unless X is $\mathbb{P}^n \hookrightarrow \mathbb{P}^n$.

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Theorem (Di Rocco, Jabbusch, L. (2014))

If X is smooth and $\mathscr{L}$ is k-jet spanned, then the Gauss map of order k is finite unless X is the k-th Veronese embedding of $\mathbb{P}^n$.

If X furthermore is toric then the Gauss map of order k is birational.

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> *The fiber of the Gauss map of order 2 need not be a Veronese embedding nor birational when finite. [Franco, Ilardri (2001)], [Piene (1981)]*

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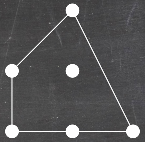
5. If the map given by the global sections of $\det(J_k(\mathscr{L}))$ has a fiber of positive dimension s then $(\det(J_k(\mathscr{L}))^{n-s} \cdot F = 0$, contradicting the ampleness of $\det(J_k(\mathscr{L}))$.

One slide on toric geometry

A finite subset $A = \{\mathbf{u_1}, \ldots, \mathbf{u_N}\} \subseteq \mathbb{Z}^n$ induces the following map

$$\begin{aligned}\phi_A : T_{\mathbb{Z}^n} \cong (\mathbb{C}^*)^n &\to \mathbb{P}^N \\ \mathbf{x} &\mapsto (\mathbf{x}^{\mathbf{u_1}} : \cdots : \mathbf{x}^{\mathbf{u_N}})\end{aligned}$$

$$\begin{aligned}(\mathbb{C}^*)^2 &\to \mathbb{P}^5 \\ (x, y) &\mapsto (1 : x : y : xy : x^2 : xy^2)\end{aligned}$$



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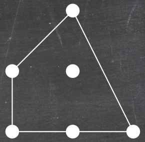


Recall that $\mathrm{Im}(\phi_A)$ is an algebraic torus $T_{\langle A-A \rangle}$, where $A - A = \{u - u' \in M \mid u, u' \in A\}$ and $\overline{\mathrm{Im}(\phi_A)}$ is a toric variety, X_A, which has $T_{\langle A-A \rangle}$ as an open dense subset.

The combinatorial description of toric varities allows for easy description of the image and fiber of the Gauss map of order k. To this end we need to define the following set of lattice points.

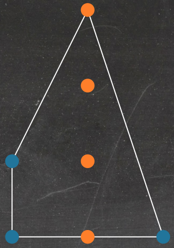
Definition

Let $A = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\} \subset \mathbb{Z}^n$ be a set of lattice points. Then define

$$B_k = \left\{ \mathbf{u}_{i_1} + \mathbf{u}_{i_2} + \ldots + \mathbf{u}_{i_{\binom{n+k}{k}}} \;\middle|\; \begin{array}{c} \{\mathbf{u}_{i_1}, \mathbf{u}_{i_2}, \ldots \mathbf{u}_{i_{\binom{n+k}{k}}}\} \text{ yeilds} \\ \text{a } k\text{-jet spanned embedding.} \end{array} \right\}$$

An example

It is straightforward to see that any 2-jet spanned subset of the lattice points to the right must consist of the three blue and three of the orange points.



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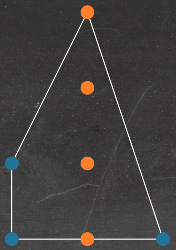

$$(0,0)+(0,1)+(2,0)+(1,0)+(1,1)+(1,2)=(5,4)$$
$$(0,0)+(0,1)+(2,0)+(1,0)+(1,1)+(1,3)=(5,5)$$
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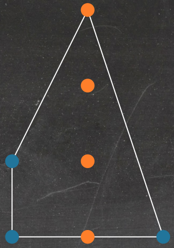

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Thus we have that $B_2 = \{(5,4),(5,5),(5,6),(5,7)\}$ and $\langle B_2 - B_2\rangle = \mathbb{Z}\langle(0,1)\rangle$.

Theorem (Fukuwara, Ito (2014) when $k = 1$),
(Di Rocco, Jabbusch, L. (2014) when $k > 1$)

Let $A \subset \mathbb{Z}^n$, $\pi_k : \mathbb{Z}^n \to \mathbb{Z}^n / (\langle B_k - B_k \rangle_{\mathbb{R}} \cap \mathbb{Z}^n)$ be the natural projection and assume that X_A is generically k-jet spanned. Then the following holds:

1. *The closure $\overline{\gamma^k(X_A)}$ of the Gauss map of order k is projectively equivalent to X_{B_k}.*

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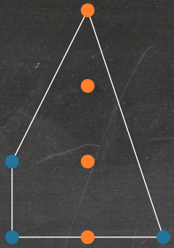
1. *The closure $\overline{\gamma^k(X_A)}$ of the Gauss map of order k is projectively equivalent to X_{B_k}.*
2. *Let F be an irreducible component of the general fiber of γ^k with the reduced structure. Then the closure $\overline{F}$ is projectively equivalent to $X_{\pi_k(A)}$. In particular the dimension of the general fiber is $n - \operatorname{rank}\langle B_k - B_k\rangle$.*

The image of γ^2 in our example



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Thus the image of the Gauss map of order 2 is projectively equivalent to the (closure of the) image of the map

$$\phi_{B_2} : (\mathbb{C}^*)^2 \to \mathbb{P}^3$$
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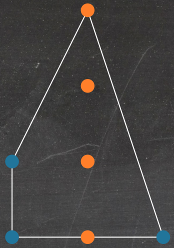

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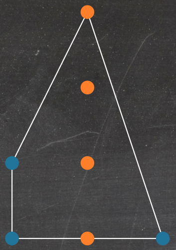
By the toric dictonary the closure of the image of this map corresponds to a rational normal curve of degree 3.

The fiber of γ^2 in our example

Furthermore $\langle B_2 - B_2 \rangle = \mathbb{Z}\langle(0,1)\rangle$.
Hence
$\mathbb{Z}^2/(\langle B_2 - B_2\rangle_{\mathbb{R}} \cap \mathbb{Z}^2) = \mathbb{Z}(1,0)$
and π_2 is the projection onto the first coordinate axis.



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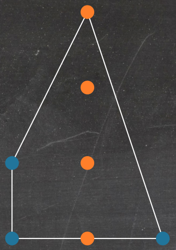

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Hence
$\mathbb{Z}^2 / (\langle B_2 - B_2 \rangle_{\mathbb{R}} \cap \mathbb{Z}^2) = \mathbb{Z}(1,0)$
and π_2 is the projection onto the first coordinate axis. By our main theorem every irreducible component of the fiber is projectively equivalent to the closure of

$$\begin{aligned} \phi_{\pi_2(A)} : \mathbb{C}^* &\to \mathbb{P}^3 \\ x &\mapsto (1, x, x^2). \end{aligned}$$

This is a rational normal curve of degree 2.

The package LatticePolytopes for Macaulay2

Joint work with Gustav Sædén Ståhl (KTH). Running the previous examples in Macaulay2 looks as follows:

```
i19 : M=matrix{{0,0,1,2},{0,1,3,0}};

               2        4
o19 : Matrix ZZ  <--- ZZ

i20 : P=convexHull(M);

i21 : A=latticePoints(P);

i22 : gausskFiber(A,2)

                 2
o22 = {1, x , x }
           0   0

o22 : List

i23 : gausskImage(A,2)

         5 4   5 5   5 6   5 7
o23 = {x x , x x , x x , x x }
         0 1   0 1   0 1   0 1
```

The package is part of the latest version of Macaulay2 (released June 10, 2015).

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