On time-reversibility of linear stochastic models

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Abstract

Reversal of the time direction in stochastic systems driven by white noise has been central throughout the development of stochastic realization theory, filtering and smoothing. Similar ideas were developed in connection with certain problems in the theory of moments, where a duality induced by time reversal was introduced to parametrize solutions. In this latter work it was shown that stochastic systems driven by arbitrary second-order stationary processes can be similarly time-reversed. By combining these two sets of ideas we present herein a generalization of time-reversal in stochastic realization theory.

I. INTRODUCTION

Time reversal of stochastic systems is central in stochastic realization theory (see, e.g., [1], [2], [3], [4], [5], [6], [7], [8]), filtering (see [9]), smoothing (see [10], [11], [12]) and system identification. The principal construction is to model a stochastic process as the output of a linear system driven by a noise process which is assumed to be white in discrete time, and orthogonal-increment in continuous time. In studying the dependence between past and future of the process, it is natural to decompose the interface between past and future in a time-symmetric manner. This gives rise to systems representations of the process running in either time direction, forward or backward in time.

In a different context (see [13]) a certain duality between the two time-directions in modeling a stochastic process was introduced in order to characterize solutions to moment problems. In this new setting the noise-process was general (not necessarily white), and the correspondence between the driving inputs to the two time-opposite models was shown to be captured by suitable dual all-pass dynamics.

In the present note, we combine these two sets of ideas to develop a general framework where two time-opposite stochastic systems model a given stochastic process. We study the relationship between these systems and the corresponding processes. In particular, we recover as a special case certain results of stochastic realization theory ([1], [5], [10]) from the 1970's using a novel procedure.

In Section II we explain how a lifting of state-dynamics into an all-pass system allows direct correspondence between sample-paths of driving generating processes, in opposite time-directions, via causal and anti-causal mappings, respectively. In Section III we utilize this mechanism in the context of general output processes and, similarly, introduce a pair of time-opposite models. Finally, in Section IV, we draw connection to literature on time reversibility and related issues in physics, and we indicate directions for future research.

II. STATE DYNAMICS AND ALL-PASS EXTENSION

In this paper we consider discrete-time as well as continuous-time stochastic linear state-dynamics. As usual, in discrete-time these take the form of a set of difference equations

$$x(t+1) = Ax(t) + Bu(t) \tag{1}$$

where $t \in \mathbb{Z}$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $n, p \in \mathbb{N}$, A has all eigenvalues in the open unit disc $\mathbb{D} = \{z \mid |z| < 1\}$, and $u(t), x(t)$ are stationary vector-valued stochastic processes. The system of equations is assumed to be reachable, i.e.,

$$\text{rank} \left[B, \, AB, \, \ldots A^{n-1}B \right] = n, \tag{2}$$

and non-trivial in the sense that B is full rank.

In continuous-time, state-dynamics take the form of a system of stochastic differential equations

$$dx(t) = Ax(t)dt + Bdu(t) \tag{3}$$

where, here, $u(t), x(t)$ are stationary continuous-time vector-valued stochastic processes. Reachability (which in this case, is equivalent to controllability) of the pair (A, B) is also assumed throughout and the condition for this is identical to the one for discrete-time given above (as is well known). In continuous time, stability of the system of equations is equivalent to A having only eigenvalues with negative real part, and will be assumed throughout along with the condition that B has full rank.

In either case, discrete-time or continuous-time, it is possible to define an output equation so that the overall system is all-pass. This is done next. The assumptions of stationarity and constant parameter matrices is made for simplicity of notation and brevity and can be easily removed.

A. All-pass extension in discrete-time

Consider the discrete-time Lyapunov equation

$$P = APA' + BB'. \tag{4}$$

Since A has all eigenvalues inside the unit disc of the complex plane and (2) holds, (4) has as solution a matrix P which is positive definite. The state transformation

$$\xi = P^{-\frac{1}{2}}x, \tag{5}$$

and

$$F = P^{-\frac{1}{2}}AP^{\frac{1}{2}}, \; G = P^{-\frac{1}{2}}B, \tag{6}$$

brings (1) into

$$\xi(t+1) = F\xi(t) + Gu(t). \tag{7}$$

For this new system, the corresponding Lyapunov equation $X = FXF' + GG'$ has I_n as solution, where I_n denotes the $(n \times n)$ identity matrix. This fact, namely, that

$$I_n = FF' + GG' \tag{8}$$

implies that this $[F, G]$ can be embedded as part of an orthogonal matrix

$$U = \left[\begin{array}{cc} F & G \\ H & J \end{array} \right], \tag{9}$$

i.e., such that $UU' = U'U = I_{n+p}$.

Define the transfer function

$$\mathbf{U}(z) := H(zI_n - F)^{-1}G + J \tag{10}$$

corresponding to

$$\xi(t+1) = F\xi(t) + Gu(t) \tag{11a}$$
$$\bar{u}(t) = H\xi(t) + Ju(t). \tag{11b}$$

This is also the transfer function of

$$x(t+1) = Ax(t) + Bu(t) \tag{12a}$$
$$\bar{u}(t) = \bar{B}'x(t) + Ju(t), \tag{12b}$$

where $\bar{B} := P^{-\frac{1}{2}}H'$, since the two systems are related by a similarity transformation. Hence,

$$\mathbf{U}(z) = \bar{B}'(zI_n - A)^{-1}B + J. \tag{13}$$

We claim that $\mathbf{U}(z)$ is an all-pass transfer function (with respect to the unit disc), i.e., that $\mathbf{U}(z)$ is a transfer function of a stable system (obvious) and that

$$\mathbf{U}(z)\mathbf{U}(z^{-1})' = \mathbf{U}(z^{-1})'\mathbf{U}(z) = I_p. \tag{14}$$

The latter claim is immediate after we observe that, since $U'U = I_{n+p}$,

$$U' \left[\begin{array}{c} \xi(t+1) \\ \bar{u}(t) \end{array} \right] = \left[\begin{array}{c} \xi(t) \\ u(t) \end{array} \right],$$

and hence,

$$\xi(t) = F'\xi(t+1) + H'\bar{u}(t) \tag{15a}$$
$$u(t) = G'\xi(t+1) + J'\bar{u}(t) \tag{15b}$$

or, equivalently,

$$x(t) = PA'P^{-1}x(t+1) + P^{\frac{1}{2}}H'u(t) \tag{16a}$$
$$u(t) = B'P^{-1}x(t+1) + J'\bar{u}(t). \tag{16b}$$

Setting

$$\bar{x}(t) := P^{-1}x(t+1), \tag{17}$$

(16) can be written

$$\bar{x}(t-1) = A'\bar{x}(t) + \bar{B}\bar{u}(t) \tag{18a}$$
$$u(t) = B'\bar{x}(t) + J'\bar{u}(t) \tag{18b}$$

with transfer function

$$\mathbf{U}(z)^* = B'(z^{-1}I_n - A')^{-1}\bar{B} + J'. \tag{19}$$

Either of the above systems inverts the dynamical relation $u \to \bar{u}$ (in (12) or (11)).

Fig. 1.   Realization (12) in the forward time-direction.

Fig. 2. Realization (18) in the backward time-direction.

B. All-pass extension in continuous-time

Consider the continuous-time Lyapunov equation

$$AP + PA' + BB' = 0. \tag{20}$$

Since A has all its eigenvalues in the left half of the complex plane and since (2) holds, (20) has as solution a positive definite matrix P. Once again, applying (5-6), the system in (3) becomes

$$d\xi(t) = F\xi(t)dt + Gdu(t). \tag{21a}$$

We now seek a completion by adding an output equation

$$d\bar{u}(t) = H\xi(t)dt + Jdu(t) \tag{21b}$$

so that the transfer function

$$\mathbf{U}(s) := H(sI_n - F)^{-1}G + J \tag{22}$$

is all-pass (with respect to the imaginary axis), i.e.,

$$\mathbf{U}(s)\mathbf{U}(-s)' = \mathbf{U}(-s)'\mathbf{U}(s) = I_p. \tag{23}$$

For this new system, the corresponding Lyapunov equation has as solution the identity matrix and hence,

$$F + F' + GG' = 0. \tag{24}$$

Utilizing this relationship we note that

$$
\begin{aligned}
&(sI_n - F)^{-1}GG'(-sI_n - F')^{-1} \\
&= (sI_n - F)^{-1}(sI_n - F - sI_n - F')(-sI_n - F')^{-1} \\
&= (sI_n - F)^{-1} + (-sI_n - F')^{-1},
\end{aligned}
$$

and we calculate that

$$
\begin{aligned}
&\mathbf{U}(s)\mathbf{U}(-s)' \\
&= (H(sI_n - F)^{-1}G + J)(G'(-sI_n - F')^{-1}H' + J') \\
&= JJ' + H(sI_n - F)^{-1}(GJ' + H') \\
&\quad (JG' + H)(-sI_n - F')^{-1}H'.
\end{aligned}
$$

For the product to equal the identity,

$$
\begin{aligned}
JJ' &= I_p \\
H &= -JG'.
\end{aligned}
$$

Thus, we may take

$$
\begin{aligned}
J &= I_p \\
H &= -G',
\end{aligned}
$$

and the forward dynamics

$$d\xi(t) = F\xi(t)dt + Gdu(t) \tag{25a}$$
$$d\bar{u}(t) = -G'\xi(t)dt + du(t). \tag{25b}$$

Substituting $F = -F' - GG'$ from (24) into (25a) we obtain the reverse-time dynamics

$$d\xi(t) = -F'\xi(t)dt + Gd\bar{u}(t) \tag{26a}$$
$$du(t) = G'\xi(t)dt + d\bar{u}(t). \tag{26b}$$

Now defining

$$\bar{x}(t) := P^{-1}x(t) \tag{27}$$

and using (5) and (6), (26) becomes

$$d\bar{x}(t) = -A'\bar{x}(t)dt + \bar{B}d\bar{u}(t) \tag{28a}$$
$$du(t) = B'\bar{x}(t)dt + d\bar{u}(t), \tag{28b}$$

with transfer function

$$\mathbf{U}(s)^* = B'(sI_n + A')^{-1}\bar{B} + I_p, \tag{29}$$

where

$$\bar{B} := P^{-1}B. \tag{30}$$

Furthermore, the forward dynamics (25) can be expressed in the form

$$dx(t) = Ax(t)dt + Bdu(t) \tag{31a}$$
$$d\bar{u}(t) = \bar{B}'x(t)dt + du(t) \tag{31b}$$

with transfer function

$$\mathbf{U}(s) = \bar{B}'(sI_n - A')^{-1}B + I_p. \tag{32}$$

III. TIME-REVERSAL OF LINEAR STOCHASTIC SYSTEMS

The development so far allows us to draw a connection between two linear stochastic systems having the same output and driven by a pair of arbitrary, but dual, stationary processes $u(t)$ and $\bar{u}(t)$, one evolving forward in time and one evolving backward in time. When one of these two processes is white noise (or, orthogonal increment process, in continuous-time), then so is the other. For this special case we recover results of [1] and [5] in stochastic realization theory.

A. Time-reversal of discrete-time stochastic systems

Consider a stochastic linear system

$$x(t + 1) = Ax(t) + Bu(t) \tag{33a}$$
$$y(t) = Cx(t) + Du(t) \tag{33b}$$

with an m-dimensional output process y, and x, u, A, B are defined as in Section II-A. All processes are stationary and the system can be thought as evolving forward in time from the remote past ($t = -\infty$). In particular,

$$\begin{pmatrix} x(t+1) \\ y(t) \end{pmatrix} \text{ is } \mathcal{F}_t^u\text{-measurable}$$

for all $t \in \mathbb{Z}$, where $\mathcal{F}_t^u$ is the σ-algebra generated by $\{u(s) \mid s \leq t\}$. Next we construct a stochastic system

$$\bar{x}(t-1) = A'\bar{x}(t) + \bar{B}\bar{u}(t) \tag{34a}$$
$$y(t) = \bar{C}\bar{x}(t) + \bar{D}\bar{u}(t), \tag{34b}$$

which evolves backward in time from the remote future $(t = \infty)$, and for which

$$\begin{pmatrix} \bar{x}(t-1) \\ y(t) \end{pmatrix} \text{ is } \bar{\mathcal{F}}_t^{\bar{u}}\text{-measurable}$$

for all $t \in \mathbb{Z}$, where $\bar{\mathcal{F}}_t^{\bar{u}}$ is the σ-algebra generated by $\{\bar{u}(s) \mid s \geq t\}$. The processes $\bar{x}, x, \bar{u}, u$ relate as in the previous section. More specifically, as shown in Section II-A,

$$\bar{u}(t) \text{ is } \mathcal{F}_t^u\text{-measurable}$$

while

$$u(t) \text{ is } \bar{\mathcal{F}}_t^{\bar{u}}\text{-measurable}$$

for all t, as examplified in Figures 1 and 2.

In fact, the all-pass extension (12) of (33a) yields

$$\bar{u}(t) = \bar{B}'x(t) + Ju(t) \tag{35}$$

It follows from (18b) that (35) can be inverted to yield

$$u(t) = B'\bar{x}(t) + J'\bar{u}(t), \tag{36}$$

where $\bar{x}(t) = P^{-1}x(t+1)$, and that we have the reverse-time recursion

$$\bar{x}(t-1) = A'\bar{x}(t) + \bar{B}\bar{u}(t). \tag{37a}$$

Then inserting (36) and

$$x(t) = P\bar{x}(t-1) = PA'\bar{x}(t) + P\bar{B}\bar{u}(t)$$

into (33b), we obtain

$$y(t) = \bar{C}\bar{x}(t) + \bar{D}\bar{u}(t), \tag{37b}$$

where $\bar{D} := DJ'$ and

$$\bar{C} := CPA' + DB'. \tag{38}$$

Then, (37) is precisely what we wanted to establish.

Moreover, the transfer functions

$$\mathbf{W}(z) = C(zI_n - A)^{-1}B + D \tag{39}$$

of (33) and

$$\bar{\mathbf{W}}(z) = \bar{C}(z^{-1}I_n - A')^{-1}\bar{B} + \bar{D} \tag{40}$$

of (34) satisfy

$$\mathbf{W}(z) = \bar{\mathbf{W}}(z)\mathbf{U}(z). \tag{41}$$

In the context of stochastic realization theory, discussed next, $\mathbf{U}(z)$ is called *structural function* ([3], [4]).

Fig. 3.   The forward stochastic system (33).

Fig. 4.   The backward stochastic system (34)

1) Time-reversal of stochastic realizations.: Given an m-dimensional stationary process y, consider a minimal stochastic realization (33), evolving forward in time, where now u is a normalized white noise process, i.e.,

$$\mathbb{E}\{u(t)u(s)'\} = I_p\delta_{t-s}.$$

Since **U**, given by (13), is all-pass, $\bar{u}$ is also a normalized white noise process, i.e.,

$$\mathbb{E}\{\bar{u}(t)\bar{u}(s)'\} = I_p\delta_{t-s}.$$

From the reverse-time recursion (34a)

$$\bar{x}(t) = \sum_{k=t+1}^{\infty} (A')^{k-(t+1)}\bar{B}\bar{u}(k).$$

Since, $\bar{u}$ is a white noise process, $\mathbb{E}\{\bar{x}(t)\bar{u}(s)'\} = 0$ for all $s \leq t$. Consequently, (34) is a backward stochastic realization in the sense of stochastic realization theory.

B. Time-reversal of continuous-time stochastic systems

We now turn to the continuous-time case. Let

$$dx = Axdt + Bdu \tag{42a}$$
$$dy = Cxdt + Ddu \tag{42b}$$

be a stochastic system with x, u, A, B as in Section II-B, evolving forward in time from the remote past ($t = -\infty$). All processes have stationary increments and

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \text{ is } \mathcal{F}_t^u\text{-measurable}$$

for all $t \in \mathbb{R}$, where $\mathcal{F}_t^u$ is the σ-algebra generated by $\{u(s) \mid s \leq t\}$.

The all-pass extension of Section II-B yields

$$d\bar{u} = du - \bar{B}'xdt \tag{43}$$

as well as the reverse-time relation

$$d\bar{x} = -A'\bar{x}dt + \bar{B}d\bar{u} \tag{44a}$$
$$du = B'\bar{x}dt + d\bar{u}, \tag{44b}$$

where $\bar{x}(t) = P^{-1}x(t)$. Inserting (44b) into

$$dy = CP\bar{x}dt + Ddu$$

yields

$$dy = \bar{C}\bar{x}dt + Dd\bar{u},$$

where

$$\bar{C} = CP + DB'. \tag{45}$$

Thus, the reverse-time system is

$$d\bar{x} = -A'\bar{x}dt + \bar{B}d\bar{u} \tag{46a}$$
$$dy = \bar{C}\bar{x}dt + Dd\bar{u}. \tag{46b}$$

From this, we deduce that

$$\begin{pmatrix} \bar{x}(t) \\ y(t) \end{pmatrix} \text{ is } \bar{\mathcal{F}}_t^{\bar{u}}\text{-measurable}$$

for all $t \in \mathbb{R}$. We also note that the transfer function

$$\mathbf{W}(s) = C(sI_n - A)^{-1}B + D$$

of (42) and the transfer function

$$\bar{\mathbf{W}}(s) = \bar{C}(sI_n + A')^{-1}\bar{B} + D$$

of (46) also satisfy

$$\mathbf{W}(s) = \bar{\mathbf{W}}(s)\mathbf{U}(s)$$

as in discrete-time.

1) Time-reversal of stochastic realizations.: In continuous-time stochastic realization theory, (42) is a forward minimal stochastic realization of an m-dimensional process y with stationary increments provided u is a normalized orthogonal-increment process satisfying

$$\mathbb{E}\{du(t)du(t)'\} = I_p dt.$$

Since $\mathbf{U}(s)$ is all-pass,

$$d\bar{u} = du - \bar{B}'xdt \tag{47}$$

also defines a stationary orthogonal-increment process $\bar{u}$ such that

$$\mathbb{E}\{d\bar{u}(t)d\bar{u}(t)'\} = I_p dt.$$

It remains to show that (46) is a backward stochastic realization, that is, at each time t the past increments of $\bar{u}$ are orthogonal to $\bar{x}(t)$. But this follows from the fact that

$$\bar{x}(t) = \int_t^{\infty} e^{-A'(t-s)}\bar{B}d\bar{u}(s)$$

and $\bar{u}$ has orthogonal increments.

IV. CONCLUDING REMARKS

The direction of time in physical laws and the fact that physical laws are symmetric with respect to time have occupied some of the most prominent minds in science and mathematics ([14], [15], [16]). These early consideration were motivated by no less an issue than that of the very nature of the quantum. Indeed, Erwin Schrödinger's aim appears to have been to draw a classical analog to his famous equation. A large body of work followed.

In particular, closer to our immediate interests, dual time-reversed models have been employed to model, in different time-directions, Brownian or Schrödinger bridges (see [17], [18]), a subject which is related to reciprocal processes ([19], [20], [21], [22]). The topic of time reversibility has also been central to thermodynamics, and in recent years studies have sought to elucidate its relation to systems theory (see [23], [24]). Possible connections between this body of work and our present paper will be the subject of future work.

The thesis of the present work is that under mild assumptions on a stochastic process, any model that consists of a linear stable dynamical system driven by an appropriate input process can be reversed in time. In fact, a reverse-time dual system along with the corresponding input process can be obtained via an all-pass extension of the state equation. The correspondence between the two input processes can be expressed in terms of each other by a causal and an anti-causal map, respectively.

The formalism of our paper can easily be extended to a non-stationary setting at a price of increased notational, but not conceptual, complexity. Informally, and in order to underscore the point, if $u(t)$ is a non-stationary process and the linear system is time-varying, under suitable conditions, a reverse-time system and a process $\bar{u}(t)$ can be similarly constructed via a time-varying orthogonal transformation.

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