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The Inverse Problem of Degree-Constrained Analytic Interpolation and Weight Selection for Control Synthesis

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Dedicated to Yutaka Yamamoto
at the occasion of his 60^{th} birthday

A man in his best age

References

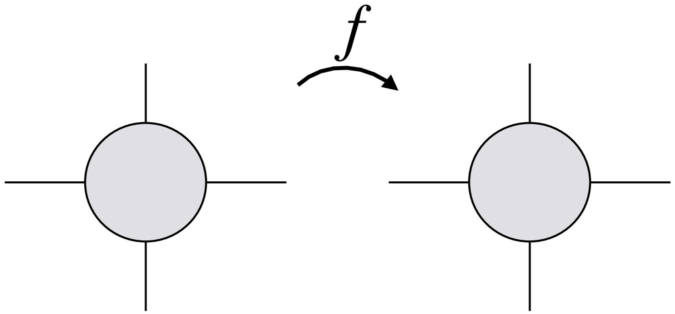
J. Karlsson, T. T. Georgiou and A. Lindquist, The inverse problem of analytic interpolation with degree constraint and weight selection for control synthesis, *IEEE Trans. Autom. Control* **AC-55** (2010), 405–418.

C. I. Byrnes, T.T. Georgiou, A. Lindquist and A. Megretski, Generalized interpolation in H^∞ with a complexity constraint, *Transactions of the American Mathematical Society* **358**(3) (2006), 965–987.

The Pick problem

Schur class:

$$\mathcal{S} = \{f \in H_\infty(\mathbb{D}) \,:\, \|f\|_\infty \leq 1\}$$



Given $z_0, z_1, \ldots, z_n \in \mathbb{D}$ and $w_0, w_1, \ldots, w_n$, find $f \in \mathcal{S}$ such that

$$f(z_k) = w_k, \quad k = 0, 1, \ldots, n$$

There exists a solution if and only if

$$P = \left[\frac{1 - w_k \bar{w}_\ell}{1 - z_k \bar{z}_\ell}\right]_{k,\ell=0}^{n} \geq 0$$ Pick matrix

Assume $P > 0$

Then infinitely many solutions

The solution is unique if and only if P is singular. Then f Blaschke product such that $\deg f = \operatorname{rank} P$.

The Nevanlinna parameterization

$$f(z) = \frac{\varphi_1(z) - \varphi_2(z)g(z)}{\varphi_3(z) - \varphi_4(z)g(z)}$$

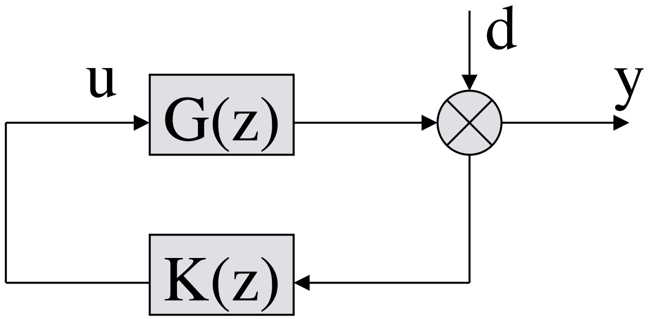
$$\varphi_j, \quad j = 1, 2, 3, 4 \quad \text{rational}$$

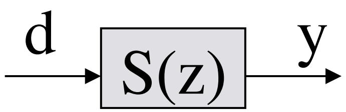
$g \in \mathcal{S}$ arbitrary parameter

$\boxed{\text{Central solution:} \quad g = 0}$ $\Rightarrow$ $\deg f = n$

This parameterization does not accommodate a simple characterization of the subfamily of solutions for which $\deg f \leq n$.

Loop shaping in robust control





$$S = (1 - GK)^{-1}$$

Sensitivity function

- Internal stability requires

  S analytic in $\mathbb{D}^c := \{z \mid |z| > 1\}$

  $S(z_k) = 0$ at all unstable poles of G

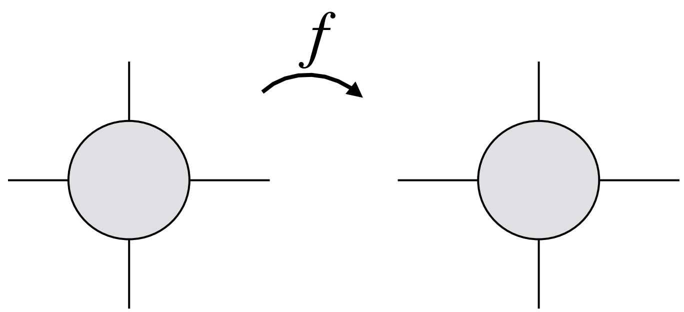
  $S(z_j) = 1$ at all zeros of G in $\mathbb{D}^c$

- Disturbance attenuation requires

  $\|S\|_\infty \leq \gamma$

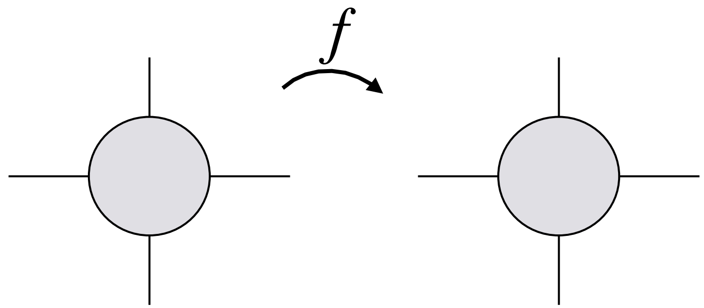
- We want $\deg S$ to be small

There is a minimum bound γ_{opt} but we choose $\gamma > \gamma_{\text{opt}}$ and define $f(z) := \frac{1}{\gamma} S(z^{-1})$



Nevanlinna-Pick interpolation
for Schur functions

$$\boxed{f(z_k) = w_k, \quad k = 0, 1, \ldots, n} \quad (\dagger)$$



class of Schur functions $\mathcal{S}$

Instead choosing the optimal bound
$\gamma = \gamma_{\text{opt}}$ would yield the unique solution
for which P singular and f Blaschke

⇒ Modulus of sensitivity constant over the spectrum

Need to use a weight, yielding a higher-degree solution (Zames 1981).

Suboptimal case: Central solution is
the unique solution of $(\dagger)$ that maximizes

$$\int_{-\pi}^{\pi} \log(1 - |f|^2) d\theta$$

Maximum entropy solution
(Mustafa–Glover)

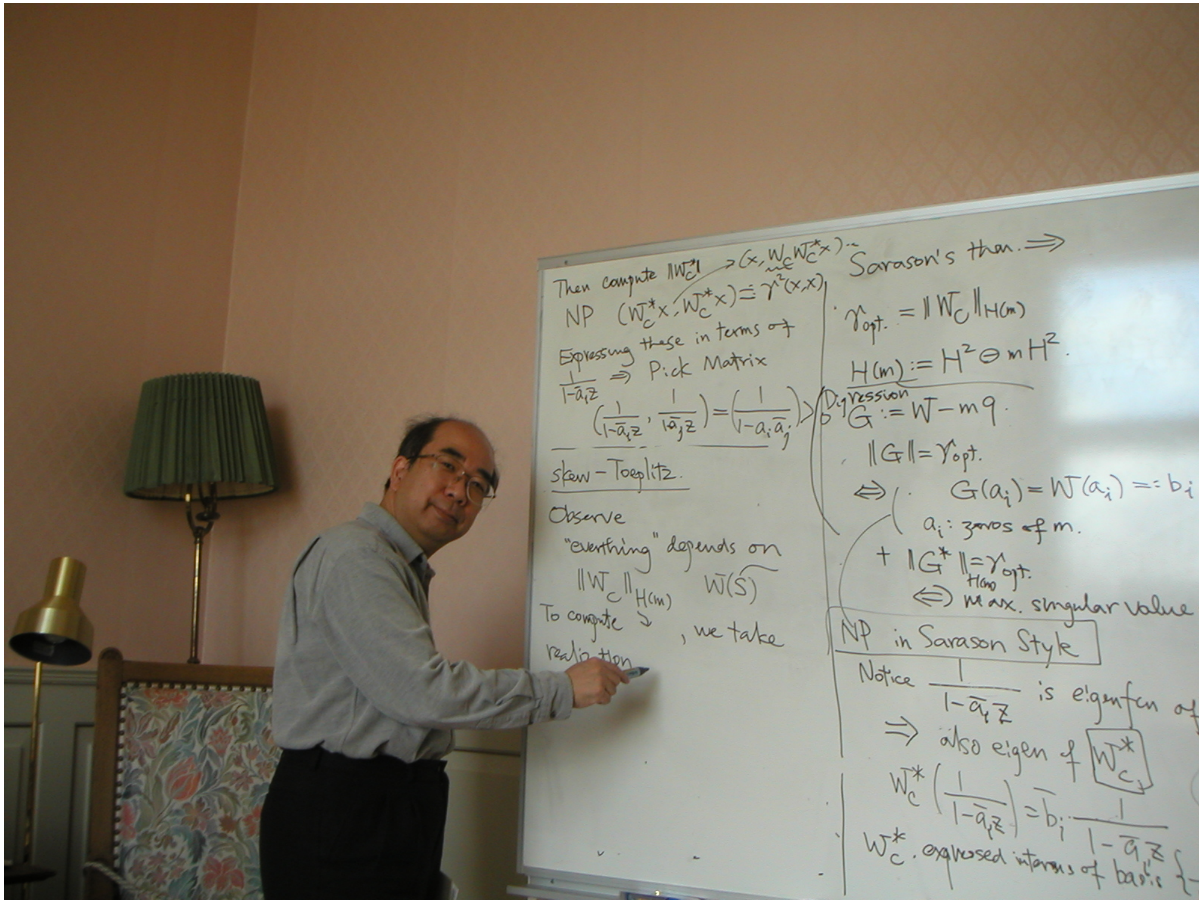
Still uniform over the spectrum.

Parametrizing solutions of degree ≤ n

$$\mathcal{K} = \left\{ \frac{p(z)}{\prod_{k=0}^{n}(1 - \bar{z}_k z)} \mid p \text{ polynomial of degree} \leq n \right\}$$

$$\mathcal{K}_0 = \{\sigma \in \mathcal{K} \mid \sigma \text{ outer (min. phase)}\}$$

The connections to Sarason interpolation is on the board



Parametrizing solutions of degree ≤ n

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THEOREM (Byrnes-Georgiou-L-Megretski) If $f \in \mathcal{S}$ is an interpolant such that $\deg f \leq n$, then there is a $\sigma \in \mathcal{K}_0$ such that f maximizes

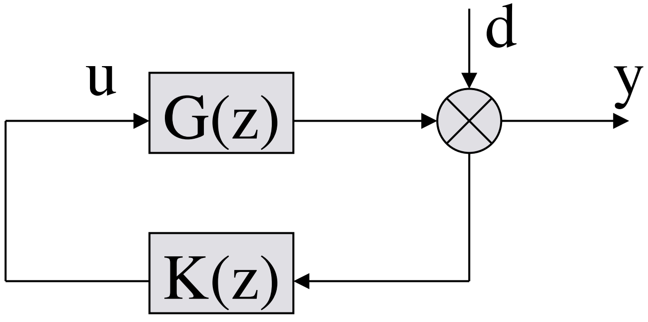
$$\mathbb{K}_\sigma(f) := \int_{-\pi}^{\pi} |\sigma(e^{i\theta})|^2 \log(1 - |f(e^{i\theta})|^2) d\theta \quad (\ddagger)$$

(uniquely) subject to the interpolation constraint. Conversely, if the interpolant $f \in \mathcal{S}$ maximizes $(\ddagger)$ for some $\sigma \in \mathcal{K}_0$, then $\deg f \leq n$.

QUESTION: How do we choose the parameter $\sigma \in \mathcal{K}_0$ to satisfy additional design specifications?

An example

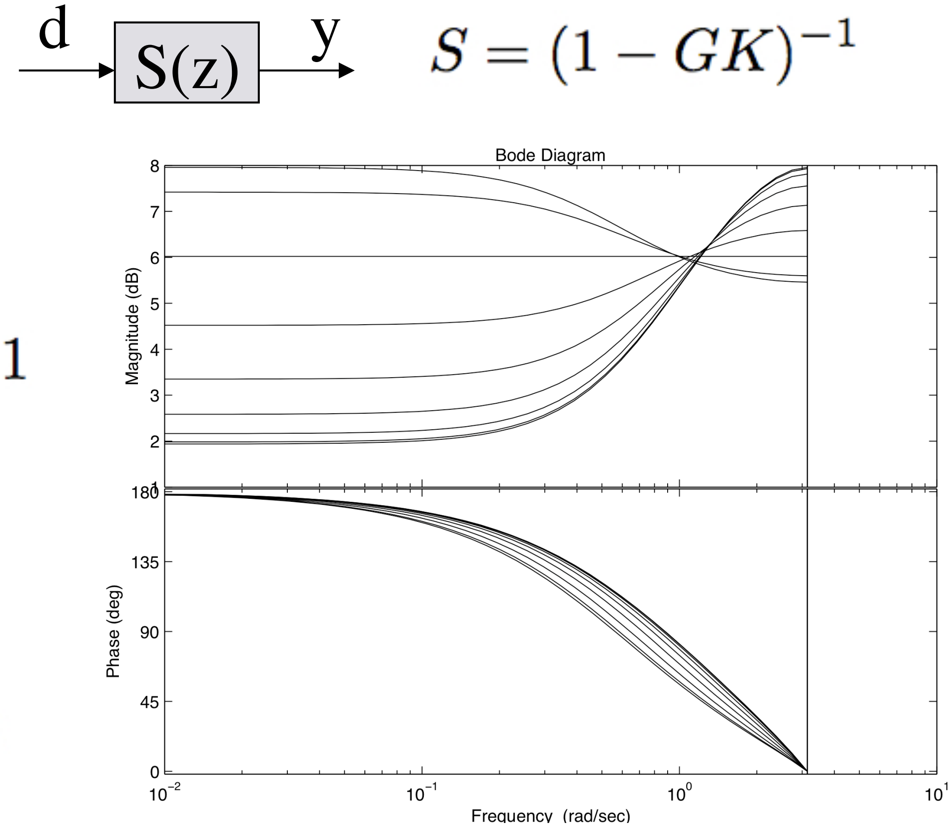
$$G(z) = \frac{1}{z-2}$$



Find all S satisfying

$S(2) = 0$ and $S(\infty) = 1$

of degree at most $n = 1$.



$$S = (1 - GK)^{-1}$$

$$S(z) = \frac{z-2}{z-a}, \quad -1 < a < 1$$

A one-parameter family with one solution for each $\sigma \in \mathcal{K}_0$.

$$\gamma = 2.5 > \gamma_{\text{opt}} = \min \|S\|_\infty = 2$$

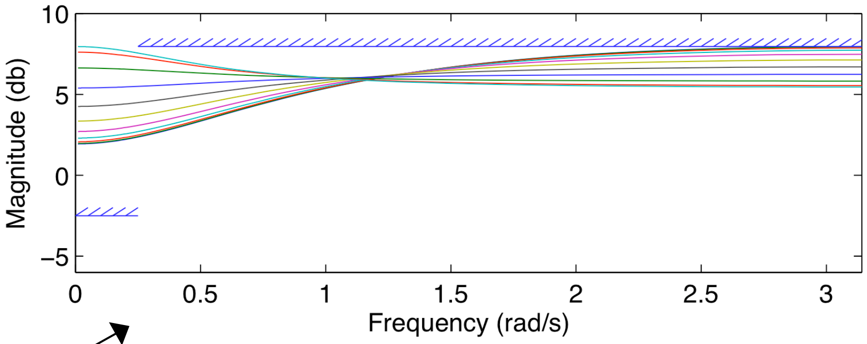
We need a procedure for determining the best $\sigma \in \mathcal{K}_0$.

What if none of the solutions satisfy the specifications?

Design specifications:

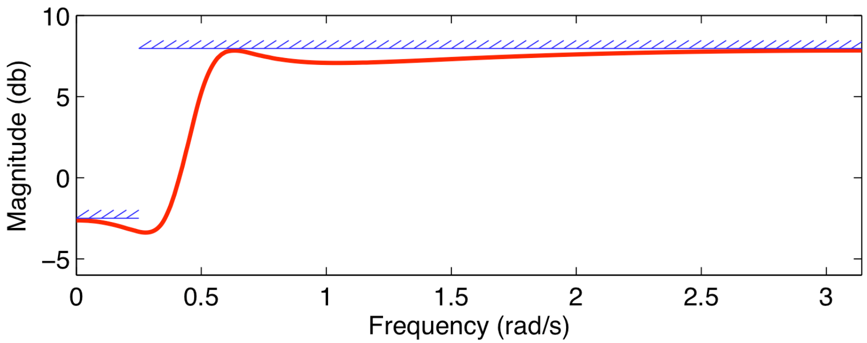
$$|S(e^{i\theta})| \leq 0.75, \qquad \text{on } (0, 0.25)$$

$$|S(e^{i\theta})| \leq 2.5, \qquad \text{on } (0.25, \pi)$$



None of these solutions satisfies the design specifications

Extend the space $\mathcal{K}_0$ to
$\mathcal{K}_m = \{\sigma = \sigma_0 \rho \mid \sigma_0 \in \mathcal{K}_0, \rho \in \mathcal{R}_m\}$,
where $\mathcal{R}_m$ is the set of outer rational functions of degree $\leq m$.



Extended parameterization

$$\mathbb{K}_\sigma(f) := \int_{-\pi}^{\pi} |\sigma(e^{i\theta})|^2 \log(1 - |f(e^{i\theta})|^2) d\theta \to \max$$

$$\text{subject to} \quad f(z_k) = w_k, \;\; k = 0, 1, \ldots, n \qquad \text{(P)}$$

THEOREM (Karlsson-Georgiou-L). Suppose $|\sigma|^2 \in L_1(\mathbb{T})$. A function f is a solution to the optimization problem (P) if and only the following three conditions hold:

(i) $f(z_k) = w_k$ for $k = 0, 1, \ldots, n$,

(ii) $f = \frac{b}{a} \in \mathcal{S}$ where $b \in \mathcal{K}$ and a is outer,

(iii) $|\sigma|^2 = |a|^2 - |b|^2$.

Any such solution is necessarily unique.

The map $\sigma \mapsto f$

- The optimization problem (P) defines a map

$$F : \Sigma \to \mathcal{S}, \qquad \sigma \mapsto f$$

where $\Sigma := \{\sigma \text{ outer} \mid \log|\sigma|^2 \in L_1(\mathbb{T})\}$.

- Define the metric

$$d(\sigma_1, \sigma_2) = \left\| \log|\sigma_1|^2 - |\log|\sigma_2|^2 \right\|_\infty$$

PROPOSITION (KGL). Suppose that $\sigma_1, \sigma_2 \in \Sigma$ are such that $d(\sigma_1, \sigma_2) = \varepsilon$, and set $f_k := F(\sigma_k)$, $k = 1, 2$. Then

$$\|\sigma_1(f_1 - f_2)\|_2^2 \le 2(e^{2\varepsilon} - 1)\mathbb{K}_{\sigma_1}(f_1).$$

The inverse problem $\boxed{F^{-1}(f) = ?}$

$$\mathbb{K}_\sigma(f) := \int_{-\pi}^{\pi} |\sigma(e^{i\theta})|^2 \log(1 - |f(e^{i\theta})|^2) d\theta \to \max$$

$$\text{subject to} \quad f(z_k) = w_k, \;\; k = 0, 1, \dots, n \qquad \text{(P)}$$

PROPOSITION (KGL). Any function $f \in \mathcal{S}$ that satisfies

(i) $f(z_k) = w_k$ for $k = 0, 1, \dots, n$,

(ii) f has at most n zeros in $\mathbb{D}$,

(iii) $\log(1 - |f|^2) \in L_1(\mathbb{T})$,

is the unique solution of (P), i.e. $f = F(\sigma)$, with

$$|\sigma|^2 = (|f|^{-2} - 1)|b|^2$$

for any $b \in \mathcal{K}$ chosen so that bf^{-1} is outer.

Shaping the interpolants

Let g be any outer function in $\mathcal{S}$. Find an interpolant f such that

$$(\dagger) \quad |f(e^{i\theta})| = |g(e^{i\theta})|, \quad \theta \in (-\pi, \pi) \quad f \text{ has the same shape as } g$$

PROPOSITION (KGL). Let $g \in \mathcal{S}$ be an outer function such that $\log(1 - |g|^2) \in L_1(\mathbb{T})$. Then there exists a pair (f, σ) such that $(\dagger)$ holds, $f = F(\sigma)$ and $\log|\sigma|^2 \in L_1(\mathbb{T})$ if and only if

$$\Pi(g) := \left[\frac{1 - w_k g(z_k)^{-1} \overline{w_\ell g(z_\ell)^{-1}}}{1 - z_k \overline{z_\ell}} \right]_{k,\ell=0}^{n}$$

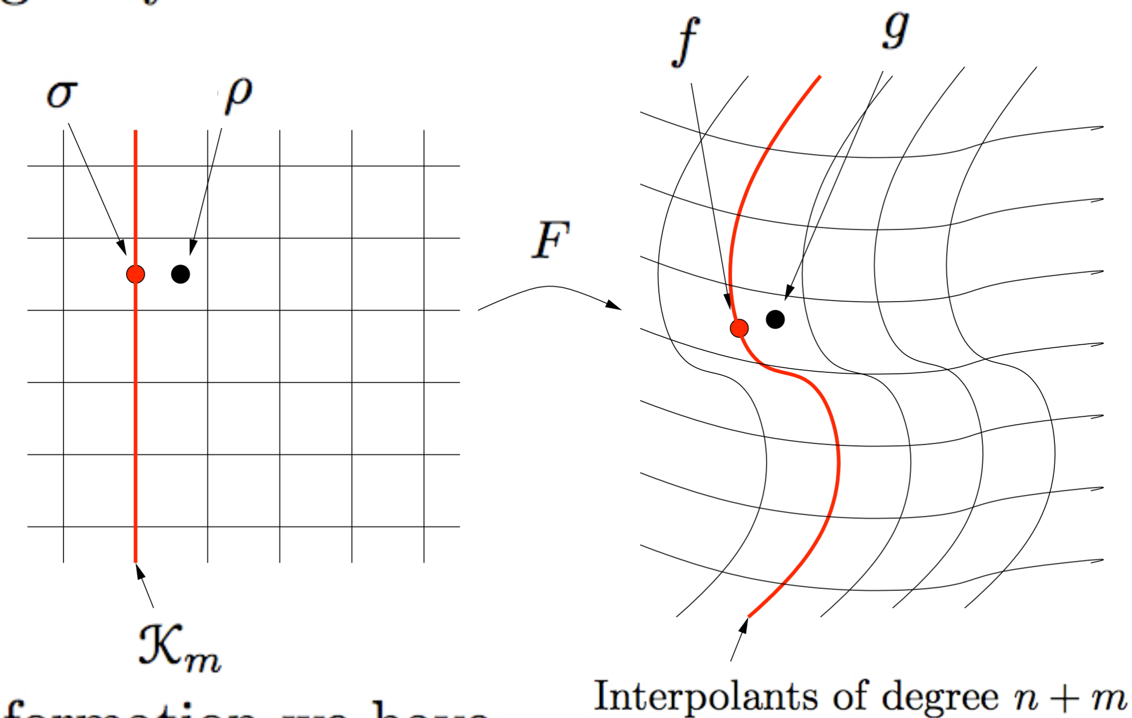
is positive semi-definite and singular. Moreover, f is uniquely determined.

Model reduction

- Modify g to be an interpolant without changing the shape $|g|$ by multiplying it by an inner factor.

- Let $\rho \in F^{-1}(g)$ and find a $\sigma \in \mathcal{K}_m$ that is close to ρ in the sense that $d(\sigma, \rho)$ is small.

- Then $f := F(\sigma)$ is close to g.



By this nonlinear transformation we have exchanged a hard non-convex problem for an easier one.

Approximation procedure

Step 1. Find an interpolant g with the desired shape without restricting the degree. (It could even be non-rational.)

Step 2. For some $m \geq 0$, find a pair (ρ, σ) with $\rho \in F^{-1}(g)$ and $\sigma \in \mathcal{K}_m$ that minimizes

$$d(\sigma, \rho) = \left\| \log |\sigma|^2 - |\log |\rho|^2 \right\|_\infty$$

This is a standard quasi-convex optimization problem. In fact, $d(\sigma, \rho) \leq \varepsilon$ if and only if

$$1 - e^\varepsilon \leq 1 - \frac{|\sigma|^2}{|\rho|^2} \leq 1 - e^{-\varepsilon} \text{ for all } z \in \mathbb{T},$$

which defines an infinite set of linear constraints.

Step 3. Find $f := F(\sigma)$ by solving the optimization problem (P).

Shaping by model reduction

$$\boxed{F:\ \Sigma \to \mathcal{S} \text{ sends } \sigma \text{ to } f\text{, the unique interpolant maximizing } \mathbb{K}_\sigma}$$

interpolant of high or infinite degree but with $|g|$ of desired shape

difficult approximation problem

interpolant of degree at most $n+m$ with a shape close to that of g

$$\begin{tikzcd}
g \arrow[r, "F^{-1}"] \arrow[d] & \rho \in \Sigma \arrow[d] \\
f & \sigma \in \mathcal{K}_m \arrow[l, "F"']
\end{tikzcd}$$

approximation by quasi-convex optimization

$$\sigma \in \mathcal{K}_m \Longrightarrow \deg f \leq n+m$$

Example $G(z) = \dfrac{1}{z-2}$

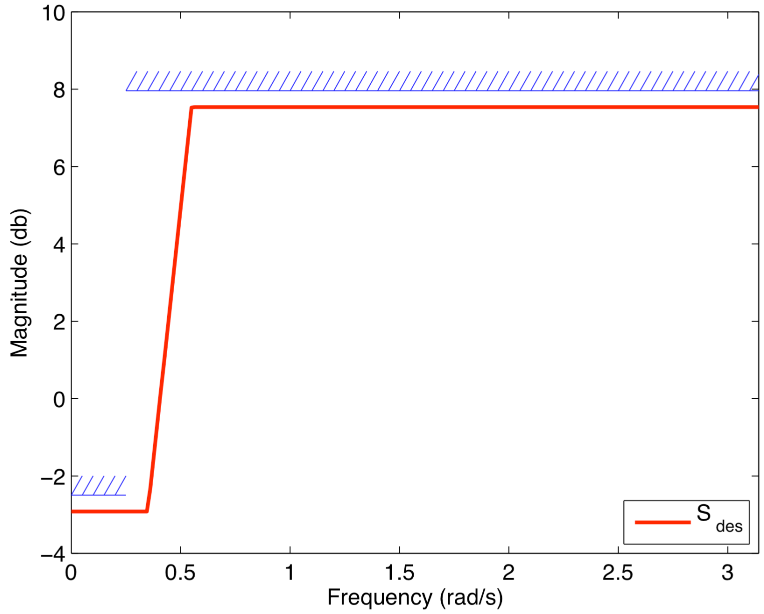
For internal stability:

$f = S/\gamma$ satisfies

$f(0) = 0.4$ and $f(0.5) = 0$

Find interpolant g with

$|g|$ as in figure (not rational)



$\rho \in F^{-1}(g)$ ➡ $|\rho|^2 = \left(|g|^{-2} - 1\right)$

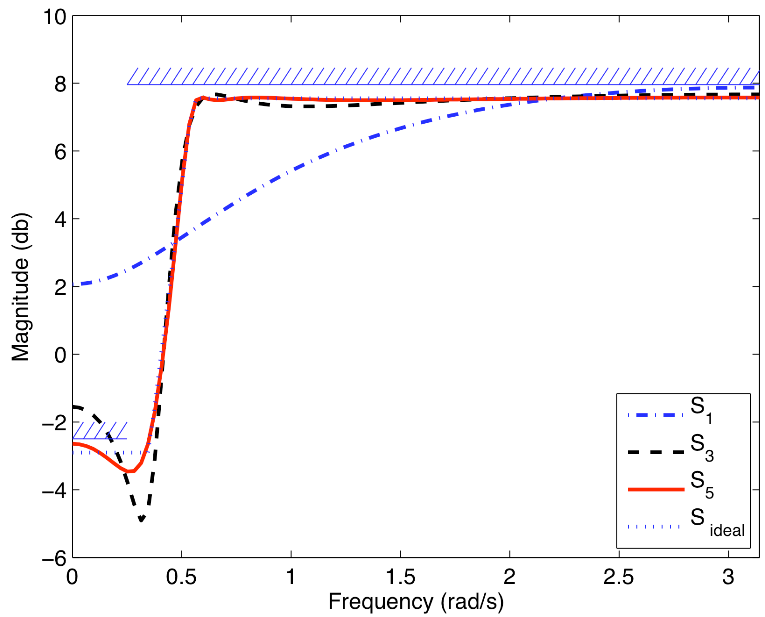
Use quasi-convex optimization

to find $\sigma \in \mathcal{K}_m$ (m=0, 2 and 4)

such that σ is close to ρ.

Sensitivity functions

of degrees 1, 3 and 5



Congratulations, Yukaka