Distribution Steering for Discrete-Time Uncertain Ensemble Systems

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***Abstract*—Ensemble systems appear frequently in many engineering applications and, as a result, they have become an important research topic in control theory. These systems are best characterized by the evolution of their underlying state distribution. Despite the work to date, few results exist dealing with the problem of directly modifying (i.e., "steering") the distribution of an ensemble system. In addition, in most existing results, the distribution of the states of an ensemble of discrete-time systems is assumed to be Gaussian. However, in case the system parameters are uncertain, it is not always realistic to assume that the distribution of the system follows a Gaussian distribution, thus complicating the solution of the overall problem. In this paper, we address the general distribution steering problem for first-order discrete-time ensemble systems, where the distributions of the system parameters and the states are arbitrary with finite first few moments. Linear system dynamics are considered using the method of power moments to transform the original infinite-dimensional problem into a finite-dimensional one. We also propose a control law for the ensuing moment system, which allows us to obtain the power moments of the desired control inputs. Finally, we solve the inverse problem to obtain the feasible control inputs from their corresponding power moments. We provide a numerical example to validate our theoretical developments.**

***Index Terms*—Distribution steering, ensemble systems, method of moments.**

I. Introduction

This paper addresses the distribution steering problem for first-order discrete-time ensemble stochastic systems. In recent years, the necessity to precisely quantify and manage uncertainty in physical systems has spurred a growing interest in the investigation of the evolution of distributions in a stochastic setting [**?**]. [Panos:] ADD some refs Distribution steering has a long history but it has been garnering increasing attention recently from researchers both from academia and industry owing to its numerous applications in handling uncertainty in a principled manner [**?**]. [Panos:] ADD some refs

The simplest case of distribution control is probably covariance control, the earliest research of which can be traced back to a series of articles by Skelton and his students in the late 1980's and early 1990's [1]–[4], which explored the assignability of the state covariance via state feedback over an infinite time horizon. The problem of controlling the covariance over a finite horizon (e.g., the "steering" problem) is much more recent. Among the numerous results in the literature, one should mention [5]–[9] for discrete-time systems or [8], [10]–[12] for continuous-time systems. In all these works, the initial and terminal distributions are assumed to be Gaussian. While the Gaussian assumption is generally acceptable when higher-order moments of the distribution are negligible or not important, the same assumption may not be suitable for many other types of distribution steering problems, where controlling higher-order moments is critical.

One of the most common applications of ensemble systems is in the area of swarm robotics. Controlling a swarm of robots with a potentially very large number of robots has become a prominent research topic having a diverse number of applications, such as environmental monitoring and autonomous construction, to name a few [13], [14]. In this context, the objective is not to steer a single robot to a specified state but rather to ensure that all agents collectively satisfy certain macroscopic properties. To achieve this objective, researchers often model the system using a fluid approximation of the multi-agent system, known as the macroscopic or mean-field model. By modeling each agent's dynamics as a Markov process, the mean-field behavior of the population is determined through the Liouville equation corresponding to a Markov process [9], [11]. Consequently, the system state is the *distribution* of all the agents, which converges weakly to a continuous distribution as the number of agents approaches infinity [15]. Most importantly, the swarm control problem differs from conventional multi-agent control due to the significantly larger number of agents involved. Therefore, graph-theoretic approaches commonly used in multi-agent and networked control problems [16], [17] do not apply to a group with a large number of agents, due to scalability concerns. Designing a control law that scales well with a large group of agents is crucial for the algorithm to be applicable in real-world settings.

Scalability is not the sole concern in swarm robotic applications, however. Another equally important issue is the absence of Gaussianity in the case of agents with non-trivial dynamics. Notably, the mean-field approximation results in non-Gaussian distributions [**?**], [18]. [Panos:] fix ref. [19] has nothing to do with MF theory Representative results of distribution steering problems that do not require a Gaussian assumption can be found in [19]–[24]. Also, a conventional feedback control strategy, in the form of a linear function of the system state, cannot be employed to address the distribution steering problem with general dynamics. For instance, as was

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shown in [25], the distribution steering problem where the initial and terminal distributions belong to different function classes, cannot be solved using deterministic feedback laws. Designing a control law for this type of distribution steering problems, therefore, poses considerable challenges.

Several attempts have been made to address the general distribution steering problem in the literature. One such approach, proposed in [19], involves using characteristic functions for discrete-time linear systems with general disturbances. Another perspective, presented in [22], treats the system evolution purely as a Markov process, where the control inputs serve as transition kernels. This allows the control inputs to be selected as random variables, with their density functions representing the transition probabilities, offering greater flexibility in control input design. In this paper, we also explore the same concept, and consider the control inputs as transition probabilities, treating them as random variables.

Ensemble control considers problems with intrinsic perturbations in the system parameters [26], [27]. However, previous results in ensemble control have primarily focused on the controllability of a single agent subject to a specific perturbation. Developing a distribution steering scheme for general ensemble systems is currently lacking in the literature.

In this paper, we address some of the aforementioned challenges. Specifically, we propose a control scheme for the general distribution steering problem of a large group of agents, where we only assume the existence and finiteness of the first few moments of the agent state distribution. Based on the moment representation of the original infinite-dimensional system, a finite-dimensional reduction is proposed. An optimal control scheme via convex optimization is then proposed, yielding the optimal power moments of the control inputs. Finally, a realization method is revisited to map the power moments of the control inputs to feasible analytic control actions for each agent. To the best of our knowledge, this is the first attempt to treat the distribution steering problem for general linear ensemble systems.

II. Distribution Steering of Linear Ensemble Systems

A. Multi-Agent Ensemble System

We consider an ensemble system consisting of N members ("agents"). The agent dynamics are linear and are subject to a perturbation in the system parameter. The perturbation is independent of the system state. Since the agents are assumed to be homogeneous, the system dynamics of the i^{th} agent take the form

$$x_i(k+1) = a_i(k)x_i(k) + u_i(k), \quad i = 1, \dots, N, \tag{1}$$

where $k = 0, 1, \dots, K$ denotes the time step, and $x_i(k), u_i(k), a_i(k)$ are all scalars. Let the initial condition be $x_i(0) \sim \chi_0$, for all $i = 1, \dots, N$. Both the state $x_i(k)$ and control $u_i(k)$ are random variables with probability distributions χ_k and ν_k, respectively, that is, for all $i = 1, \dots, N$, $x_i(k) \sim \chi_k$ and $u_i(k) \sim \nu_k$ for all $k = 0, 1, 2, \dots, K$. The initial state distribution χ_0 is arbitrary with the first $2n$ power moments being finite. We also assume that, for each $i = 1, \dots, N$, $a_i(k)$ are random variables independent of $x_i(k)$ and $u_i(k)$, with realizations drawn from a known common distribution given by α_k, that is, $a_i(k) \sim \alpha_k$. In the distribution steering problem formulation considered in this work the identity of each agent is ignored. In other words, all agents follow the same statistics, although their particular realizations may differ. Moreover, we also assume that the agents are non-interacting and that the size of each agent is negligible, following the standard modeling assumptions of mean-field theory [28].

B. Problem Statement

We first give the definition of the general distribution steering problem we consider in this paper. Provided with an initial probability density function χ_0 of $x_i(0)$ and a final probability density function χ_f of $x_i(K)$, along with the system equation (1), we wish to determine ν_k for $k = 0, \dots, N-1$, and then a control sequence $(u_i(0), \dots, u_i(K-1))$ for each agent $i \in \{1, \dots, N\}$ such that the terminal state distribution χ_K satisfies $\chi_K = \chi_f$. Unlike the conventional distribution steering problem **[?]**, [Panos:] ADD REF we do not assume that χ_0 and χ_f are necessarily Gaussian. This lack of Gaussianity severely complicates the ensemble distribution steering problem. Henceforth, and without a great loss of generality, we assume that $x_i(k), a_i(k)$ and $u_i(k)$ are supported on the whole $\mathbb{R}$.

Denote, as usual, by $\mathbb{E}[\,\cdot\,]$ the expectation operator. Since $a_i(k)$ is independent of $x_i(k)$ and $u_i(k)$, the power moments of the state $x_i(k)$ up to order $2n$ obey the equation $\mathbb{E}\left[x_i^\ell(k+1)\right] = \mathbb{E}\left[(a_i(k)x_i(k) + u_i(k))^\ell\right] = \sum_{j=0}^{\ell} \binom{\ell}{j} \mathbb{E}[a_i^j(k)]\, \mathbb{E}[x_i^j(k) u_i^{\ell-j}(k)]$. We note that it is difficult to treat the term $\mathbb{E}[x_i^j(k) u_i^{\ell-j}(k)]$ using conventional methods, in which the control input $u_i(k)$ is given as a function of the current state $x_i(k)$. In [29]–[31], we proposed a scheme where the control inputs are independent of the current system states, so that we can write $\mathbb{E}[x_i^j(k) u_i^{\ell-j}(k)] = \mathbb{E}[x_i^j(k)]\, \mathbb{E}[u_i^{\ell-j}(k)]$. Using the algorithms in [29]–[31], we were able to steer an arbitrary probability distribution to another one, by only assuming the existence of the first few moments for both distributions. However, these algorithms require that the system dynamics be stable, i.e., $|a_i(k)| < 1$ for all $k = 0, \dots, K$, thus limiting their applicability.

C. Density Steering of Ensemble Dynamics

In this section, we consider the density steering of the first-order discrete-time ensemble system (1) that is not always stable. Instead of the open-loop control inputs of the schemes presented in [29]–[31], we propose to use a *feedback control law*. Specifically, we choose the control input at each time step k as

$$u_i(k) = -c(k)a_i(k)x_i(k) + \tilde{u}_i(k), \tag{2}$$

where $x_i(k)$ and $\tilde{u}_i(k)$ are independent random variables, and $c(k)$ is a constant such that $c(k) \in [0, 1]$. The problem now becomes one of determining $c(0), \dots, c(K-1)$, and $\tilde{u}_i(0), \dots, \tilde{u}_i(K-1)$. We emphasize that in the proposed control law (2) $\tilde{u}_i(k)$ is neither a constant nor a function of $x_i(k)$.

By denoting $\tilde{a}_i(k) = a_i(k) - c(k)a_i(k)$, the system equation can now be written as

$$x_i(k+1) = \tilde{a}_i(k)x_i(k) + \tilde{u}_i(k). \quad (3)$$

In practice, directly implementing (2) may not be feasible, as it is often impractical to measure the state $x_i(k)$ accurately for each agent i in a large population. Given this practical challenge, we turn our attention to the statistics of the whole ensemble. During implementation, the control $u_i(k)$ is determined by sampling from the distribution of the "group control" $u(k)$, which is determined by the "group state" $x(k)$. Below we provide the details for deriving group state $x(k)$ and group control $u(k)$.

The control law in (2) leads to a convenient characterization of the group of agents using occupation measures as follows. The use of occupation measures, allows us to introduce a single equation that conveniently characterizes the "average" behavior of the ensemble. For more details, see also [15], [32].

D. Ensemble Group Dynamics and Moment System

To this end, let the measure $\mathrm{d}\mu_k : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \mapsto \mathcal{M}_+$, given by $\mathrm{d}\mu_k(\mathrm{x}, \mathrm{v}, \mathrm{w}) = \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{\{(x_i(k)\}}(\mathrm{x})\mathbf{1}_{\{\tilde{u}_i(k)\}}(\mathrm{v})\mathbf{1}_{\{\tilde{a}_i(k))\}}(\mathrm{w})\,\mathrm{dx}\,\mathrm{dv}\,\mathrm{dw}$, where $\mathbf{1}_{\boldsymbol{A}} : \mathbb{R} \mapsto \{0, 1\}$ denotes the indicator function, that is, $\mathbf{1}_{\boldsymbol{A}}(\mathrm{x}) = 1$ with $\mathrm{x} \in \boldsymbol{A} \subseteq \mathbb{R}$, and $\mathbf{1}_{\boldsymbol{A}}(\mathrm{x}) = 0$ otherwise, and $\mathcal{M}_+$ denotes the set of positive Radon measures. Then, the marginal $\mathrm{d}\kappa_k(\mathrm{x}) = \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{\{(x_i(k)\}}(\mathrm{x})\,\mathrm{dx}$ describes the probability of the state of each agent being x, at time step k. Similarly, $\mathrm{d}\mu_k$ induces the following marginals for the control and system parameter $\mathrm{d}\beta_k(\mathrm{v}) = \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{\{\tilde{u}_i(k)\}}(\mathrm{v})\,\mathrm{dv}$ and $\mathrm{d}\eta_k(\mathrm{w}) = \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{\{\tilde{a}_i(k))\}}\,\mathrm{dw}$. In order to characterize the aggregate behavior of the agents, we need to re-write the system equation (3), for all $i \in \{1, \ldots, N\}$, as a group system equation. To this end, let $\mathbb{P}\{x(k) = \mathrm{x}\} = \mathrm{d}\kappa_k(\mathrm{x})$, $\mathbb{P}\{\tilde{u}(k) = \mathrm{v}\} = \mathrm{d}\beta_k(\mathrm{v})$, and $\mathbb{P}\{\tilde{a}(k) = \mathrm{w}\} = \mathrm{d}\eta_k(\mathrm{w})$. [Panos:] $x(k)$ is a continuous RV, so $\mathbb{P}\{x(k) = \mathrm{x}\} = 0$? Do you mean $\mathbb{P}\{\mathrm{x} \le x(k) \le \mathrm{x} + d\kappa_k(\mathrm{x})\} = d\kappa_k(\mathrm{x})$? Hence, we have that

$$\begin{aligned}\mathbb{P}\{x(k+1) = r\} &= \mathrm{d}\kappa_{k+1}(r) = \int_{\mathbb{R}} \mathbf{1}_{\{r\}}(\mathrm{x})\,\mathrm{d}\kappa_{k+1}(\mathrm{x})\\ &= \frac{1}{N}\int_{\mathbb{R}} \mathbf{1}_{\{r\}}(\mathrm{x}) \sum_{i=1}^{N} \mathbf{1}_{\{x_i(k+1)\}}(\mathrm{x})\,\mathrm{dx}\\ &= \begin{cases} 1/N, & r = \tilde{a}_i(k)x_i(k) + \tilde{u}_i(k),\\ 0, & \text{otherwise.}\end{cases}\end{aligned} \quad (4)$$

Similarly,

$$\begin{aligned}\mathbb{P}\{\tilde{a}(k)x(k) + \tilde{u}(k) = r\} &= \int_{\mathbb{R}} \mathbf{1}_{\{r\}}(\mathrm{xw} + \mathrm{v})\,\mathrm{d}\mu_k(\mathrm{x}, \mathrm{v}, \mathrm{w})\\ &= \frac{1}{N}\int_{\mathbb{R}} \mathbf{1}_{\{r\}}(\mathrm{xw} + \mathrm{v}) \sum_{i=1}^{N} \mathbf{1}_{\{x_i(k)\}}(\mathrm{x})\\ &\quad \times \mathbf{1}_{\{\tilde{u}_i(k)\}}(\mathrm{v})\mathbf{1}_{\{\tilde{a}_i(k)\}}(\mathrm{w})\,\mathrm{dx}\,\mathrm{dv}\,\mathrm{dw}\\ &= \begin{cases} 1/N, & r = \tilde{a}_i(k)x_i(k) + \tilde{u}_i(k),\\ 0, & \text{otherwise.}\end{cases}\end{aligned} \quad (5)$$

Therefore, for all $r \in \mathbb{R}$, we have that $\mathbb{P}\{x(k+1) = r\} = \mathbb{P}\{\tilde{a}(k)x(k) + \tilde{u}(k) = r\}$. Since the probability values of the random variables $x(k+1)$ and $\tilde{a}(k)x(k) + \tilde{u}(k)$ are the same, these two random variables obey identical probability laws, which leads to the following system equation of the swarm group

$$x(k+1) = \tilde{a}(k)x(k) + \tilde{u}(k), \quad (6)$$

where the random variables $x(k), \tilde{u}(k)$ and $\tilde{a}(k)$ represent the system state, the control input, and the system parameter of the swarm group at time step k, respectively.

Here we note that we do not take into account the situation where a single point in $\mathbb{R}$ is occupied by more than one agent. This is because each agent is assumed to have zero volume, making the event of the overlaps of agents to have measure zero. Hence, the probability value of each occupation measure at any point on $\mathbb{R}$ cannot exceed $1/N$.

When steering a large group of homogeneous agents, controlling directly the state of each agent is challenging, and may be computationally very expensive. Instead of controlling each agent individually, we propose to control the moments of the distribution of agents, which encode the macroscopic statistics of the ensemble. Using the previous occupation measures, the moments of the system state and the control inputs can be calculated as follows.

Define $X(k) := \{x_1(k), \ldots, x_N(k)\}$ and $\tilde{U}(k) := \{\tilde{u}_1(k), \ldots, \tilde{u}_N(k)\}$. It then follows that [Panos:] fixed the equations, please check

$$\begin{aligned}&\mathbb{E}\left[x^{\ell}(k)\right]\\ &= \int_{\mathbb{R}} \mathrm{x}^{\ell}\mathrm{d}\kappa_k(\mathrm{x}) = \int_{\mathbb{R}} \mathrm{x}^{\ell}\frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{\{(x_i(k)\}}(\mathrm{x})\,\mathrm{dx}\\ &= \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} x_i^{\ell}(k)\right] = \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left[x_i^{\ell}(k)\right]\\ &= \frac{1}{N}\sum_{i=1}^{N}\int_{\mathbb{R}} x_i^{\ell}\chi_k(x_i)\,\mathrm{d}x_i = \int_{\mathbb{R}} x^{\ell}\chi_k(x)\,\mathrm{d}x,\end{aligned} \quad (7)$$

where $\ell = 0, 1, \ldots$ is a nonnegative integer, such that $\ell \le 2n$. The fourth equality in (7) stems from the fact that $x_i(k) \sim \chi_k$ for each $i \in \{1, \cdots, N\}$.

Next, denote the probability distribution of $\tilde{u}(k)$ as $\tilde{\nu}_k$. Then, and similarly to (7), we obtain that $\mathbb{E}\left[\tilde{u}^{\ell}(k)\right] = \int_{\mathbb{R}} \tilde{u}^{\ell}\tilde{\nu}_k(\tilde{u})\,\mathrm{d}\tilde{u}$, and $\mathbb{E}\left[x^{j}(k)\tilde{u}^{\ell-j}(k)\right]$ as in (8). The fifth equation in (8) is owing to the fact $\tilde{u}_i(k) \sim \tilde{\nu}_k$ for all $i \in \{1, \cdots, N\}$. Since $a_i(k)$ is not measured, we integrate $a(k)$ out when calculating the power moments.

$$
\begin{aligned}
&\mathbb{E}\left[x^j(k)\tilde{u}^{\ell-j}(k)\right]\\
=&\mathbb{E}\left[\int_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}} \mathrm{x}^j \mathrm{v}^{\ell-j}\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}_{\{(x_i(k)\}}(\mathrm{x})\mathbf{1}_{\{\tilde{u}_i(k)\}}(\mathrm{v})\mathbf{1}_{\{\tilde{a}_i(k))\}}(\mathrm{w})\,\mathrm{dx}\,\mathrm{dv}\,\mathrm{dw}\right]\\
=&\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}x_i^j(k)\tilde{u}_i^{\ell-j}(k)\right]=\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[x_i^j(k)\right]\mathbb{E}\left[\tilde{u}_i^{\ell-j}(k)\right]\\
=&\frac{1}{N}\sum_{i=1}^{N}\int_{\mathbb{R}}x_i^j\chi_k(x_i)\,\mathrm{d}x_i\int_{\mathbb{R}}\tilde{u}_i^{\ell-j}\nu_k(\tilde{u}_i)\,\mathrm{d}\tilde{u}_i=\int_{\mathbb{R}}x^j\chi_k(x)\,\mathrm{d}x\int_{\mathbb{R}}\tilde{u}^{\ell-j}\nu_k(\tilde{u})\,\mathrm{d}\tilde{u}=\mathbb{E}\left[x^{\ell}(k)\right]\mathbb{E}\left[\tilde{u}^{\ell-j}(k)\right].
\end{aligned}
\tag{8}
$$

Since each $a_i(k)$ is independent of $u_i(k)$ and $x_i(k)$, by following a similar treatment as in (8), we have that $a(k)$ is independent of $u(k)$ and $x(k)$. Using (2) and (8), we have that

$$
\mathbb{E}\left[u^{\ell}(k)\right]=\sum_{i=0}^{\ell}\left(-c(k)\right)^i\mathbb{E}\left[a^i(k)\right]\mathbb{E}\left[x^i(k)\right]\mathbb{E}\left[\tilde{u}^{\ell-i}(k)\right]. \tag{9}
$$

Then, the dynamics of the moments can be written as the linear matrix equation

$$
\mathcal{X}(k+1)=\tilde{\mathcal{A}}(\tilde{\mathcal{U}}(k))\mathcal{X}(k)+\tilde{\mathcal{U}}(k), \tag{10}
$$

which we call the moment counterpart of the original system (1), where the system matrix $\tilde{\mathcal{A}}(\tilde{\mathcal{U}}(k))$ is given in (13). Accordingly, the state vector of (10) is given by

$$
\mathcal{X}(k)=\begin{bmatrix}\mathbb{E}[x(k)] & \mathbb{E}[x^2(k)] & \cdots & \mathbb{E}[x^{2n}(k)]\end{bmatrix}^{\intercal}, \tag{11}
$$

and the control vector is given by

$$
\tilde{\mathcal{U}}(k)=\begin{bmatrix}\mathbb{E}[\tilde{u}(k)] & \mathbb{E}[\tilde{u}^2(k)] & \cdots & \mathbb{E}[\tilde{u}^{2n}(k)]\end{bmatrix}^{\intercal}. \tag{12}
$$

We note that the form of the moment system (10) is similar to the one that we proposed in our previous work [30]. The only difference is that the parameters in (13) are the power moments of $\tilde{a}(k)$. We note that only finitely many orders of power moments appear in (13). Hence, using the moment representation (10), the original problem which is infinite-dimensional, is approximated with a finite-dimensional one. Using (10), the original distribution steering problem is reduced to a problem of steering the corresponding moment system, which is formulated as follows. Given an arbitrary initial density χ_0, determine the control sequences $c(0),\dots,c(K-1)$ and $u(0),\dots,u(K-1)$ such that, for all $i=1,\dots,N$, the first $2n$ order moments of the final density χ_K of $x_i(K)$ are identical to the moments of a specified final probability density, that is, for $\ell=1,\cdots,2n$, $\int_{\mathbb{R}}x^{\ell}\chi_K(x)\,\mathrm{d}x=\int_{\mathbb{R}}x^{\ell}\chi_f(x)\,\mathrm{d}x$.

In this paper, we consider maximizing the smoothness of the state transition in terms of the power moments [31], which leads to the following optimization problem

$$
\min_{\mathcal{X}(1),\dots,\mathcal{X}(K-1)}\mathcal{L}\left(\mathcal{X}(1),\dots,\mathcal{X}(K-1)\right). \tag{14}
$$

where $\mathcal{L}\left(\mathcal{X}(1),\dots,\mathcal{X}(K-1)\right):=\sum_{k=0}^{K-1}\left(\mathcal{X}(k+1)-\mathcal{X}(k)\right)^{\intercal}\left(\mathcal{X}(k+1)-\mathcal{X}(k)\right)$. The directional derivative of $\delta\mathcal{L}$ along $\delta\mathcal{X}(k)$ then reads

$$
\begin{aligned}
&\delta\mathcal{L}\big(\mathcal{X}(1),\dots,\mathcal{X}(K-1);\delta\mathcal{X}(k)\big)\\
&\quad=2\left(\mathcal{X}(k)-\mathcal{X}(k-1)\right)-2\left(\mathcal{X}(k+1)-\mathcal{X}(k)\right).
\end{aligned}
$$

It has to be zero at a minimum for all variations $\delta\mathcal{X}(k)$, for each $k\in\{0,\cdots,K-1\}$. Therefore, for all $k=0,\dots,K-1$, we have $\mathcal{X}(k)-\mathcal{X}(k-1)=\mathcal{X}(k+1)-\mathcal{X}(k)$. It is easy to verify that

$$
\mathcal{X}(k)=\frac{K-k}{K}\mathcal{X}(0)+\frac{k}{K}\mathcal{X}(K). \tag{15}
$$

It is important to note that (14) is not the sole approach for determining the trajectory of moments; however, it has demonstrated promising performance in various distribution steering tasks. Further research will focus on refining metrics to enhance its performance. The power moments of the system states of the original system (1) up to order $2n$ are then determined for all $k=1,\dots,K-1$ from (15). However, the existence of $x(k)$ for $k=1,\dots,K-1$ given the moments in (15) still remains to be shown. Next, we provide a proof of the existence of such an $x(k)$.

Lemma II.1. *Given the moment sequence $\mathcal{X}(0),\dots,\mathcal{X}(K-1)$ satisfying* (15)*, there always exists a state sequence of the original system $x(0),\dots,x(K-1)$, not necessarily unique, which corresponds to this moment sequence.*

Proof. The statement is equivalent to proving that, for all $\mathcal{X}(k)$ satisfying (15), there exists $x(k)$ satisfying (11) for all $k=1,\dots,K-1$. To this end, first, define the Hankel matrix

$$
[\mathcal{X}(k)]_H:=\begin{bmatrix}1 & \mathbb{E}[x(k)] & \cdots & \mathbb{E}[x^n(k)]\\ \mathbb{E}[x(k)] & \mathbb{E}[x^2(k)] & \cdots & \mathbb{E}[x^{n+1}(k)]\\ \vdots & \vdots & \ddots & \\ \mathbb{E}[x^n(k)] & \mathbb{E}[x^{n+1}(k)] & & \mathbb{E}[x^{2n}(k)]\end{bmatrix}. \tag{16}
$$

From [33, Theorem 3.8], it suffices to prove that, for all $1\le k\le K-1$, $[\mathcal{X}(k)]_H\succ 0$. Since χ_0,χ_K are specified initial and terminal densities, $\mathcal{X}(0),\mathcal{X}(K)$ exist. It follows that $[\mathcal{X}(0)]_H,[\mathcal{X}(K)]_H$ are both positive definite. Using (15), by rearranging the elements of $\mathcal{X}(k)$, we have

$$
[\mathcal{X}(k)]_H=\frac{K-k}{K}[\mathcal{X}(0)]_H+\frac{k}{K}[\mathcal{X}(K)]_H. \tag{17}
$$

Since the scalars $(K-k)/K$ and k/K are both positive for all $k=1,\dots,K-1$, it follows that $[\mathcal{X}(k)]_H$, being the sum of two positive definite matrices, is also positive definite, thus completing the proof. □

Remark. We should also note that

$$
\chi_k(x)=\frac{K-k}{K}\chi_0(x)+\frac{k}{K}\chi_K(x) \tag{18}
$$

$$\tilde{\mathcal{A}}(\tilde{\mathcal{U}}(k)) = \begin{bmatrix} \mathbb{E}\left[\tilde{a}(k)\right] & 0 & 0 & \cdots & 0 \\ 2\mathbb{E}\left[\tilde{a}(k)\right]\mathbb{E}[\tilde{u}(k)] & \mathbb{E}\left[\tilde{a}^2(k)\right] & 0 & \cdots & 0 \\ 3\mathbb{E}\left[\tilde{a}(k)\right]\mathbb{E}[\tilde{u}^2(k)] & 3\mathbb{E}\left[\tilde{a}^2(k)\right]\mathbb{E}[\tilde{u}(k)] & \mathbb{E}\left[\tilde{a}^3(k)\right] & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{2n}{1}\mathbb{E}\left[\tilde{a}(k)\right]\mathbb{E}[\tilde{u}^{2n-1}(k)] & \binom{2n}{2}\mathbb{E}\left[\tilde{a}^2(k)\right]\mathbb{E}[\tilde{u}^{2n-2}(k)] & \binom{2n}{3}\mathbb{E}\left[\tilde{a}^3(k)\right]\mathbb{E}[\tilde{u}^{2n-3}(k)] & \cdots & \mathbb{E}\left[\tilde{a}^{2n}(k)\right] \end{bmatrix} \quad (13)$$

is a feasible distribution of $x(k)$. By denoting the ℓ^{th} element of $\mathcal{X}(k)$ as $\mathcal{X}_\ell(k)$ we have, for $\ell = 1,\dots,2n$, that $\mathcal{X}_\ell(k) = \int_{\mathbb{R}} x^\ell \chi_k(x)\,\mathrm{d}x = \int_{\mathbb{R}} x^\ell\,\frac{K-k}{K}\chi_0(x)\,\mathrm{d}x + \int_{\mathbb{R}} x^\ell\,\frac{k}{K}\chi_K(x)\,\mathrm{d}x = \frac{K-k}{K}\mathcal{X}_\ell(0) + \frac{k}{K}\mathcal{X}_\ell(K)$, which leads to (17). Therefore, the distribution in (18) satisfies the moment condition (15). However, the problem we treat is an infinite-dimensional one, since the initial distribution, namely χ_0, and the terminal one, namely χ_K, are both, in general, infinite-dimensional. Hence, so is χ_k by (18). However, there seldom exist feasible $u(k)$ for $k = 0,\dots,K-1$, given the choice of χ_k in (18). Indeed, since we are given a limited number of moments, determining the probability density function of the system state given the moment conditions is an ill-posed problem. There will always be an infinite number of feasible $x(k)$ corresponding to a given $\mathcal{X}(k)$. However, for distribution steering tasks, we desire the distribution of the system state at each time step to be analytic and the parameter space to be finite-dimensional [34], which yields a feasible control input $u(k)$. Finding such an analytic χ_k for each $[\mathcal{X}(k)]_H \succ 0$ is treated in the next section.

III. Optimal Controller Design

A. Existence of Density Steering Controller

We have proposed a novel feedback control law, given in (2), for solving the distribution steering task. We note that designing such a control is considerably easier than the treatments in [29]–[31], since $c(k) = 1$ is always a feasible solution. By setting $c(k) = 1$ for all $k = 0,\dots,K-1$, it is always possible to obtain a feasible control input $u(k)$ given $\mathcal{X}(k)$. Based on this observation, we propose an optimization scheme for determining the control sequences $\mathbf{c} = (c(0),\dots,c(K-1))$ and $\tilde{\boldsymbol{\mathcal{U}}} = \left(\tilde{\mathcal{U}}(0),\dots,\tilde{\mathcal{U}}(K-1)\right)$.

Theorem III.1. *The following optimization problem is convex.*

$$\min_{0\le c(k)\le 1} \mathbb{E}\left[\,(-c(k)a(k)x(k) + \tilde{u}(k))^2\,\right], \quad (19a)$$

$$\tilde{\mathcal{U}}(k) = \mathcal{X}(k+1) - \tilde{\mathcal{A}}(\tilde{\mathcal{U}}(k))\mathcal{X}(k), \quad [\tilde{\mathcal{U}}(k)]_H \succ 0. \quad (19b)$$

Proof. We first prove that the cost function (19a) is convex. The second-order power moment of $u(k)$ yields $\mathbb{E}\left[u^2(k)\right] = c^2(k)\mathbb{E}\left[a^2(k)\right]\mathbb{E}\left[x^2(k)\right] - 2c(k)\mathbb{E}\left[a(k)\right]\mathbb{E}\left[x(k)\right]\mathbb{E}\left[\tilde{u}(k)\right] + \mathbb{E}\left[\tilde{u}^2(k)\right]$. Noting that $\frac{\mathrm{d}^2\mathbb{E}\left[u^2(k)\right]}{\mathrm{d}c(k)^2} = \mathbb{E}\left[a^2(k)\right]\mathbb{E}\left[x^2(k)\right] \ge 0$, yields that the cost function is convex [35]. Next, we prove that the domain is a convex set. We need to prove that the feasible domain of $c(k)$ under the constraints in (19) is convex. First, note that $[\tilde{\mathcal{U}}(k)]_H$ is a continuous matrix function of $c(k)$. Hence, there exists $0 < \epsilon < 1$ such that, for all $c(k) \in (\epsilon, 1]$, $[\tilde{\mathcal{U}}(k)]_H \succ 0$. However, there may be several subintervals of $[0,1]$ that are not path-connected that satisfy $[\tilde{\mathcal{U}}(k)]_H \succ 0$. Under this circumstance, the domain of $c(k)$ is not convex. Therefore, we need to prove that there exists an $\epsilon > 0$, such that $[\tilde{\mathcal{U}}(k)]_H \succ 0$ for $c(k) > \epsilon$, and $[\tilde{\mathcal{U}}(k)]_H \not\succ 0$ for $c(k) < \epsilon$. This is equivalent to showing that there exists a feasible $\tilde{u}(k)$ for $c(k) > \epsilon$, while there is no feasible $\tilde{u}(k)$ for $c(k) < \epsilon$.

Assume that $\tilde{u}_1(k), \tilde{u}_2(k)$ exist given $c_1(k), c_2(k)$. Then, we need to prove that, for any $c_3(k) \in [c_1(k), c_2(k)]$, $\tilde{u}_3(k)$ exists. First, write $c_3(k) = \lambda c_1(k) + (1-\lambda)c_2(k), 0 \le \lambda \le 1$. From equations (1) and (2), [Panos:] these equations are for $x_i(k)$ not $x(k)$ we have $x(k+1) = (1 - c_1(k))\,a(k)x(k) + \tilde{u}_1(k) = (1 - c_2(k))\,a(k)x(k) + \tilde{u}_2(k)$.

Therefore, we can write

$$\begin{aligned} x(k+1) =& \lambda\left((1 - c_1(k))\,a(k)x(k) + \tilde{u}_1(k)\right) \\ &+ (1-\lambda)\left((1 - c_2(k))\,a(k)x(k) + \tilde{u}_2(k)\right) \\ =& \left(1 - \lambda c_1(k) - (1-\lambda)\,c_2(k)\right)a(k)x(k) \\ &+ \lambda\tilde{u}_1(k) + (1-\lambda)\,\tilde{u}_2(k) \\ =& \left(1 - c_3(k)\right)a(k)x(k) + \lambda\tilde{u}_1(k) \\ &+ (1-\lambda)\,\tilde{u}_2(k). \end{aligned} \quad (20)$$

It follows that $\tilde{u}_3(k) = \lambda\tilde{u}_1(k) + (1-\lambda)\,\tilde{u}_2(k)$ is the control corresponding to $c_3(k)$ and hence the set of all feasible $c(k)$ is convex. We also conclude that there exists an $\epsilon > 0$ such that for any $c(k) \in (\epsilon, 1]$, $u(k)$ exists, i.e., $[\tilde{\mathcal{U}}(k)]_H \succ 0$. □

It remains to prove the existence of a solution to the optimization problem (19). In case the conditions of (19b) are satisfied, the feasible set of $c(k)$ is closed [Panos:] why closed? and convex, which ensures the existence of a solution to the optimization problem. Let the feasible domain of $c(k)$ be $(\epsilon, 1]$, where $0 < \epsilon < 1$. In this case, we may relax the second condition in (19b) to $[\tilde{\mathcal{U}}(k)]_H \succeq 0$. Since $[\tilde{\mathcal{U}}(k)]_H$ is a continuous matrix function of $c(k)$, the feasible domain of $c(k)$ is the closed and convex set $[\epsilon, 1]$. Hence, the existence of a solution to the optimization problem follows. [Panos:] next sentence is not clear - where does it come from? Furthermore, when $c(k) = \epsilon$ is the optimal solution to the optimization problem (19), $\tilde{\nu}(k)$ is an atomic distribution supported on n discrete points on $\mathbb{R}$, rather than a continuous distribution.

Theorem III.1 along with the proof of the existence of a solution allows us to obtain an optimal control $\tilde{\mathcal{U}}(k)$ for each $k = 0,\dots,K-1$. However, $\tilde{\mathcal{U}}(k)$ consists of the statistics of the random variable $\tilde{u}(k)$. The problem now becomes one of determining a control $\tilde{u}(k)$ given the $\tilde{\mathcal{U}}(k)$ obtained by the solution to the optimization problem (19). In our previous work [29]–[31] this step is called the *realization problem* of the random variables $\tilde{u}(k)$ for $k = 0,\dots,K-1$. Here we adopt the same treatment found in [30].

For the sake of simplicity, henceforth, we omit the index k if there is no danger of ambiguity. The problem now becomes one of designing an algorithm to estimate the probability density supported on $\mathbb{R}$ of which the power moments are given. This is known in the literature as the Hamburger moment problem [33]. Often, the Kullback-Leibler divergence is a widely used measure in the literature to characterize the difference between the reference density and the density estimate [36]–[38]. A convex optimization scheme for density estimation using the Kullback-Leibler divergence has been proposed in [39] for the Hamburger moment problem. We adopt this strategy to realize the control inputs.

Let $\mathcal{P}$ be the space of probability density functions defined and having support on the real line, and let $\mathcal{P}_{2n}$ be the subset of all $p \in \mathcal{P}$ that have at least $2n$ finite moments. The Kullback-Leibler divergence between the probability density functions $p, r \in \mathcal{P}$ is defined as $\mathsf{KL}(r\|p) := \int_{\mathbb{R}} r(\tilde{u}) \log \frac{r(\tilde{u})}{p(\tilde{u})} \mathrm{d}\tilde{u}$. Define the linear operator $\Gamma : \mathcal{P}_{2n} \to \mathbb{R}^{(n+1)\times(n+1)}$ as $\Gamma(\tilde{p}) = \Sigma := \int_{\mathbb{R}} G(\tilde{u})\tilde{p}(\tilde{u})G^{\intercal}(\tilde{u})\mathrm{d}\tilde{u}$, where $G(\tilde{u}) = \begin{bmatrix} 1 & \tilde{u} & \cdots & \tilde{u}^n \end{bmatrix}^{\intercal}$. It can be easily shown that

$$\Sigma = \begin{bmatrix} 1 & \mathbb{E}[\tilde{u}] & \cdots & \mathbb{E}[\tilde{u}] \\ \mathbb{E}[\tilde{u}] & \mathbb{E}[\tilde{u}^2] & \cdots & \mathbb{E}[\tilde{u}^{n+1}] \\ \vdots & \vdots & \ddots & \\ \mathbb{E}[\tilde{u}^n] & \mathbb{E}[\tilde{u}^{n+1}] & & \mathbb{E}[\tilde{u}^{2n}] \end{bmatrix}, \tag{21}$$

where $\mathbb{E}[\tilde{u}^\ell]$, $(\ell = 1, \ldots, 2n)$ are obtained from $\tilde{\mathcal{U}}$ using (9). [Panos:] using (9)? is this correct?

Since $\mathcal{P}_{2n}$ is convex, $\mathrm{range}(\Gamma) = \Gamma\mathcal{P}_{2n}$ is also convex.

Given $r \in \mathcal{P}$ and $\Sigma \succ 0$, then there is a unique $\nu^* \in \mathcal{P}_{2n}$ that minimizes the Kullback-Leibler distance [Panos:] check $\mathsf{KL}(r\|\nu^*)$ subject to $\Gamma(\nu^*) = \Sigma$, which is given by

$$\nu^* = \frac{r}{G^{\intercal}\Lambda^* G}, \tag{22}$$

where Λ^* is the unique solution to the minimization problem [39]

$$\min_{\Lambda \in \mathcal{L}_+} \mathcal{J}_r(\Lambda) := \mathrm{tr}(\Lambda\Sigma) - \int_{\mathbb{R}} r(\tilde{u}) \log\left[G(\tilde{u})^{\intercal}\Lambda G(\tilde{u})\right] \mathrm{d}\tilde{u}, \tag{23}$$

where $\mathcal{L}_+ := \{\Lambda \in \mathrm{range}(\Gamma) \mid G(\tilde{u})^{\intercal}\Lambda G(\tilde{u}) > 0, \tilde{u} \in \mathbb{R}\}$.

The probability density function of the random variable u can now be estimated by solving the convex optimization problem (23). From Theorem III.1, we obtain the values of $c(k)$ for all $k = 0, \ldots, K-1$ by solving the convex optimization problem (19). Therefore, the control input $u(k)$ can be uniquely determined by solving the two convex optimization problems (19) and (23). It is worth noting that the power moments of the proposed density estimate align exactly with the specified moments. This property distinguishes the proposed approach from other similar moment-matching methods in the literature [40]. Consequently, the proposed approach can be used to realize the control inputs. Since both the prior density $r(\tilde{u})$ and the density $\nu^*(\tilde{u})$ are supported on the real line, one can usually select a Gaussian distribution for $r(\tilde{u})$ when χ_f is a sub-Gaussian distribution [39], or a Cauchy distribution when χ_f is heavy-tailed.

B. Algorithm for Density Steering

We now propose an algorithm for steering a large, but finite, group of agents. The distribution of the system state is discrete, representing the individual agents. We do not aim to steer a specific agent to a specific state. Instead, we target the terminal discrete system state distribution to be the desired one.

It should be noted that the control law is not a purely state feedback control law. Instead, each control input is the sum of a state feedback function and a random variable that is independent of the current system state. A similar idea has appeared in [22], where the evolution of the state distribution is regarded as a Markov process with the control input serving as the transition rate or probability.

An algorithm for solving the discrete distribution steering is given in Algorithm 1. Since the control input $\tilde{u}(k)$ and the current system state $x(k)$ are independent, we may obtain each $\tilde{u}_i(k)$ by drawing i.i.d samples from the realized distribution $\nu_k^*(\tilde{u})$. By doing this, $\tilde{u}(k)$ serves as a transition probability of a Markov process.

Algorithm 1 Discrete distribution steering of a large group of agents

Input: Number of agents N; maximal time step K; distribution of system parameter $a(k)$ for $k = 0, \ldots, K-1$; initial discrete distribution χ_0; specified terminal discrete distribution χ_f

Output: Control inputs of the i^{th} agent $u_i(k)$, $k = 0, \ldots, K-1$, $i = 1, \ldots, N$

```
k ← 0
Calculate X(0) from (11)
while 0 ≤ k < K do
    Calculate X(k+1) from (15).
    Solve optimization problem (19), obtain c(k) and corresponding Ũ(k) and Σ from (21)
    Optimize cost function (23) and obtain the density ν*_k from (22)
    Draw N i.i.d. samples ũ_i(k) for i = 1, ..., N, from the distribution ν*_k
    Calculate control inputs u_i(k) for each agent i = 1, ..., N from (2)
    k ← k + 1.
end while
```

IV. A Numerical Example

In this section, we provide a numerical example to demonstrate the proposed approach to solve ensemble distribution steering problems. We first simulate the discrete-time Liouville control problem, where an infinite number of agents is assumed, and the control inputs $u(k)$ are continuous functions. However, since $x(0)$ and $u(k)$ for all $k = 0, 1, \ldots, K$ are not assumed to fall within an exponential family (e.g., Gaussian), the probability density function of $x(k)$ for $k = 1, \ldots, K$ does not always have an analytic form. This makes comparing the results using our algorithm to the desired distribution a difficult task. To validate the performance of the proposed algorithm, we simulated a large group of $2,000$ agents, based on the results of the Liouville control problem. The solution is a sufficiently fine discrete distribution, which makes it possible to compare it to the desired continuous distribution. We simulate a steering problem in four steps ($K = 4$) where the system parameter follows a Laplace distribution. The initial density is chosen as

$$\chi_0(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{2}}.$$

The terminal density function is specified as a multi-modal density which is a mixture of two Gaussian densities as follows

$$\chi_f(x) = \frac{0.5}{\sqrt{2\pi}} e^{\frac{(x+2)^2}{2}} + \frac{0.5}{\sqrt{2\pi}} e^{\frac{(x-3)^2}{2}}. \quad (24)$$

The system parameter $a(k)$ follows the Laplace distribution $\mathcal{C}\,(0.5, 0.1)$ for all $k = 0, 1, 2, 3$. The results are given in Figures 1-5. The optimal values of $c(k) = 0$ for $k = 0, 1, 2$, and $c(3) = 0.16$. Since $c(3) \neq 0$, the corresponding $\tilde{\mathfrak{U}}(3)$ is not positive definite, with its determinant being zero. [Panos:] why? This causes the control input $u(3)$ to be a discrete distribution supported on two points $-2.38, 2.69$, with corresponding probability values $0.472, 0.528$, respectively. From Figure 5, we note that the terminal discrete distribution using the proposed algorithm is close to the desired continuous one.

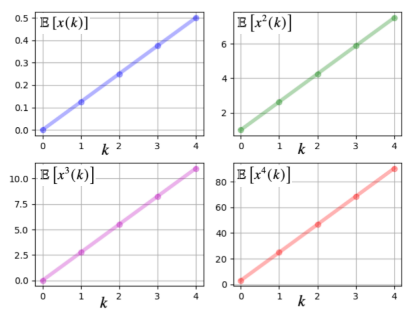

Fig. 1. $\mathfrak{X}(k)$ at time steps $k = 0, 1, 2, 3, 4$.

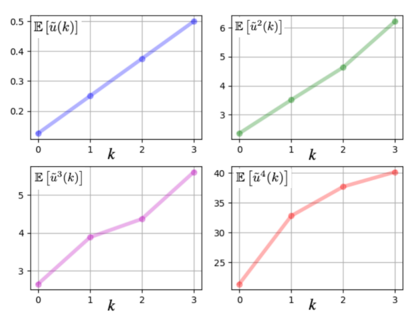

Fig. 2. $\tilde{\mathfrak{U}}(k)$ at time steps $k = 0, 1, 2, 3$.

V. Conclusions

In this paper, we have tackled the control of a large group of agents through a generalized distribution steering problem. We present a novel formulation for the dynamics of the ensemble system. Unlike traditional approaches, this distribution steering problem does not rely on assuming Gaussian distributions for the system states or the parameters. Instead, we only require

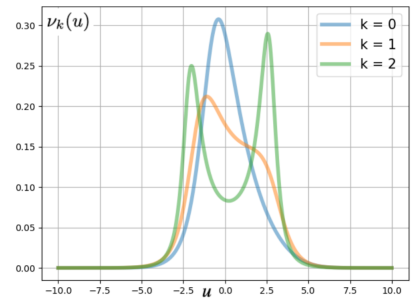

Fig. 3. Realized distributions of control inputs $\nu_k(u)$ by $\tilde{\mathfrak{U}}(k)$ for $k = 0, 1, 2, 3$, which are obtained by our proposed control scheme.

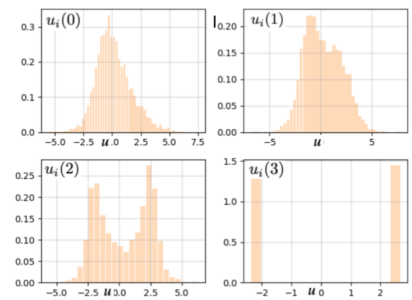

Fig. 4. The histograms of $u_i(k)$ at time step k for each agent i. The upper left and right figures are $u_i(0)$ and $u_i(1), i = 1, \cdots, 2000$ respectively. The lower left and right figures are $u_i(2)$ and $u_i(3)$ respectively.

the existence of finite power moments up to order $2n$, where n is some positive integer. To our knowledge, this algorithm is the first viable solution for the distribution steering of a large swarm robot group governed by linear ODE systems with uncertainty in the system parameter.

The findings of this paper highlight the effectiveness of power moments in tackling infinite-dimensional stochastic control problems. By utilizing power moments, we represent both the uncertainty of the system parameter and the agents' distribution as a whole. This leads to a significant simplification in propagating the system state during the control process. Power moments also serve as compact representations of the distribution's macroscopic properties. As evidenced in the

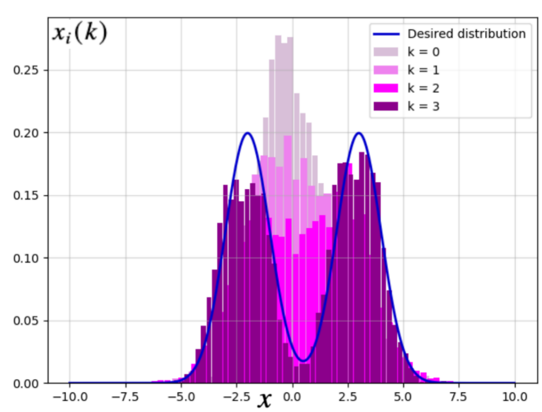

Fig. 5. The histogram of the terminal distribution of $x_i(k), i = 1, \ldots, 2000$ compared to the desired distribution. It is close to the specified terminal distribution (24).

preceding numerical examples, the moment system of order $2n = 4$ provides an accurate approximation of the original system, with our algorithm yielding a terminal distribution that closely matches the desired distribution.

Looking forward, future research will aim to extend the results of this paper to multiple-dimensional and/or nonlinear systems. However, such an extension poses major challenges. Ensuring the existence of the control inputs, namely, control inputs with non-negative probability density functions, poses a significant hurdle necessitating the application of tools from real algebraic geometry, e.g., Positivstellensatz.

[Panos:] Are there updated versions of [29] and [31]?

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