



# Canonical Correlation Analysis, Approximate Covariance Extension, and Identification of Stationary Time Series

ANDERS LINDQUIST<sup>†</sup> and GIORGIO PICCI<sup>‡</sup>

*Certain subspace identification methods cannot be expected to work for generic data since the positive-real requirement is ignored. The algorithms are analyzed in the context of stochastic realization theory. The original model reduction procedure of Desai and Pal is partially justified.*

**Key Words** — Identification; balanced realizations; approximate covariance extension; canonical correlation analysis; stochastic model reduction.

**Abstract**—In this paper we analyze a class of state space identification algorithms for time-series, based on canonical correlation analysis, in the light of recent results on stochastic systems theory. In principle, these so called “subspace methods” can be described as covariance estimation followed by stochastic realization. The methods offer the major advantage of converting the nonlinear parameter estimation phase in traditional ARMA models identification into the solution of a Riccati equation but introduce at the same time some nontrivial mathematical problems related to positivity. The reason for this is that an essential part of the problem is equivalent to the well-known rational covariance extension problem. Therefore, the usual deterministic arguments based on factorization of a Hankel matrix are not valid for generic data, something that is habitually overlooked in the literature. We demonstrate that there is no guarantee that several popular identification procedures based on the same principle will not fail to produce a positive extension, unless some rather stringent assumptions are made which, in general, are not explicitly reported. In this paper the statistical problem of stochastic modeling from estimated covariances is phrased in the geometric language of stochastic realization theory. We review the basic ideas of stochastic realization theory in the context of identification, discuss the concept of stochastic balancing and of stochastic model reduction by principal subsystem truncation. The model reduction method of Desai and Pal (1982) [A realization approach to stochastic model reduction. *Proc. 1st Decision and Control Conf.*, pp. 1105–1112.], based on truncated balanced stochastic realizations, is partially justified, showing that the reduced system structure has a positive covariance sequence but is in general not balanced. As a byproduct of this analysis we obtain a theorem prescribing conditions un-

der which the ‘subspace identification’ methods produce *bona fide* stochastic systems. Copyright ©1996 Elsevier Science Ltd.

## 1. INTRODUCTION

Recently there has been a renewed interest in state space identification algorithms for time series based on a two steps procedure which in principle can be described as estimation of a rational covariance model from observed data followed by stochastic realization. The method offers the major advantage of converting the nonlinear parameter estimation phase which is necessary in traditional ARMA models identification into a partial realization problem, involving a Hankel matrix of estimated covariances, and the solution of a Riccati equation, both much better understood problems for which efficient numerical solution techniques are available. In this framework we can naturally accommodate multivariate processes and there are indications that the algorithms may work also with data containing purely deterministic components (van Overshee and De Moor, 1993). A drawback, however, to be emphasized in this paper, is that, unlike, say, least-squares identification of ARMA models, these methods do not work for arbitrary data.

This type of procedure was apparently first advocated by Faurre (1969); see also Faurre and Chataigner (1971) and Faurre and Marmorat (1969). More recent work, based on canonical correlation analysis (Akaike, 1975) (or some other singular-value decomposition) and the Ho–Kalman algorithm (Kalman *et al.*, 1969), is due to Aoki (1990), Larimore (1990), and van Overshee and De Moor (1993). In the modern versions of the algorithm canonical correlation analysis is performed directly on the observed data without

Received 21 September 1994; revised 16 May 1995; received in final form 13 October 1995. This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Bo Wahlberg under the direction of Editor Torsten Söderström. Corresponding author Professor Anders Lindquist. Tel.: +44 8 790 7311; Fax: +46 8 202 398; E-mail: alq@math.kth.se.

<sup>†</sup> Division of Optimization and Systems Theory, Royal Institute of Technology, 100 44 Stockholm, Sweden.

<sup>‡</sup> Dipartimento di Elettronica e Informatica, Università di Padova, 35131 Padova, Italy.

computing the covariance estimates (van Overshee and De Moor, 1993). Numerical experience shows that the computation time needed to get the final model parameters estimates compares very favorably with traditional iterative prediction error methods for ARMA models.

On the other hand there is a price to be paid for this simplification. These methods introduce some nontrivial mathematical problems related to positivity. The reason for this is that an essential part of the problem is equivalent to the well-known "rational covariance extension problem". Therefore the usual deterministic realization arguments based on factorization of a Hankel matrix are not valid for generic data, something that is habitually overlooked in the literature. Note that positivity is the natural condition insuring solvability of the Riccati equation required to compute state space models of the signal from the covariance estimates.

Central in the procedures described above is the following classical problem of identification of a covariance sequence. Let

$$\{\Lambda_0, \Lambda_1, \dots, \Lambda_\nu\} \tag{1}$$

be a finite set of sample  $m \times m$  covariance matrices estimated in some unspecified way from a certain  $m$ -dimensional sequence of observations

$$\{y_0, y_1, y_2, \dots, y_T\}, \tag{2}$$

and consider the problem of finding a minimal\* triplet of matrices  $(A, C, \bar{C})$  such that

$$CA^{k-1}\bar{C}' = \Lambda_k \quad k = 1, 2, \dots, \nu \tag{3}$$

and such that the infinite sequence

$$\{\Lambda_0, \Lambda_1, \Lambda_2, \dots\}, \tag{4}$$

obtained from (1) by setting  $\Lambda_k := CA^{k-1}\bar{C}'$  for  $k = \nu + 1, \nu + 2, \dots$ , is a *bona fide* covariance sequence.

In the literature the last condition is generally ignored. The remaining problem of finding a minimal triplet  $(A, C, \bar{C})$  satisfying (3) is called the "minimal partial realization problem". The triplet  $(A, C, \bar{C})$  is usually computed by minimal factorization of a block Hankel matrix corresponding to the data (1) as follows:

$$H = \begin{bmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 & \cdots & \Lambda_j \\ \Lambda_2 & \Lambda_3 & \Lambda_4 & \cdots & \Lambda_{j+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Lambda_i & \Lambda_{i+1} & \Lambda_{i+2} & \cdots & \Lambda_{i+j-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-1} \end{bmatrix} \begin{bmatrix} \bar{C} \\ \bar{C}A' \\ \vdots \\ \bar{C}(A')^{j-1} \end{bmatrix}' \tag{5}$$

\* Here  $(A, C, \bar{C})$  is minimal if  $(A, C)$  is completely observable and  $(A, \bar{C}')$  is completely reachable.

where  $i + j - 1 = \nu$  and the Hankel matrix  $H$  is chosen as close to square as possible by taking  $|i - j| \leq 1$ . In fact, (3) holds if and only if (5) holds for all  $(i, j)$  such that  $i + j - 1 = \nu$ , and hence the minimal factorization must be made for a choice of  $(i, j)$  in which the Hankel matrix (5) has maximal rank. The infinite sequence  $\{\Lambda_0, \Lambda_1, \Lambda_2, \dots\}$  obtained in this way by setting  $\Lambda_k := CA^{k-1}\bar{C}'$  for  $k = \nu + 1, \nu + 2, \dots$  is called a "minimal rational extension" of the finite sequence (1) and is in general not a covariance sequence. The dimension  $r$  of a minimal rational extension is called the "(algebraic) degree" of the partial sequence (1). Clearly the degree  $r$  is also equal to the McMillan degree of the  $m \times m$  rational matrix

$$Z(z) = C(zI - A)^{-1}\bar{C}' + \frac{1}{2}\Lambda_0. \tag{6}$$

and the elements of the infinite sequence (4) are the coefficients of the Laurent expansion

$$Z(z) = \frac{1}{2}\Lambda_0 + \Lambda_1z^{-1} + \Lambda_2z^{-2} + \dots \tag{7}$$

about  $z = \infty$ .

The underlying identification problem is however a great deal more complicated than the classical partial realization problem. In fact, the requirement that (4) be a *bona fide* covariance sequence amounts to (4) being a positive sequence in the sense that, for every  $t \in \mathbb{Z}_+$ , the block Toeplitz matrices  $T_t$ ,

$$T_t = \begin{bmatrix} \Lambda_0 & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_t \\ \Lambda_1' & \Lambda_0 & \Lambda_1 & \cdots & \Lambda_{t-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Lambda_t' & \Lambda_{t-1}' & \Lambda_{t-2}' & \cdots & \Lambda_0 \end{bmatrix}, \tag{8}$$

formed from the infinite sequence (4), be positive definite or, equivalently, that the matrix function

$$\Phi(z) := Z(z) + Z(1/z)' \tag{9}$$

be positive semidefinite on the unit circle, i.e.

$$\Phi(e^{i\theta}) \geq 0 \quad \theta \in [0, 2\pi). \tag{10}$$

This property is equivalent to  $\Phi$  being a spectral density matrix. In fact, it will be the spectral density of the covariance sequence (4). Clearly (1) cannot be a partial covariance sequence unless  $T_\nu > 0$ , but this is not enough.

From the point of view of identification there seem to be two possible routes to determine a model  $(A, C, \bar{C})$  from the finite covariance sequence (1). One that has been proposed in the literature is do minimal factorization (5) of a finite block Hankel matrix in balanced form (Aoki, 1990; van Overshee and De Moor, 1993). This yields a solution to the minimal partial realization problem, and, as will be shown in this paper, there is no a priori guarantee that this method will yield a positive extension. This fact has nothing to do with sample variability

(random fluctuations) of the covariance estimates (1), and to emphasize this point we initially assume that all strings of data (2) are infinitely long. A theoretically sounder identification method, which will not be considered in this paper, could instead be to perform positive extension first and then to use a stochastic model reduction procedure on the triplet  $(A, C, \bar{C})$  of the positive extended sequence.

The issues regarding positive extension are discussed in Section 2, where the nontrivial nature of the positivity constraints are explained. The failure to take this difficulty into consideration has been pointed out by the authors of this paper at many scientific meetings in the last ten years. This has had no apparent effect, except for two recent papers, Heij *et al.* (1992) and Vaccaro and Vukina (1993), in which these problems are mentioned. Consequently this point will be strongly emphasized. We illustrate our point on the identification procedure of Aoki (1990) and demonstrate that there is a hidden, and not easily tested, assumption without which the procedure will not be guaranteed to succeed. The punch line is that none of the subspace identification methods under consideration can be expected to always work for generic data but that some not entirely natural conditions on the data are needed.

The analysis of the basic theoretical issues behind subspace identification is carried out in the geometric framework of stochastic realization theory; see, e.g. Lindquist and Picci (1985), Lindquist and Picci (1991). In Section 3 we introduce some basic concepts from this theory and adapt them to the problem of identification. To this end, we first discuss an idealized situation in which the time series (2) is infinitely long, i.e.  $T = \infty$ , and the available covariance data are given by the ergodic limit

$$\lim_{T \rightarrow \infty} \frac{1}{T+1} \sum_{t=0}^T y_{t+k} y'_{t+j} = \Lambda_{k-j} \quad (11)$$

for all  $k$  and  $j$ . Then the sample estimates in the sequence (1) are *bona fide* covariance matrices and the Toeplitz matrix  $T_v$  formed from the data will be positive definite and symmetric. We introduce a Hilbert space of observed (infinite) strings of data  $\{y_t\}$ , allowing us to use the geometric concepts and machinery of linear stochastic system theory as developed in Lindquist and Picci (1985) and Lindquist and Picci (1991) also for the statistical problem of identification. In this way we establish a correspondence which turns operations on random quantities defined on abstract probability spaces into prototypes of statistical algorithms involving computations based on the observed data. Canonical correlations and balanced stochastic realizations are then analyzed in this setting in Section 4, and the basic concepts and principles used in the subspace identification methods, as well as in the model re-

duction procedures of Desai and Pal, are translated into the more natural context of geometric stochastic realization theory.

Although the explicit computation of covariance sequences can be avoided completely in the methods discussed in this paper, it is useful to think in terms of such objects. The realization theory developed in Sections 3 and 4 deals with an idealized situation which admits the construction of an exact infinite covariance sequence (4). Consequently, the difficult question of positivity is not an issue here. Nor is it the finite sample size *per se* which is the problem, but the fact that only a finite covariance sequence (1) could be constructed from the data (2) when  $T$  is finite. Therefore, we separate these issues by discussing stochastic realization theory from finite covariance data in Section 5 and subspace identification in Section 6. In this framework we show that the method of van Overshee and De Moor (1993) is valid under some rather stringent assumptions. We stress that we are only concerned with identification procedures for state space modeling of time series. "Subspace identification" methods for deterministic systems with measurable inputs or for spectral factors do not involve positivity, but stability may still be a problem. However, the algorithms of van Overshee and De Moor (1994a) and van Overshee and De Moor (1994b) also have a stochastic part, so the problem of positivity arises here too.

Another idea behind the subspace identification methods considered in this paper is to disregard modes corresponding to "small" canonical correlation coefficients. This is called "balanced truncation" and is in fact a stochastic model reduction procedure. In all such procedures there must be a guarantee that the reduced-degree matrix function (6) is positive real, and therefore the preservation of positivity in such reductions is a main concern of this paper. Section 7 is devoted to such issues. The model reduction procedure of Desai and Pal (1982) was never theoretically justified in their work or in their subsequent work Desai *et al.* (1985) and Desai (1986).<sup>\*</sup> Here we shall demonstrate that this reduction procedure produces a positive real, but not in general balanced, reduced model structure. In fact, the singular values of the truncated system are usually not equal to the  $r$  first singular values of the original system.

It is an interesting fact that the procedure of Desai and Pal does produce balanced truncations for continuous-time stochastic systems. A partial result in this direction was given by Harshavaradhana *et al.* (1984), who showed that the truncated function

<sup>\*</sup> In Desai *et al.* (1985) a different model reduction procedure, which is not relevant to subspace identification, is considered, namely "deterministic" model reduction of the minimum phase spectral factors.

is positive real and conjectured that it is balanced. We shall demonstrate that it is indeed balanced, a result that is actually already contained in the work of Ober (1991). The problem with the Desai–Pal procedure in discrete time depends on the fact that the spectral factors of the truncated approximate spectrum behave differently than in continuous time. While in continuous time the realizations of the reduced spectral factors are proper subsystems, obtained by partitioning the matrices of the realizations of the factors of  $\Phi$ , this is *not* the case in discrete time, contrary to early claims of Desai and Pal. As indicated in Ober (1991), a balanced truncation procedure is available in discrete time, but the systems matrices are no longer submatrices of those of the original system, and therefore it is not equivalent to the truncation procedure used in subspace identification.

Several of the results of this paper have previously been announced in Lindquist and Picci (1994a)\* and in Lindquist and Picci (1994b).

2. POSITIVE, NONPOSITIVE AND APPROXIMATE FACTORIZATIONS OF THE HANKEL MATRIX OF COVARIANCES

The solution to the minimal partial realization problem, i.e. the problem to find the triplet  $(A, C, \bar{C})$  satisfying (1) is in general not unique. This lack of uniqueness, studied in, for example, Kalman *et al.* (1969), Kalman (1979) and Gragg and Lindquist (1983), is not an issue in this paper. Therefore, to avoid this question altogether, we shall make the standard assumption that the algebraic degree of (1) equals that of

$$\{\Lambda_0, \Lambda_1, \dots, \Lambda_{\nu-1}\} \tag{12}$$

so that we can use a Hankel matrix (5) based on this data, i.e. with  $i + j = \nu$ , allowing us to define the shifted Hankel matrix

$$\sigma(H) = \begin{bmatrix} \Lambda_2 & \Lambda_3 & \Lambda_4 & \cdots & \Lambda_{j+1} \\ \Lambda_3 & \Lambda_4 & \Lambda_5 & \cdots & \Lambda_{j+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Lambda_{i+1} & \Lambda_{i+2} & \Lambda_{i+3} & \cdots & \Lambda_{\nu} \end{bmatrix} \tag{13}$$

uniquely. In this case the classical Ho–Kalman algorithm (Kalman *et al.*, 1969) produces a minimal solution  $(A, C, \bar{C})$  which is unique up to a similarity transformation.

As first pointed out by Zeiger and McEwen (1974), the minimal factorization on which the Ho–Kalman procedure is based may be performed

by “Singular-value decomposition”, thereby fixing  $(A, C, \bar{C})$  uniquely; see also Kung (1978). In fact, the Hankel matrix  $H$  may be factored as

$$H = U\Sigma V' \quad U'U = I = V'V, \tag{14}$$

where  $\Sigma$  is the square  $n \times n$  diagonal matrix of the *nonzero* singular values taken in decreasing order. Setting  $\Omega := U\Sigma^{1/2}$  and  $\bar{\Omega} := V\Sigma^{1/2}$  this leads to a factorization

$$H = \Omega\bar{\Omega}' \quad \Omega'\Omega = \Sigma = \bar{\Omega}'\bar{\Omega} \tag{15}$$

of the type (5). Then a minimal realization  $(A, C, \bar{C})$  is obtained by solving

$$\Omega A \bar{\Omega}' = \sigma(H), \quad C \bar{\Omega}' = \rho_1(H), \quad \bar{C} \Omega' = \rho_1(H'),$$

where  $\sigma(H)$  is the shifted Hankel matrix (13) and  $\rho_1(H)$  is the first block row of  $H$ . It follows that the triplet  $(A, C, \bar{C})$  must be given by

$$A = \Sigma^{-1/2} U' \sigma(H) V \Sigma^{-1/2}, \tag{16}$$

$$C = \rho_1(H) V \Sigma^{-1/2}, \tag{17}$$

$$\bar{C} = \rho_1(H') U \Sigma^{-1/2}, \tag{18}$$

a form to which we refer as “finite-interval balanced”, since it is balanced in the sense that  $\Omega'\Omega$  and  $\bar{\Omega}'\bar{\Omega}$  are both equal to  $\Sigma$ , and that

$$\Omega = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-1} \end{bmatrix}, \quad \bar{\Omega} = \begin{bmatrix} \bar{C} \\ \bar{C}A' \\ \vdots \\ \bar{C}(A')^{j-1} \end{bmatrix}. \tag{19}$$

Aoki (1990) has proposed that this procedure be used also for identification of time series. The problem with such a strategy is that this algorithm is a deterministic realization procedure and hence does not a priori insure that (6) is positive real, or even stable for that matter, even if the Toeplitz matrix  $T_{\nu}$  is positive definite. In fact, it is shown in Byrnes and Lindquist (1982) that there are open subsets of the space of covariance data (1) for which  $A$  is not stable, and a fortiori the same holds for positivity. In fact, like that in van Overshee and De Moor (1993), the procedure in Aoki (1990) is based on the following hidden assumption which is not entirely natural.

*Assumption 1.* The covariance data (1) can be generated exactly by some (unknown) stochastic system of dimension equal to rank  $H$ .

Therefore, not only must we know that there exists an underlying finite-dimensional system, but we must also have some upper bound for its dimension. A conservative upper bound which will always suffice is  $\lfloor \frac{\nu}{2} \rfloor$ .

Is this assumption natural? If the covariance data are really generated exactly from a “true” stochastic

\* We warn the reader that a preliminary version of Lindquist and Picci (1994a), containing some erroneous statements, was accidentally published in place of the paper finally submitted for publication. The correct version can be obtained from the authors.

system and there is a reliable estimate of its order which is no more than half of the length of the covariance sequence, then the assumption will hold. However, and this is an important point of this paper, one cannot expect Assumption 1 to hold for an arbitrary covariance sequence (1).

To clarify this point, let us agree to call  $\{\Lambda_0, \Lambda_1, \Lambda_2, \dots\}$  a “minimal rational extension” of  $\{\Lambda_0, \Lambda_1, \dots, \Lambda_\nu\}$  if the rational function (7) has minimal degree. By definition this is the algebraic degree of  $\{\Lambda_0, \Lambda_1, \dots, \Lambda_\nu\}$ . A rational extension is called *positive* if, for every  $\mu > \nu$ , the block Toeplitz matrices  $T_\mu$  formed from the corresponding infinite sequence (4) are positive definite. An extension with this property is called a “positive rational extension”. It is well known that the extension  $\{\Lambda_0, \Lambda_1, \Lambda_2, \dots\}$  is positive if and only if (7) is *positive real*, i.e. the rational function  $Z(z)$  is analytic in the closed unit disc and the matrix function

$$\Phi(z) = Z(z) + Z(1/z)' \quad (20)$$

is nonnegative definite on the unit circle, making  $\Phi$  a spectral density matrix. A *minimal* positive rational extension of the finite sequence (1) is one for which the dimension of the triplet  $(A, C, \bar{C})$  in (6) is as small as possible.

**Definition 1.** The *positive degree*  $p$  of the finite covariance sequence  $\{\Lambda_0, \Lambda_1, \dots, \Lambda_\nu\}$  is the dimension of any minimal positive extension.

A well-known example of a positive extension is the “maximum entropy extension” (Whittle, 1963) corresponding to the spectral density  $\Phi(z) := W(z)W(1/z)'$ , where the spectral factor  $W(z)$  is (modulo a multiplicative constant matrix) the inverse of the Levinson-Szegö matrix polynomial of order  $\nu$  corresponding to the finite covariance sequence (1). Since the rational function  $W(z)$  generically has the McMillan degree equal to  $m\nu$ , it follows from spectral factorization theory (Anderson, 1958) that  $Z(z)$  has also degree  $m\nu$ . Consequently, the positive degree  $p$  is bounded from below by the algebraic degree  $r$  and from above by  $m\nu$ .

As already pointed out, it is very common in the literature (Aoki (1990), van Overshee and De Moor (1993) and others) to disregard the positivity constraint and to use algebraic rather than positive extensions, usually computed by minimal factorization a block Hankel matrix such as (5), or by methods which in principle are equivalent to this, even if the Hankel matrix is not explicitly computed. In fact, Assumption 1 may also be formulated in the following way.

**Assumption 2.** The positive degree of (1) equals the algebraic degree.

This assumption prescribes a property of the covariance sequence (1) which is not generic. We can illustrate this point by considering the rational extension problem for a finite scalar covariance sequence (1). The positive degree  $p$  lies between the algebraic degree  $r$  and  $\nu$ . Note that neither the case  $p = \nu$  nor the case  $p < \nu$  are “rare events”, because there are open sets of covariance sequences (1) of both categories. In fact, it was shown in Byrnes and Lindquist (1996) that for each  $\mu$  such that  $\frac{\nu}{2} \leq \mu \leq \nu$  there is an open set of covariance data in  $\mathbb{R}^\nu$  for which  $p = \mu$ . If the upper limit  $p = \nu$  is attained there are infinitely many nonequivalent minimal triplets  $(A, C, \bar{C})$  providing a positive extension, one of which is the maximum entropy extension. In fact, it can be shown that these  $\nu$ -dimensional extensions form an Euclidean space (Byrnes and Lindquist, 1989). This shows that the finite data (1) never contains enough information to establish a “true” underlying system. A similar statement can be made in the case when  $p < \nu$ .

**Example 1.** Consider the case  $m = 1$  and  $\nu = 2$ , i.e. consider a scalar partial covariance sequence  $\{\Lambda_0, \Lambda_1, \Lambda_2\}$ . If  $\Lambda_1 = \Lambda_2 = 0$ , we have  $r = p = 0$ . Otherwise, we always have  $r = 1$ , whereas the positive degree can be either one or two. In fact, setting  $\gamma_0 := \Lambda_1/\Lambda_0$  and  $\gamma_1 := (\Lambda_1^2 + \Lambda_2)/(1 - \Lambda_1^2)$ , it can be shown (Georgiou (1987); also see Byrnes and Lindquist (1996), where other examples are also given) that  $p = 1$  if and only if

$$|\gamma_1| < \frac{|\gamma_0|}{1 + |\gamma_0|}$$

and  $p = 2$  otherwise.

In fact, it is not hard to construct examples for which the gap between algebraic and positive rank is arbitrarily large, as the following theorem shows.

**Theorem 3.** Let  $n \in \mathbb{Z}_+$  be fixed. Then for an arbitrarily large  $\nu$  there is a stable rational function  $Z(z)$  of degree  $n$ , such that the Toeplitz matrix  $T_\nu$  formed as in (8) from the coefficients of the Laurent expansion (7), is positive definite while  $T_{\nu+1}$  is indefinite.

Consequently, you cannot test the positivity of a rational extension of (1) by checking a finite Toeplitz matrix, however large is its dimension. The proof of Theorem 3 is given in Appendix A.

Let us now return to the identification procedure of Aoki (1990). In practice the rank of  $H$  will always be full, and to compute a partial realization of reasonable dimension the basic idea is to partition  $\Sigma$  as

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \quad (21)$$

where the singular values in  $\Sigma_2$  are smaller than those in  $\Sigma_1$ , perhaps close to zero, and then take  $\Sigma_2 = 0$  so that  $H$  is approximated by

$$H_1 = U \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} V' = U_1 \Sigma_1 V_1'. \quad (22)$$

The matrix  $H_1$  is a best approximation (given the rank) of  $H$  in (the induced)  $\ell^2$ -norm, but it is in general not Hankel and hence cannot be used to determine a reduced order system. Of course, one may instead use Hankel-norm approximation (Adamjan *et al.*, 1971), which produces another best approximation of  $H$  in  $\ell^2$ -norm that is Hankel and has the same rank as  $H_1$ . However, if  $\Sigma_2$  is "very small" compared to  $\Sigma_1$ , then  $H_1$  is close to  $H$  and hence approximately Hankel. For this reason, Aoki's (1990) procedure is based on the original data  $H$  and  $\sigma(H)$ . Thus identifying  $H_1$  with  $H$  in (22) and noting that  $U_1 U_1' = I$  and  $V_1 V_1' = I$ , the same type of calculation as above yields the reduced triplet  $(A_r, C_r, \bar{C}_r)$  given by

$$A_r = \Sigma_1^{-1/2} U_1' \sigma(H) V_1 \Sigma_1^{-1/2}, \quad (23)$$

$$C_r = \rho_1(H) V_1 \Sigma_1^{-1/2}, \quad (24)$$

$$\bar{C}_r = \rho_1(H') U_1 \Sigma_1^{-1/2}. \quad (25)$$

It is not hard to see, and it is shown in Aoki (1990), that (23)–(25) is a "principal subsystem truncation in the sense that, if  $H$  is produced by a finite-dimensional system with  $(A, C, \bar{C})$  having finite-interval balanced form (16)–(18), we have

$$A_r = A_{11}, \quad C_r = C_1, \quad \bar{C}_r = \bar{C}_1, \quad (26)$$

where

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \\ C &= \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \\ \bar{C} &= \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix}. \end{aligned} \quad (27)$$

In fact, since  $U_1 U_1' = V_1 V_1' = [I, 0]$ , this is seen by merely solving (16)–(18) for  $\sigma(H)$ ,  $\rho_1(H)$  and  $\rho_1(H')$  and inserting in (23)–(25).

However, it must be shown that (26) corresponds to a stochastic system, i.e. that

$$Z_1(z) = C_1(zI - A_{11})^{-1} \bar{C}_1' + \frac{1}{2} \Lambda_0 \quad (28)$$

is positive real, provided of course that  $Z$ , defined by (6), is positive real. The question of stability was answered in the affirmative in Pernebo and Silverman (1982) and is addressed in Aoki (1990). The crucial question of positivity, however, is not discussed in Aoki (1990) and its validity is in doubt.

Positivity will, however, be proven for a somewhat modified procedure described below.

In fact, following Akaike (1975), Desai and Pal (1984) and Desai *et al.* (1985), instead of  $H$  we shall consider a normalized Hankel matrix

$$\hat{H} = L_+^{-1} H L_-^{-T}, \quad (29)$$

where  $L_-$  and  $L_+$  are lower triangular Cholesky factors of the Toeplitz matrices  $T_-$  and  $T_+$  of (1) and the corresponding sequence of transposed covariances respectively; see Section 4 below. This is also the Hankel matrix considered in van Overshee and De Moor (1993). Taking the singular value decomposition of  $\hat{H}$  instead of  $H$ , the singular values become the "canonical correlation coefficients", i.e. the cosines of the angles between the past and the future of the process  $y$ . The systems matrices can be determined in a manner analogous to (16)–(18), but now

$$\Omega' T_+^{-1} \Omega = \hat{\Sigma} = \bar{\Omega}' T_-^{-1} \bar{\Omega} \quad (30)$$

instead of (15) so the realization is not balanced in the same (deterministic) way as above. To see this, consider the singular value decomposition  $\hat{H} = \hat{U} \hat{\Sigma} \hat{U}'$  so that  $H = (L_+ \hat{U}) \hat{\Sigma} (L_- \hat{U}')'$ . Since  $H = \Omega \bar{\Omega}'$  and this factorization is unique modulo coordinate transformation in state space, we may take  $\Omega = L_+ \hat{U} \hat{\Sigma}^{1/2}$  and  $\bar{\Omega} = L_- \hat{U} \hat{\Sigma}^{1/2}$ . Then, (30) follows from  $\hat{U}' \hat{U} = I = \hat{V}' \hat{V}$ . As we shall see next, (30) corresponds to a more natural type of balancing corresponding to a Hankel operator describing the interface between the past and the future of the time series  $y$ .

### 3. STOCHASTIC REALIZATION THEORY IN THE HILBERT SPACE OF A SAMPLE FUNCTION

In this section we introduce a mathematical framework which is suitable for the identification problem described above. We define a Hilbert space of observed (infinite) strings of data  $\{y_t\}$ . This framework turns out to be isomorphic to that of geometric stochastic realization theory, thus allowing us to use the geometric concepts and machinery of linear stochastic system theory as developed in Lindquist and Picci (1985) and Lindquist and Picci (1991) also for the statistical problem of identification. In this way we also establish a correspondence which converts operations on random quantities defined on abstract probability spaces into prototypes of statistical algorithms involving computations based on the observed data.

In identification we have access only to a finite string of data

$$\{y_0, y_1, y_2, \dots, y_T\}. \quad (31)$$

Here  $T$  may be quite large but, of course, always finite. To begin with, we shall, however, consider

the idealized situation that we are given a doubly infinite sequence of  $m$ -dimensional data

$$\{\dots, y_{-3}, y_{-2}, y_{-1}, y_0, y_1, y_2, y_3, \dots\} \quad (32)$$

together with a corresponding covariance sequence  $\{\Lambda_k\}_{k \geq 0}$ , each matrix  $\Lambda_k$  of the sequence being computed from the data (32) by an ergodic limit of the type (11). In Section 5 we then modify the theory to handle the situation of finite data (31).

For each  $k \in \mathbb{Z}$  define the  $m \times \infty$  matrix

$$y(t) := [y_t, y_{t+1}, y_{t+2}, \dots] \quad (33)$$

and consider the sequence  $y := \{y(t)\}_{t \in \mathbb{Z}}$ . This object will be referred to as the  $m$ -dimensional *stationary time series* constructed from the data (32). The space  $Y$  of all finite linear combinations

$$\sum a'_k y(t_k); \quad a_k \in \mathbb{R}^m, t_k \in \mathbb{Z}$$

is a real vector space and can be equipped with an inner product defined by linear extension of the bilinear form

$$\begin{aligned} \langle a' y(k), b' y(j) \rangle &:= \lim_{T \rightarrow \infty} \frac{1}{T+1} \sum_{t=t_0}^{t_0+T} a' y_{t+k} y'_{t+j} b \\ &= a' \Lambda_{k-j} b, \end{aligned} \quad (34)$$

which clearly does not depend on  $t_0$ . This inner product is nondegenerate if the Toeplitz matrix  $T_k$ , constructed from the covariance data  $\{\Lambda_0, \Lambda_1, \dots, \Lambda_k\}$ , is a positive definite symmetric matrix for all  $k$ . Here we shall assume that the sequence  $\{T_k\}_{k \geq 0}$  is actually *coercive*, i.e.  $T_k > cI$  for some  $c > 0$  and all  $k \geq 0$ . (See Assumption 2 below for an alternative characterization.) We also define a "shift operator"  $U$  on the family of semi-infinite matrices (33), by setting

$$U a' y(t) = a' y(t+1) \quad t \in \mathbb{Z}, \quad a \in \mathbb{R}^m,$$

defining a linear map which is isometric with respect to the inner product (34) and extendable by linearity to all of  $Y$ . In particular, the sequence of matrices  $\{y(t)\}$  corresponding to the time series  $y$  is propagated in time by the action of the operator  $U$ , i.e.

$$y_i(t) = U^t y_i(0), \quad i = 1, 2, \dots, m, \quad t \in \mathbb{Z}, \quad (35)$$

where  $y_i$  denotes the  $i$ th row component of  $y$ . Then, closing the vector space  $Y$  in the inner product (34), we obtain a Hilbert space  $H(y) := \text{cl } Y$ . The shift operator  $U$  is extended by continuity to all of  $H(y)$  and is a unitary operator there.

As explained in more detail in Appendix B, this Hilbert space framework is isomorphic to the one described in Lindquist and Picci (1985) and Lindquist and Picci (1991), and hence all results in the geometric theory of stochastic realization can

be carried over to the present framework by merely identifying the time series  $y$  with a stationary stochastic process  $y$ . In particular, the subspaces  $H^-$  and  $H^+$  of  $H(y)$  generated by the elements (33) for  $t < 0$  and  $t \geq 0$ , respectively, can be regarded as the past and future subspaces of the stationary process  $y$ . For reasons of uniformity of notation the inner product (34) will also be denoted

$$\langle \xi, \eta \rangle = E \{ \xi \eta' \} \quad (36)$$

as the frameworks are completely equivalent. Here we allow  $E \{ \cdot \}$  to operate on matrices of time series, taking inner products component-wise. Moreover, the coercivity condition introduced above insures that  $\cap_{t \in \mathbb{Z}} U^t H^- = 0$  and  $\cap_{t \in \mathbb{Z}} U^t H^+ = 0$ , i.e.  $y$  is a purely nondeterministic sequence.

As we have pointed out above, the subspace identification methods of Aoki (1990) and van Overshee and De Moor (1993) are based on the assumption that the available data is generated by an underlying stochastic system of finite dimension. More specifically, using the notations introduced above, we assume that the data are generated by a linear system of the type

$$\begin{cases} x(t+1) = Ax(t) + Bw(t), \\ y(t) = Cx(t) + Dw(t) \end{cases} \quad (37)$$

defined for all  $t \in \mathbb{Z}$ , where  $w$  is some vector-valued normalized *white noise* time series\* (say, of dimension  $p$ ), and  $(A, B, C, D)$  are constant matrices with  $A$  a stability matrix. Throughout this paper we shall assume (without restriction) that  $(A, B, C)$  is a minimal triplet and that the matrix  $\begin{bmatrix} B \\ D \end{bmatrix}$  has linearly independent columns.

The system is assumed to be in statistical steady state so that the  $n$ -dimensional state  $x$  and the  $m$ -dimensional output  $y$  are uniquely defined by (37) as linear causal functionals of the past input  $w$ . This clearly implies that  $x$  and  $y$  are jointly stationary time series so that in particular, the cross-covariance matrices of  $x(t)$  and  $y(s)$  will depend only on the difference  $t - s$ . We shall think of the system (37) as a representation of the output time series  $y$ . The state and input variables  $x$  and  $w$  are introduced in order to display the special structure of the dynamic model of  $y$  and are by no means unique. Such a representation is called a state space realization of  $y$ .

*Remark 1.* Despite the fact that the model (37) is defined in terms of sample sequences, all equalities must be understood in the sense of Hilbert space metric, just as in the case of models based on random variables.

\* This means that  $E\{w(t)w(s)'\} = I\delta_{t,s}$  where  $\delta_{t,s}$  is the Kronecker delta.

The number of state variables  $n$  is called the *dimension* of the realization. A realization is *minimal* if there is no other realization of  $y$  of smaller dimension. In this case the covariance matrix of the state vector,

$$P = E\{x(t)x(t)'\} \tag{38}$$

is positive definite. Moreover as the matrix  $\begin{bmatrix} B \\ D \end{bmatrix}$  is taken with linearly independent columns, the number of (scalar) white noise inputs  $p$  is also as small as possible. Clearly, the covariance sequence  $\{\Lambda_0, \Lambda_1, \Lambda_2, \dots\}$  of the output  $\{y(t)\}$  of a minimal model (37) is a rational sequence of degree  $n$ , i.e. represented as

$$\begin{cases} \Lambda_k = CA^{k-1}\bar{C}' & k = 0, 1, 2, \dots, \\ \text{where } \bar{C}' = APC' + BD', \\ \Lambda_0 = CPC' + DD'. \end{cases} \tag{39}$$

In the following we shall need to assume that the corresponding spectral density  $\Phi(z)$  satisfies the following condition.

*Assumption 4.* The spectral density  $\Phi$  of the output process of the underlying system (37) is *coercive* in the sense that

$$\Phi(e^{i\theta}) > 0 \quad \text{for all } \theta \in [0, 2\pi]. \tag{40}$$

In particular,  $y$  is a *full-rank* process, i.e. its components are linearly independent sequences. Recall that a positive real function  $Z$  such that  $\Phi(z) := Z(z) + Z(z^{-1})'$  satisfies (40) is called strictly positive real.

Let  $H(w)$  be the Hilbert space generated by  $w$ , i.e. the closure of the linear space spanned by the family  $\{w_i(t), i = 1 \dots p, t \in \mathbb{Z}\}$  with respect to the metric induced by the inner product  $\langle \xi, \eta \rangle = E\{\xi\eta\}$ , where  $E\{\cdot\}$  is defined by (36). Let  $H^+$  and  $H^-$  be the subspaces of  $H(w)$  generated by the components of future  $\{y(0), y(1), y(2) \dots\}$  and past outputs  $\{y(-1), y(-2), y(-3) \dots\}$ , respectively.

The subspace

$$X := \{a'x(0) \mid a \in \mathbb{R}^n\} \tag{41}$$

is invariant under coordinate changes of the type  $(A, B, C) \rightarrow (TAT^{-1}, TB, CT^{-1})$  and is a coordinate-free representation of the realization (37). Such an object is called a Markovian splitting subspace in Lindquist and Picci (1985) and Lindquist and Picci (1991). Next, define the stationary Hankel operator of  $y$ ,  $\mathcal{H} : H^+ \rightarrow H^-$  as

$$\mathcal{H} := E^{H^-} |_{H^+}, \tag{42}$$

where  $E^{H^-} \lambda$  is the orthogonal projection of  $\lambda$  onto  $H^-$ . The splitting subspace property of  $X$  is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} H^+ & \xrightarrow{\mathcal{H}} & H^- \\ \mathcal{O}^* \searrow & & \nearrow C \\ & X & \end{array}$$

i.e. to the factorization

$$\mathcal{H} = C\mathcal{O}^*, \tag{43}$$

where the operators  $\mathcal{O} := E^{H^+} |_X$  and  $C := E^{H^-} |_X$  are the observability and constructibility operators, respectively, relative to the splitting subspace  $X$ . It can be shown that the splitting subspace  $X$  is minimal if and only if  $\mathcal{O}$  and  $C$  are both injective (see, e.g. Lindquist and Picci, 1991).

The system (37) is a *forward* or causal realization of  $y$  in the sense that the subspace  $H^+(w)$ , generated by the future of  $w$ , is orthogonal to  $X$  and  $H^-$ , i.e. to the present state and past output. Corresponding to (37) there is another realization

$$\begin{cases} \bar{x}(t-1) = A'\bar{x}(t) + \bar{B}\bar{w}(t-1), \\ y(t-1) = \bar{C}\bar{x}(t) + \bar{D}\bar{w}(t-1), \end{cases} \tag{44}$$

which is *backward* or anticausal in the sense that the subspace  $H^-(\bar{w})$ , generated by the past of  $\bar{w}$ , is orthogonal to  $X$  and  $H^+$ . Like  $x(0)$ ,  $\bar{x}(0)$  is a basis in  $X$ , i.e.

$$X := \{a'\bar{x}(0) \mid a \in \mathbb{R}^n\}. \tag{45}$$

In fact,  $\bar{x}(0)$  is the *dual basis* of  $x(0)$  in the sense that  $E\{x(0)\bar{x}(0)'\} = I$ . Hence

$$\bar{P} = P^{-1} \quad \bar{x}(0) = P^{-1}x(0). \tag{46}$$

The particular notations used in (37) and (44) reflect the special meaning of the parameters  $(A, C, \bar{C})$ . Computing the covariance matrix of the output using the dual realizations (37) and (44), it is in fact readily seen that  $(A, C, \bar{C})$  is precisely a triplet realizing the positive real part (6) of the spectral density matrix  $\Phi(z)$  of the time series  $y$ . There are infinitely many minimal factorizations (43), one for each Markovian splitting subspace, but the basis in each state space  $X$  can be chosen so that the triplets  $(A, C, \bar{C})$  are the same for each minimal  $X$ . This is called a uniform choice of bases (Lindquist and Picci, 1991).

Important examples of minimal splitting subspaces are the *forward* and *backward* predictor spaces

$$X_- = E^{H^-} H^+ \quad X_+ = E^{H^+} H^-, \tag{47}$$

which are the orthogonal complements of the null space of the Hankel operator (42) and of its adjoint, respectively.

Fixing a uniform choice of bases, and thus the triplets  $(A, C, \bar{C})$ , the splitting subspace  $X_-$  has the forward stochastic realization

$$\begin{cases} x_-(t+1) = Ax_-(t) + B_-w_-(t), \\ y(t) = Cx_-(t) + D_-w_-(t), \end{cases} \tag{48}$$



with state covariance  $P_-$ , and  $X_+$  has the backward realization

$$\begin{cases} \bar{x}_+(t-1) = A' \bar{x}_+(t) + \bar{B}_+ \bar{w}_+(t-1), \\ y(t-1) = \bar{C} \bar{x}_+(t) + \bar{D}_+ \bar{w}_+(t-1), \end{cases} \quad (49)$$

with state covariance  $\bar{P}_+$ .

These two stochastic realizations will play an important role in what follows. In fact, an important interpretation of these realizations is that

$$x_-(t+1) = Ax_-(t) + B_- D_-^{-1} [y(t) - Cx_-(t)] \quad (50)$$

is the unique steady-state Kalman filter of any minimal realization (37) of  $y$  in the fixed uniform choice of bases. Moreover, if  $P_+$  is the state covariance matrix (38) corresponding to the forward counterpart of (49), i.e.  $P_+ = (\bar{P}_+)^{-1}$ , then

$$P_- \leq P \leq P_+ \quad (51)$$

for the state covariance of any minimal realization (37).

In the same way

$$\bar{x}_+(t-1) = A' \bar{x}_+(t) + \bar{B}_+ \bar{D}_+^{-1} [y(t-1) - C\bar{x}_+(t)]$$

is the backward steady-state Kalman filter of all minimal backward realizations (44), and

$$\bar{P}_+ \leq \bar{P} \leq \bar{P}_-$$

for an arbitrary backward minimal realization (44), where  $\bar{P}_-$  is the backward counterpart of  $P_-$ .

#### 4. CANONICAL CORRELATIONS AND BALANCED STOCHASTIC REALIZATION

In this section we characterize the properties of minimal factorizations of the (stationary) Hankel operator (42) of a time series admitting a finite-dimensional realization of the type (37). Equivalently, we study certain factorizations of the infinite Hankel matrix of the corresponding infinite covariance sequence  $\{\Lambda_0, \Lambda_1, \Lambda_2, \dots\}$ . Some portions of this section can be found in an equivalent but somewhat different setting in Section 2 of Desai *et al.* (1985). Here we need to recall the basic concepts and set notations. This will be done in the geometric framework of Section 3, thereby providing several new insights.

To obtain a convenient matrix representation of the Hankel operator  $\mathcal{H}$  we shall introduce orthonormal bases in  $H^-$  and  $H^+$ . To this end it will be useful to represent past and future outputs as infinite vectors in the form

$$y_- = \begin{bmatrix} y(-1) \\ y(-2) \\ y(-3) \\ \vdots \end{bmatrix}, \quad y_+ = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \end{bmatrix}. \quad (52)$$

Let  $L_-$  and  $L_+$  be the lower triangular Cholesky factors of the infinite block Toeplitz matrices

$$T_- := E\{y_- y_-'\} = L_- L_-', \quad T_+ := E\{y_+ y_+'\} = L_+ L_+' \quad (53)$$

and let

$$v := L_-^{-1} y_-, \quad \bar{v} := L_+^{-1} y_+ \quad (53)$$

be the corresponding orthonormal bases in  $H^-$  and  $H^+$ , respectively. Now, (39) implies that

$$\begin{aligned} H_\infty := E\{y_+ y_-'\} &= \begin{bmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 & \dots \\ \Lambda_2 & \Lambda_3 & \Lambda_4 & \dots \\ \Lambda_3 & \Lambda_4 & \Lambda_5 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ &= \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} \begin{bmatrix} \bar{C} \\ \bar{C}A' \\ \bar{C}(A')^2 \\ \vdots \end{bmatrix}', \end{aligned} \quad (54)$$

and therefore we have the following representation result, which can be found in Desai *et al.* (1985).

*Proposition 1.* Let  $y$  be realized by a finite dimensional model of the form (37). Then, in the orthonormal basis (53) the matrix representation of the Hankel operator  $\mathcal{H}$  is

$$\hat{H}_\infty = L_+^{-1} E\{y_+ y_-'\} L_-^{-T} = L_+^{-1} \Omega \bar{\Omega}' L_-^{-T}, \quad (55)$$

where

$$\Omega = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} \quad \text{and} \quad \bar{\Omega} = \begin{bmatrix} \bar{C} \\ \bar{C}A' \\ \bar{C}(A')^2 \\ \vdots \end{bmatrix}. \quad (56)$$

Note that, with a uniform choice of bases, we obtain the same matrix factorization (54) for  $H_\infty$ , irrespective of which  $X$  (i.e. which minimal realization of  $y$ ) is chosen.

Recall that the adjoint  $\mathcal{O}^*$  of the observability operator  $\mathcal{O}$  is defined as the unique linear operator  $H^+ \rightarrow X$  such that  $\langle \mathcal{O}\xi, \lambda \rangle = \langle \xi, \mathcal{O}^*\lambda \rangle$  for all  $\xi \in X$  and  $\lambda \in H^+$ . Orthogonality implies that

$$\langle E^{H^+} \xi, \lambda \rangle = \langle \xi, \lambda \rangle = \langle \xi, E^X \lambda \rangle,$$

and therefore  $\mathcal{O}^* = E^X|_{H^+}$ . In the same way, we see that  $C^* = E^X|_{H^-}$ . The finite-rank linear operators  $\mathcal{O}^*\mathcal{O}$  and  $C^*C$  are defined on  $X$  and are the coordinate-free representations of the observability and constructibility gramians. The splitting subspace  $X$  is observable if and only if  $\mathcal{O}^*\mathcal{O}$  is full rank and constructible if and only if  $C^*C$  is full rank. The following representations show that these gramians are related to  $P_-$  and  $\bar{P}_+$ , the state covariances of the forward and backward steady-state Kalman filters (Picci and Pinzoni, 1994).

*Proposition 2.* Let  $x(0)$  and  $\bar{x}(0)$  be the conjugate basis vectors in a minimal splitting subspace  $X$  as defined above. Then, in a uniform choice of bases,

$$\mathcal{O}^* \mathcal{O} a' \bar{x}(0) = a' \bar{P}_+ x(0) \tag{57}$$

and

$$C^* C a' x(0) = a' P_- \bar{x}(0), \tag{58}$$

i.e.  $C^* C$  and  $\mathcal{O}^* \mathcal{O}$  have matrix representations  $P_-$  and  $\bar{P}_+$ , respectively, independently of  $X$ .

*Proof.* It is shown in Lindquist and Picci (1991) that, since  $X$  is minimal,

$$E^{H^-} a' x(0) = a' x_-(0),$$

and therefore

$$C^* C a' x(0) = E^X a' x_-(0) = E^X a' P_- \bar{x}_-(0).$$

But, since the bases  $\bar{x}(0)$  and  $\bar{x}_-(0)$  are chosen uniformly

$$E^X a' \bar{x}_-(0) = a' \bar{x}(0) \quad a \in \mathbb{R}^n,$$

and consequently (58) follows. The proof of (57) is analogous.

The factorization (55) can also be derived from (43) and the following useful matrix representations of the observability and constructibility operators.

*Proposition 3.* Let  $x(0)$  and  $\bar{x}(0)$  be basis vectors for the minimal splitting subspace  $X$  given by (41) and (45). Then

$$\mathcal{O} a' \bar{x}(0) = a' \Omega' L_+^{-T} \bar{v}, \quad \mathcal{O}^* b' \bar{v} = b' L_+^{-1} \Omega x(0) \tag{59}$$

and

$$C a' x(0) = a' \bar{\Omega}' L_-^{-T} v, \quad C^* b' v = b' L_-^{-1} \bar{\Omega} \bar{x}(0), \tag{60}$$

where  $\Omega$  and  $\bar{\Omega}$  are given by (56).

*Proof.* Since, in view of (37)

$$y_+ = \Omega x(0) + \text{terms which are orthogonal to } X$$

and  $\bar{v} = L_+^{-1} y_+$ , we have

$$E \{ \bar{v} x(0)' \} = L_+^{-1} \Omega P. \tag{61}$$

Consequently, for any  $a \in \mathbb{R}^n$ , the usual projection formula \* yields

$$\mathcal{O} a' x(0) = E^{H^+} a' x(0) = a' E \{ x(0) \bar{v}' \} \bar{v}$$

and

$$\mathcal{O}^* b' \bar{v} = E^X b' \bar{v} = b' E \{ \bar{v} x(0)' \} P^{-1} x(0)$$

\* If  $\xi \in H(w)$  and the subspace  $Z \subset H(w)$  is spanned by the components of the full-rank random vector  $z$ , then  $E^Z \xi = E \{ \xi z' \} (E \{ z z' \})^{-1} z$ .

from which (59) follows. A symmetric argument yields (60).

To interpret this result in the context of balanced realization theory one should note that the matrix representations of  $\mathcal{O}^*$  and  $C^*$  are the transposes of those of  $\mathcal{O}$  and  $C$  if and only if  $x(0)$  is an orthogonal basis, i.e.  $P = \bar{P} = I$ . Moreover, it follows from (59) that

$$\mathcal{O}^* \mathcal{O} a' \bar{x}(0) = a' \Omega' T_+^{-1} \Omega x(0),$$

showing that  $\Omega' T_+^{-1} \Omega$  is a matrix representation of  $\mathcal{O}^* \mathcal{O}$ , in harmony with the analysis at the end of Section 2. In the same way, (60) yields

$$C^* C a' x(0) = a' \bar{\Omega}' T_-^{-1} \bar{\Omega} \bar{x}(0),$$

and hence  $\bar{\Omega}' T_-^{-1} \bar{\Omega}$  is a matrix representation of  $C^* C$ . Together with Proposition 2 this yields the following explicit formulas for  $P_-$  and  $\bar{P}_+$ :

$$\Omega' T_+^{-1} \Omega = \bar{P}_+, \quad \bar{\Omega}' T_-^{-1} \bar{\Omega} = P_- \tag{62}$$

Now, let  $\{\sigma_1, \sigma_2, \sigma_3, \dots\}$  be the singular values of the Hankel operator  $\mathcal{H}$ . Since  $\text{rank } \mathcal{H} = n$ ,  $\sigma_i = 0$  for  $i > n$ . The nonzero singular values

$$1 \geq \sigma_1 \geq \sigma_2 \geq \sigma_3 \dots \geq \sigma_n > 0 \tag{63}$$

are the cosines of the angles between the subspaces  $H_-$  and  $H_+$ ; they are known as the *canonical correlation coefficients* of  $y$  (Hotelling, 1936; Anderson, 1958). Obviously  $\sigma_1 < 1$  if and only if  $H_- \cap H_+ \neq 0$ . The squares of the canonical correlation coefficients are the eigenvalues of  $\mathcal{H}^* \mathcal{H}$ , i.e.

$$\mathcal{H}^* \mathcal{H} \xi_i = \sigma_i^2 \xi_i,$$

which, in view of (43) may be written

$$\mathcal{O}^* \mathcal{O} C^* C (\mathcal{O}^* \xi_i) = \sigma_i^2 (\mathcal{O}^* \xi_i),$$

and, therefore, as was also demonstrated in Picci and Pinzoni (1994)

$$\lambda \{ \mathcal{O}^* \mathcal{O} C^* C \} = \{ \sigma_1^2, \sigma_2^2, \dots, \sigma_n^2 \}, \tag{64}$$

i.e.  $\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2$  are the eigenvalues of  $\mathcal{O}^* \mathcal{O} C^* C$ . But, in view of Proposition 2, this is precisely the coordinate-free version of the invariance condition

$$\{ \sigma_1^2, \sigma_2^2, \dots, \sigma_n^2 \} = \lambda \{ P_- \bar{P}_+ \} \tag{65}$$

of Desai and Pal (1984).

This suggests that an appropriate uniform choice of bases would be the one that makes  $P_-$  and  $\bar{P}_+$  equal and equal to the diagonal matrix of nonzero canonical correlation coefficients.

In fact, in view of Proposition 1, the infinite normalized Hankel matrix  $\bar{H}_\infty$  is the matrix representation of the operator  $\mathcal{H}$  in the orthonormal bases

(53). Therefore,  $\hat{H}_\infty$  has the singular-value decomposition

$$\hat{H}_\infty = U_\infty \Sigma_\infty V_\infty' = U \Sigma V', \quad (66)$$

where  $\Sigma$  is the diagonal  $n \times n$  matrix consisting of the canonical correlation coefficients

$$\Sigma = \text{diag}\{\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n\}, \quad (67)$$

and  $\Sigma_\infty$  is the infinite matrix

$$\Sigma_\infty = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}.$$

Moreover  $U_\infty$  and  $V_\infty$  are infinite orthogonal matrices, and  $U$  and  $V$  are  $\infty \times n$  submatrices of  $U_\infty$  and  $V_\infty$  with the property that

$$U'U = I = V'V. \quad (68)$$

We now rotate the orthonormal bases (53) in  $H^+$  and  $H^-$  to obtain  $u := U_\infty' \bar{v}$  and  $v := V_\infty' v$ , respectively. Note that  $E\{uv'\} = \Sigma_\infty$ . What makes these orthonormal bases useful is that they are adapted to the orthogonal decomposition \*

$$H^- \vee H^+ = [H^- \cap (H^+)^{\perp}] \oplus H^\square \oplus [H^+ \cap (H^-)^{\perp}], \quad (69)$$

where  $H^\square := X_- \vee X_+$  is the so-called *frame space* (Lindquist and Picci, 1985; Lindquist and Picci, 1991), in the sense that

$$X_- = \text{span}\{v_1, v_2, \dots, v_n\} \\ X_+ = \text{span}\{u_1, u_2, \dots, u_n\}.$$

This is true since  $X_-$  is precisely the subspace of random variables in  $H^-$  having nonzero correlation with the future  $H^+$  and, dually,  $X_+$  is the subspace of random variables in  $H^+$  having nonzero correlation with the past  $H^-$ . Since therefore  $\{v_{n+1}, v_{n+2}, v_{n+3}, \dots\}$  and  $\{u_{n+1}, u_{n+2}, u_{n+3}, \dots\}$  span  $H^- \cap (H^+)^{\perp}$  and  $H^+ \cap (H^-)^{\perp}$ , respectively, these spaces will play no role in what follows.

Now define the  $n$ -dimensional vectors

$$z = \begin{bmatrix} \sigma_1^{1/2} v_1 \\ \sigma_2^{1/2} v_2 \\ \vdots \\ \sigma_n^{1/2} v_n \end{bmatrix} = \Sigma^{1/2} V' L_-^{-1} y_-, \\ \bar{z} = \begin{bmatrix} \sigma_1^{1/2} u_1 \\ \sigma_2^{1/2} u_2 \\ \vdots \\ \sigma_n^{1/2} u_n \end{bmatrix} = \Sigma^{1/2} U' L_+^{-1} y_+. \quad (70)$$

From what we have seen before,  $z$  is a basis in  $X_-$  and  $\bar{z}$  is a basis in  $X_+$ , and they have the properties

$$E\{zz'\} = \Sigma = E\{\bar{z}\bar{z}'\}. \quad (71)$$

\* The symbols  $\vee$  and  $\oplus$  denote vector sum and orthogonal vector sum of subspaces.

In fact, we even have more as seen from the following amplification <sup>†</sup> of a theorem by Desai and Pal (1984, Theorem 1).

*Theorem 5.* The basis vectors

$$x_-(0) = z, \quad \bar{x}_+(0) = \bar{z} \quad (72)$$

in  $X_-$  and  $X_+$ , respectively, belong to the same uniform choice of basis, i.e. to the same choice of triplets  $(A, C, \bar{C})$ , and in this uniform choice

$$P_- = \Sigma = \bar{P}_+. \quad (73)$$

If the canonical correlation coefficients  $\{\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n\}$  are distinct, this is, modulo multiplication with a signature matrix <sup>‡</sup>, the only uniform choice of bases for which (73) holds.

Such a choice of  $(A, C, \bar{C})$  is known as “stochastically balanced”, and, in the case of distinct canonical correlation coefficients, it defines a canonical form with respect to state space isomorphism in (6) by fixing the sign in, say, the first element in each row of  $C$ . Such canonical forms have also been studied by Ober (1991).

*Proof.* It follows from (55) and (66) that

$$E\{\bar{z}z'\} = \Sigma^2. \quad (74)$$

Now, choose  $(A, C, \bar{C})$  so that  $\bar{x}_+(0) = \bar{z}$ , and let the bases in the other splitting subspaces be chosen accordingly so that the choice of bases is uniform. We want to show that  $x_-(0) = z$ . To this end, first note that  $x_+(0) = \Sigma^{-1} \bar{x}_+(0)$  and that  $x_-(0) = E^{X_-} x_+(0)$ ; see Lindquist and Picci (1991). Then, by usual projection formula and the fact that  $z$  is a basis in  $X_-$

$$x_-(0) = \Sigma^{-1} E\{\bar{z}z'\} \Sigma^{-1} z,$$

which, in view of (74), yields  $x_-(0) = z$  as claimed. Hence (73) follows from (71).

Next, suppose that  $(QAQ^{-1}, CQ^{-1}, \bar{C}Q')$  is another uniform choice of bases which is also stochastically balanced. Since then  $x_-(0) = Qz$  and, as is readily seen from the backward system (44),  $\bar{x}_+(0) = Q^{-T} \bar{z}$  so that  $P_- = Q\Sigma Q'$  and  $\bar{P}_+ = Q^{-T} \Sigma Q^{-1}$ , (73) yields

$$Q\Sigma Q' = \Sigma \quad \text{and} \quad Q^{-T} \Sigma Q^{-1} = \Sigma,$$

from which we have

$$Q\Sigma^2 = \Sigma^2 Q.$$

Since  $\Sigma$  has distinct entries, it follows from Corollary 2, p. 223 in Gantmacher (1959) that there is

<sup>†</sup> A priori there is no reason why choosing bases in  $X_-$  and  $X_+$  would lead to the same  $(A, C, \bar{C})$ . This important property is explicitly mentioned in Theorem 5.

<sup>‡</sup> A signature matrix is a diagonal matrix of  $\pm 1$ .

a scalar polynomial  $\varphi(z)$  such that  $Q = \varphi(\Sigma^2)$ . Hence  $Q$  is diagonal and commutes with  $\Sigma$  so that, by  $Q\Sigma Q' = \Sigma$ , we have

$$QQ' = I.$$

Consequently, since  $Q$  is diagonal, it must be a signature matrix.

In view of (72) and (46), the first of relations (60) and (59), respectively, yield

$$z = \bar{\Omega}' T_-^{-1} y_- \quad \bar{z} = \Omega' T_+^{-1} y_+. \quad (75)$$

Consequently, in view of (71), (30) holds also for the case of an infinite Hankel matrix. This can, of course, also be seen from (62).

Note that the normalization of the block Hankel matrix  $H_\infty$  is necessary in order for the singular values to become the canonical correlation coefficients, i.e. the singular values of  $\mathcal{H}$ . In fact, if we were to use the unnormalized matrix representation (54) of  $\mathcal{H}$  instead, as may seem simpler and more natural, the transpose of (54) would not be the matrix representation of  $\mathcal{H}^*$  in the same bases, a property which is crucial in the singular value decomposition above. This is because (54) corresponds to the bases  $y_-$  in  $H^-$  and  $y_+$  in  $H^+$ , which are not orthogonal. As we shall see in the next section, this holds also in applicable parts for the finite-dimensional case studied in Section 2, and therefore the normalized Hankel matrix  $\hat{H}$ , defined in Section 2, is preferable to the unnormalized  $H$ .

Formulas, such as (16)–(18), expressing  $A, C, \bar{C}$  in terms of the Hankel matrix  $H_\infty$ , can be easily derived from basic principles. In fact, standard calculations based on the forward model (37) and the backward model (44) yield

$$A = E\{x(1)x(0)'\}P^{-1}, \quad (76)$$

$$C = E\{y(0)x(0)'\}P^{-1}, \quad (77)$$

$$\bar{C} = E\{y(-1)\bar{x}(0)'\}\bar{P}^{-1} = E\{y(-1)x(0)'\} \quad (78)$$

for any dual pair of bases  $x(0)$  and  $\bar{x}(0)$ .

*Proposition 4.* The triplet (76)–(78) corresponding to the stochastically balanced bases (70) can be computed by means of the formulas

$$A = \Sigma^{-1/2}U'L_+^{-1}\sigma(H_\infty)L_-^T V \Sigma^{-1/2}, \quad (79)$$

$$C = \rho_1(H_\infty)L_-^T V \Sigma^{-1/2}, \quad (80)$$

$$\bar{C} = \rho_1(H'_\infty)L_+^{-T}U\Sigma^{-1/2}, \quad (81)$$

where  $H_\infty$  is the unnormalized Hankel matrix (54),  $\sigma(H_\infty)$  is obtained from  $H_\infty$  by deleting the first block row, and  $\rho_1(H_\infty)$  is the first block row.

*Proof.* First, in (76) and (77), we take  $x(0)$  to be  $x_-(0)$ . By the Kalman filter representation

$a'[x_+(1) - x_-(1)] \perp UH^- \supset H^-$  for all  $a \in \mathbb{R}^n$ , we have

$$\begin{aligned} E\{x_-(1)x_-(0)'\} &= E\{x_+(1)x_-(0)'\} \\ &= \bar{P}_+^{-1}E\{\bar{x}_+(1)x_-(0)'\}. \end{aligned}$$

But,  $(A, C, \bar{C})$  is stochastically balanced, and therefore, by Theorem 5 and (70),  $P_- = \Sigma = \bar{P}_+$ ,  $x_-(0) = \Sigma^{1/2}V'L_-^{-1}y_-$  and  $\bar{x}_+(1) = \Sigma^{1/2}U'L_+^{-1}\sigma(y_+)$ , where  $\sigma(y_+)$  is obtained from  $y_+$  by deleting the subvector corresponding to time  $t = 0$ . Consequently, in view of (76)

$$A = \Sigma^{-1/2}U'L_+^{-1}E\{\sigma(y_+)y_-\}'L_-^T V \Sigma^{-1/2},$$

which is identical to (79). Likewise, from (80)

$$C = E\{y(0)y_-\}'L_-^T V \Sigma^{-1/2},$$

which yields (80). Finally, taking  $\bar{x}(0)$  to be  $\bar{x}_+(0)$  in (78), a symmetric argument yields (81).

Note that (79)–(81) are obtained by applying the Ho–Kalman algorithm to  $H_\infty$  factorized corresponding to the singular-value decomposition (66).

#### 5. STOCHASTIC REALIZATION FROM FINITE COVARIANCE DATA

In this section we modify the realization theory of Section 4 to the case that only a finite segment

$$\{y(0), y(1), y(2), \dots, y(\nu)\}, \quad (82)$$

of the time series  $\{y(t)\}$  is available. We still define each  $y(t)$  as the semi-infinite string (33) of data, and therefore we can form, via the ergodic limit (11), an exact partial covariance sequence

$$\{\Lambda_0, \Lambda_1, \Lambda_2, \dots, \Lambda_\nu\}. \quad (83)$$

The corresponding realization problem, which is purely theoretical and is intended to prepare for the more realistic identification situation with finite strings of observed data (Section 6), is therefore the partial stochastic realization problem mentioned in Section 2. We retain the crucial Assumption 1, implying that the data (82) is the output of some minimal “true” system (37) of dimension  $n$  and that  $\nu$  is large enough for  $n$  to equal the positive degree of the partial sequence (83).

Now, suppose that  $\nu = 2\tau - 1$ , and partition the data into two matrices

$$y_\tau^- = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(\tau-1) \end{bmatrix}, \quad y_\tau^+ = \begin{bmatrix} y(\tau) \\ y(\tau+1) \\ \vdots \\ y(2\tau-1) \end{bmatrix}, \quad (84)$$

representing the past and the future, respectively, and define the corresponding (finite-dimensional) subspaces  $Y_\tau^-$  and  $Y_\tau^+$  spanned by the rows of  $y_\tau^-$

and  $y_\tau^+$ , respectively, as explained in Section 3. Since the data size  $\tau$  will be important in the considerations that will follow, we denote the finite block Hankel matrix  $H$  of Section 2, relative to the data (84), by  $H_\tau$ , i.e.

$$H_\tau = E\{y_\tau^+(y_\tau^-)'\}. \tag{85}$$

Let  $\tau_0$  be the smallest integer  $\tau$  such that  $\text{rank } H_\tau = n$ . It is well-known that  $\tau_0$  is the maximum of the observability and constructibility indices of  $(A, C, \bar{C})$ , so  $n$  is an upper bound for  $\tau_0$ . As pointed out in the beginning of Section 2, we need  $\tau > \tau_0$  to be certain that the factorization of  $H_\tau$  yields a unique  $(A, C, \bar{C})$ .

Next, we shall consider the class of minimal splitting subspaces for  $Y_\tau^-$  and  $Y_\tau^+$ , i.e. the subspaces  $X_\tau$  admitting a canonical factorization

$$\begin{array}{ccc} Y_\tau^+ & \xrightarrow{H_\tau} & Y_\tau^- \\ O_\tau^* \setminus & & / C_\tau \\ & X_\tau & \end{array}$$

of the finite-interval Hankel operator

$$\mathcal{H}_\tau := E^{Y_\tau^-} |_{Y_\tau^+}. \tag{86}$$

It is standard (Lindquist and Picci, 1985; Lindquist and Picci, 1991) to show that the forward and backward predictor spaces

$$\hat{X}_{\tau-} = E^{Y_\tau^-} Y_\tau^+ \quad \text{and} \quad \hat{X}_{\tau+} = E^{Y_\tau^+} Y_\tau^-$$

are such minimal splitting subspaces. The proof of the following theorem is deferred to Appendix D.

*Theorem 6.* Let  $X$  be a minimal Markovian splitting subspace for the stationary time series  $\{y(t)\}$ . Then, if  $\tau > \tau_0$

$$X_\tau := U^\tau X \tag{87}$$

is a minimal splitting subspace for  $Y_\tau^-$  and  $Y_\tau^+$ , and

$$\hat{X}_{\tau-} = E^{Y_\tau^-} X_\tau, \quad \hat{X}_{\tau+} = E^{Y_\tau^+} X_\tau. \tag{88}$$

Conversely, any basis  $\hat{x}(\tau)$  in  $\hat{X}_{\tau-}$  has a unique representation\*

$$\hat{x}(\tau) = E^{Y_\tau^-} x(\tau), \tag{89}$$

where  $x(\tau)$  is a basis in  $X_\tau$ , and any basis  $\hat{\bar{x}}(\tau)$  in  $\hat{X}_{\tau+}$  has a unique representation

$$\hat{\bar{x}}(\tau) = E^{Y_\tau^+} \bar{x}(\tau), \tag{90}$$

with  $\bar{x}(\tau)$  a basis in  $X_\tau$ . As  $X$  varies over the family  $\mathcal{X}$  of all minimal Markovian splitting subspaces, the corresponding  $x(0)$  [ $\bar{x}(0)$ ] constitute a uniform choice of bases.

The stochastic realizations corresponding to the finite-interval predictor spaces  $\hat{X}_{\tau-}$  and  $\hat{X}_{\tau+}$  are

\* With slight misuse of notations, the orthogonal projection operator applied to a vector will denote the vector of the projections of the components.

nonstationary. However, taking advantage of the representations (89) and (90), we shall be able to express these realizations in such a way that they can be parameterized by the stationary triplet  $(A, C, \bar{C})$  corresponding to one uniform choice of bases, both for the forward and the backward settings. In fact, if the bases  $\hat{x}(\tau)$  and  $\hat{\bar{x}}(\tau)$  are chosen so that  $x(\tau)$  and  $\bar{x}(\tau)$  in representations (89) and (90) are dual bases in  $X_\tau$ , i.e.  $E\{x(\tau)\bar{x}(\tau)\} = I$ , then the same choice of  $(A, C, \bar{C})$  is used for all  $X \in \mathcal{X}$ . Such a choice of bases in  $\hat{X}_{\tau-}$  and  $\hat{X}_{\tau+}$  is called *coherent*.

The realizations generated by these coherent bases are precisely the (transient) forward and backward Kalman filters. In fact, the vector  $\hat{x}(\tau)$  is the one-step predictor of  $x(\tau)$  based on  $Y_\tau^-$  and, as shown in Appendix C, it evolves in time as the Kalman filter

$$\begin{aligned} \hat{x}(t+1) &= A\hat{x}(t) + K(t)[y(t) - C\hat{x}(t)]; \\ \hat{x}(0) &= 0, \end{aligned} \tag{91}$$

where the gain  $K(t)$  is given by

$$K(t) = (\bar{C}' - AP_-(t)C')(\Lambda_0 - CP_-(t)C')^{-1} \tag{92}$$

and the filter estimate covariance

$$P_-(t) = E\{\hat{x}(t)\hat{x}(t)'\} \tag{93}$$

is the solution of the matrix Riccati equation

$$\begin{cases} P_-(t+1) = AP_-(t)A' + (\bar{C}' - AP_-(t)C') \\ \quad (\Lambda_0 - CP_-(t)C')^{-1}(\bar{C}' - AP_-(t)C')', \\ P_-(0) = 0. \end{cases} \tag{94}$$

Symmetrically, in terms of the backward system (44) corresponding to (37), the components of

$$\hat{\bar{x}}(\tau) = E^{Y_\tau^+} \bar{x}(\tau) \tag{95}$$

form a basis in  $\hat{X}_{\tau+}$  and are generated by the backward Kalman filter

$$\begin{aligned} \hat{\bar{x}}(t-1) &= A'\hat{\bar{x}}(t) + \bar{K}(t)[y(t-1) - \bar{C}\hat{\bar{x}}(t)], \\ \hat{\bar{x}}(2\tau-1) &= 0, \end{aligned} \tag{96}$$

with

$$\bar{K}(t) = (C' - A'\bar{P}_+(t)\bar{C}')(\Lambda_0 - \bar{C}\bar{P}_+(t)\bar{C}')^{-1}, \tag{97}$$

where

$$\bar{P}_+(t) = E\{\hat{\bar{x}}(t)\hat{\bar{x}}(t)'\} \tag{98}$$

is obtained by solving the matrix Riccati equation

$$\begin{cases} \bar{P}_+(t-1) = A'\bar{P}_+(t)A + (C' - A'\bar{P}_+(t)\bar{C}') \\ \quad (\Lambda_0 - \bar{C}\bar{P}_+(t)\bar{C}')^{-1}(C' - A'\bar{P}_+(t)\bar{C}')' \\ \bar{P}_+(2\tau-1) = 0. \end{cases} \tag{99}$$

Now, it is well known that both

$$v(t) = (\Lambda_0 - CP_-(t)C')^{-1/2}[y(t) - C\hat{x}(t)] \tag{100}$$

and

$$\tilde{v}(t) = (\Lambda_0 - \bar{C}\bar{P}_+(t)\bar{C}')^{-1/2}[y(t-1) - \bar{C}\hat{x}(t)] \tag{101}$$

are normalized white noises, called the forward and backward (*transient*) *innovation processes*, respectively. Consequently, we may write the Kalman filter (91) as

$$\begin{cases} \hat{x}(t+1) = A\hat{x}(t) + B_-(t)v(t), \\ y(t) = C\hat{x}(t) + D_-(t)v(t), \end{cases} \tag{102}$$

where  $D_-(t) := (\Lambda_0 - CP_-(t)C')^{1/2}$  and  $B_-(t) := K(t)D_-(t)$ . Likewise, the backward Kalman filter (91) may be written

$$\begin{cases} \hat{x}(t-1) = A'\hat{x}(t) + \bar{B}_+(t)\tilde{v}(t-1), \\ y(t-1) = \bar{C}\hat{x}(t) + \bar{D}_+(t)\tilde{v}(t-1), \end{cases} \tag{103}$$

where  $\bar{D}_+(t) := (\Lambda_0 - \bar{C}\bar{P}_+(t)\bar{C}')^{1/2}$  and  $\bar{B}_+(t) := \bar{K}(t)\bar{D}_+(t)$ . Comparing with (37) and (44), we see that (102) and (103) are stochastic realizations, which unlike (37) and (44) are time-varying and describe the output  $y$  only on the interval  $[0, 2\tau - 1]$ . In fact, since

$$P - P_-(t) = E\{[x(t) - \hat{x}(t)][x(t) - \hat{x}(t)]'\} \geq 0,$$

and, for the same reason,  $\bar{P} - \bar{P}_+(t) \geq 0$ , we have

$$P_-(t) \leq P \leq P_+(t) := \bar{P}_+(t)^{-1}, \tag{104}$$

so we see that the predictor spaces  $\hat{X}_{\tau-}$  and  $\hat{X}_{\tau+}$  are extremal splitting subspaces, just as  $X_-$  and  $X_+$  in (51).

It is now immediately seen that the finite-interval counterparts of equations (76)–(78) are given by

$$A = E\{\hat{x}(\tau+1)\hat{x}(\tau)'\}P_-(\tau)^{-1}, \tag{105}$$

$$C = E\{y(\tau)\hat{x}(\tau)'\}P_-(\tau)^{-1}, \tag{106}$$

$$\begin{aligned} \bar{C} &= E\{y(\tau-1)\hat{x}(\tau)'\}\bar{P}_+(\tau)^{-1} \\ &= E\{y(\tau-1)\hat{x}(\tau)'\}. \end{aligned} \tag{107}$$

In complete analogy with the stationary framework in Section 4, the canonical correlation coefficients

$$1 \geq \sigma_1(\tau) \geq \sigma_2(\tau) \geq \dots \geq \sigma_n(\tau) > 0 \tag{108}$$

between the finite past  $Y_{\tau-}$  and the finite future  $Y_{\tau+}$  are now defined as the singular values of the operator  $\mathcal{H}_{\tau}$  given by (86). To determine these we need a matrix representation of  $\mathcal{H}_{\tau}$  in some orthonormal bases. Using the pair (100)–(101) of transient innovation processes for this purpose, we obtain the *normalized* matrix (29), which we shall here denote  $\hat{H}_{\tau}$ . Singular value decomposition yields

$$\hat{H}_{\tau} = U_{\tau}\Sigma_{\tau}V_{\tau}', \tag{109}$$

where  $U_{\tau}U_{\tau}' = I = V_{\tau}V_{\tau}'$ , and  $\Sigma_{\tau}$  is the diagonal matrix of canonical correlation coefficients. As in Section 4 it is seen that

$$\begin{cases} z(\tau) = \Sigma_{\tau}^{1/2}V_{\tau}'(L_{\tau}^-)^{-1}y_{\tau}^-, \\ \bar{z}(\tau) = \Sigma_{\tau}^{1/2}U_{\tau}'(L_{\tau}^+)^{-1}y_{\tau}^+ \end{cases} \tag{110}$$

are bases in  $\hat{X}_{\tau-}$  and  $\hat{X}_{\tau+}$ , respectively, and that

$$E\{z(\tau)z(\tau)'\} = \Sigma_{\tau} = E\{\bar{z}(\tau)\bar{z}(\tau)\}. \tag{111}$$

Here  $L_{\tau}^-$  and  $L_{\tau}^+$  are the finite-interval counterparts of  $L_-$  and  $L_+$ , respectively, and they are of course submatrices of these. Note that  $H_{\tau}$ , as defined by (85), is now given by

$$H_{\tau} = L_{\tau}^+\hat{H}_{\tau}(L_{\tau}^-)'. \tag{112}$$

We observe that, in analogy to Theorem 5,  $z(\tau)$  and  $\bar{z}(\tau)$  are coherent bases, and the corresponding triplet  $(A, C, \bar{C})$  is a finite-interval stochastically balanced realization, i.e.

$$P_-(\tau) = \Sigma_{\tau} = \bar{P}_+(\tau). \tag{113}$$

The following finite-interval modification of Proposition 4 is essentially the canonical singular-value decomposition version of the Ho–Kalman algorithm applied to the finite block Hankel matrix  $H_{\tau}$ , and the proof is analogous.

*Proposition 5.* The finite-interval stochastically balanced triplet  $(A_{\tau}, C_{\tau}, \bar{C}_{\tau})$ , obtained from (105)–(107) by choosing the bases  $\hat{x}(\tau) = z(\tau)$  and  $\hat{\bar{x}}(\tau) = \bar{z}(\tau)$ , is given by

$$\begin{aligned} A_{\tau} &= \Sigma_{\tau}^{-1/2}U_{\tau}'(L_{\tau}^+)^{-1}\sigma(H_{\tau})(L_{\tau}^-)^{-T}V_{\tau}\Sigma_{\tau}^{-1/2}, \\ C_{\tau} &= \rho_1(H_{\tau})(L_{\tau}^-)^{-T}V_{\tau}\Sigma_{\tau}^{-1/2}, \\ \bar{C}_{\tau} &= \rho_1(H_{\tau}')(L_{\tau}^+)^{-T}U_{\tau}\Sigma_{\tau}^{-1/2}, \end{aligned} \tag{114}$$

where the operators  $\sigma(\cdot)$  and  $\rho_1(\cdot)$  are defined as in Section 2 and in Proposition 4.

Note that the triplet  $(A_{\tau}, C_{\tau}, \bar{C}_{\tau})$  actually varies with  $\tau$ , but that, for each  $\tau$ , it is similar to the stochastically balanced triplet  $(A, C, \bar{C})$  of Section 4, i.e. there is a nonsingular matrix  $Q_{\tau}$  so that

$$(A_{\tau}, C_{\tau}, \bar{C}_{\tau}) = (Q_{\tau}AQ_{\tau}^{-1}, CQ_{\tau}^{-1}, \bar{C}Q_{\tau}'). \tag{115}$$

It is easy to check that, in the uniform choice of bases corresponding (115), the stationary predictor spaces  $X_-$  and  $X_+$  will have the state covariances

$$P_- = Q_{\tau}\Sigma Q_{\tau}' \quad \text{and} \quad \bar{P}_+ = Q_{\tau}^{-T}\Sigma Q_{\tau}^{-1}, \tag{116}$$

analogously to the situation in the proof of Theorem 5. The fact that these state covariances are not diagonal and equal is a manifestation of the fact that the triplet  $(A_{\tau}, C_{\tau}, \bar{C}_{\tau})$  is not stochastically balanced in the sense of Section 4. It is well known

that  $P_-(t)$  and  $\bar{P}_+(t)$  tend monotonically to  $P_-$  and  $\bar{P}_+$ , respectively, as  $t \rightarrow \infty$ , and therefore we have the following ordering:

$$P_-(\tau) := \Sigma_\tau \leq P_- \leq (\bar{P}_+)^{-1} \leq (\bar{P}_+(\tau))^{-1} := \Sigma_\tau^{-1}.$$

Since the number  $n$  of nonzero singular values (108) is in general too large to yield a reasonable model, we must consider what happens when some of the smallest singular values are set equal to zero. The truncation procedure employed by van Oversee and De Moor (1993) is equivalent to the principal subsystem truncation presented in Section 2, except that, and this is very important, the singular-value decomposition is performed on the normalized block Hankel matrix  $\hat{H}_\tau$ , which is the natural matrix representation of the operator  $\mathcal{H}_\tau$ . It will be shown in Section 7 that such a truncation will preserve positivity in the stationary case (Theorem 10). In order to carry this result over to the case of finite  $\tau$ , we need to assume that the spectral density  $\Phi$  of the time series  $\{y(t)\}$  is *coercive* so that Assumption 2 is fulfilled, i.e. that the function  $Z$  is strictly positive real.

The following theorem is a corollary of Theorem 10, to be proved in Appendix D, shows that principal subsystem truncation preserves positivity provided  $\tau$  is chosen large enough.

*Theorem 7.* Suppose that the spectral density  $\Phi$  of the time series  $\{y(t)\}$  is coercive. Then, there is an integer  $\tau_1 > \tau_0$  such that, for  $\tau \geq \tau_1$ , the principal subsystem truncation  $((A_\tau)_{11}, (C_\tau)_1, (\bar{C}_\tau)_1)$  of  $(A_\tau, C_\tau, \bar{C}_\tau)$  is a minimal realization of a strictly positive real function (28).

### 6. SUBSPACE IDENTIFICATION

The analysis in Sections 3–5 is based on the idealized assumption that we have access to an infinite sequence (32) of data. In reality we will have a *finite* string of observed data

$$\{y_0, y_1, y_2, \dots, y_N\}, \tag{117}$$

where, however,  $N$  may be quite large. More specifically, we assume that  $N$  is sufficiently large that replacing the ergodic limits (11) by truncated sums yields good approximations of

$$\{\Lambda_0, \Lambda_1, \Lambda_2, \dots, \Lambda_\nu\}, \tag{118}$$

where, of course,  $\nu \ll N$ . This is equivalent to saying that  $T := N - \nu$  is sufficiently large for

$$\frac{1}{T+1} \sum_{i=0}^T a' y_{i+k} y'_{i+j} b \tag{119}$$

to be essentially the same as the inner product (34). In this section, therefore, we shall use the finite-

interval realization theory of Section 5 as if we had a finite time series

$$\{y(0), y(1), y(2), \dots, y(\nu)\}, \tag{120}$$

while substituting the semi-infinite string (33) of data by

$$y(t) = [y_t, y_{t+1}, \dots, y_{T+t}] \quad \text{for } t = 0, 1, \dots, \nu. \tag{121}$$

In particular, in this case the inner product becomes merely that of a finite-dimensional Euclidean space so that the block Hankel matrix  $H_\tau$  can be written

$$H_\tau = \frac{1}{T+1} y_\tau^+ (y_\tau^-)',$$

where

$$y_\tau^- = \begin{bmatrix} y_{\tau-1} & y_\tau & \dots & y_{T+\tau-1} \\ y_{\tau-2} & y_{\tau-1} & \dots & y_{T+\tau-2} \\ \vdots & \vdots & \ddots & \vdots \\ y_0 & y_1 & \dots & y_T \end{bmatrix} \quad \text{and}$$

$$y_\tau^+ = \begin{bmatrix} y_\tau & y_{\tau+1} & \dots & y_{T+\tau} \\ y_{\tau+1} & y_{\tau+2} & \dots & y_{T+\tau+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_{2\tau-1} & y_{2\tau} & \dots & y_{T+2\tau-1} \end{bmatrix}.$$

Consequently, the identification of a minimal stationary state space innovation model describing the data (117) can be performed in the following steps:

- (1) Perform canonical correlation analysis on the data  $y_\tau^-, y_\tau^+$  to obtain, from (110), the state vectors  $\hat{x}_-(\tau) = z(\tau)$  and  $\hat{x}_+(\tau) = \bar{z}(\tau)$  and, from (109), the corresponding common state covariance matrix  $\Sigma_\tau$ , i.e. the diagonal matrix of the (finite interval) canonical correlation coefficients (108).
- (2) Given the singular value decomposition (109), compute via (114) a minimal realization  $(A, C, \bar{C})$ . This realization will be in *finite-interval balanced* form, i.e. (113) will hold instead of (73).
- (3) To obtain a state space model (37) for  $y$  we need to compute the matrices  $B$  and  $D$ . Note that such matrices will exist if and only if  $(A, C, \bar{C}, \Lambda_0)$  defines a positive real function (6), or, in other words, if and only if there is a symmetric positive definite  $P = P'$ , such that

$$M(P) := \begin{bmatrix} P - APA' & \bar{C}' - APC' \\ \bar{C} - CPA' & \Lambda_0 - CPC' \end{bmatrix} \geq 0. \tag{122}$$

(See, e.g. Faurre *et al.* (1979) or Willems (1971).) For each  $P$  satisfying (122),  $B$  and  $D$  can be determined (in a nonunique way) by a full rank factorization of  $M(P)$ , i.e.

$$\begin{bmatrix} B \\ D \end{bmatrix} \begin{bmatrix} B' & D' \end{bmatrix} = M(P). \tag{123}$$

(4) In particular, the (stationary) forward innovation model (48) can be determined in this way once the state covariance  $P_- = E\{x_-(t)x_-(t)'\}$  has been determined. Obtaining  $P_-$  amounts to finding the minimal solution of the algebraic Riccati equation

$$P = APA' + (\bar{C}' - APC') \\ (\Lambda_0 - CPC')^{-1}(\bar{C}' - APC')' \quad (124)$$

or, alternatively, taking the limit in the Riccati equation (94) as  $t \rightarrow \infty$  with initial condition  $P_-(\tau) = \Sigma_\tau$ . (The corresponding dual procedures yield  $\bar{P}_+$ .) Again, in both cases, a positive definite  $P_-$  can be found if and only if  $(A, C, \bar{C}, \Lambda_0)$  defines a positive real function (6). In fact, in general,  $\{P_-(t)\}_{t \geq 0}$  may not even converge unless this positivity condition is fulfilled and may in fact exhibit dynamical behavior with several of the characteristics of chaotic dynamics (Byrnes *et al.*, 1991; Byrnes *et al.*, 1994).

Assuming that Assumption 1 holds, this procedure is consistent in the sense that, for  $\tau$  fixed but sufficiently large (see Section 2), we will have  $\text{rank} H_\tau = n$  as  $T \rightarrow \infty$ , and the triplet  $(A, C, \bar{C})$  will be uniquely determined from the data and similar to the triplet  $(A, C, \bar{C})$  of the “true” generating system. Hence, in particular, in the limit as  $T \rightarrow \infty$ , at least in theory positivity will be guaranteed. If  $\hat{n}$  is an upper bound for the order of the “true” system, we may choose  $\tau$  to be any integer larger than  $\hat{n}$ .

In practice, however,  $T$  is finite, and even if we had a true system generating exact data, the spectral estimate  $\Phi_T$ , although converging to the true spectrum  $\Phi$  as  $T \rightarrow \infty$  may in principle fail to be positive for any finite  $T$  if there are frequencies  $\omega$  for which  $\Phi(e^{i\omega}) = 0$ . Positivity for a suitably large  $T$  can however be guaranteed if the “true” spectrum is coercive. The following proposition, which also applies to Aoki’s (1990) method discussed in Section 2, is proved in Appendix D.

*Proposition 6.* Suppose that the conditions of Assumptions 1 and 4 are fulfilled. Then, there is a  $T_0 \in \mathbb{Z}_+$  such that, for  $T \geq T_0$ , the triplet  $(A, C, \bar{C})$  defined by (114) yields a function (6) which is strictly positive real.

However, in practice,  $\text{rank} H_\tau$  normally will keep increasing with  $\tau$ , even for very large  $T$ , so that one must resort to some kind of truncation of the Hankel singular values. As we have pointed out in Section 5, setting all canonical correlation coefficients  $\sigma_{r+1}(\tau), \sigma_{r+2}(\tau), \dots$  equal to zero for some suitable  $r$ , as is done in, for example, van Overshee and De Moor (1993), is equivalent to principal sub-

system truncation. An important issue is therefore under what conditions such a procedure will insure positivity. Here we must distinguish between problems generated by the sample fluctuations of the data due to finite sample size  $T$ , as considered in Proposition 6, and the system theoretical question of preserving positivity under truncation, as considered in Theorem 7. Even if we had an infinite string of data generated by a “true” high-dimensional system, such a truncation procedure may fail if  $\tau$  is smaller than that dimension.

Combining Theorem 7 with Proposition 6, we immediately obtain the following result, which justifies this approximation procedure, provided the rather stringent Assumption 1 holds and we have coercivity, and provided  $T$  and  $\tau$  are sufficiently large.

*Theorem 8.* Suppose that the conditions of Assumptions 1 and 4 are fulfilled. Then, there are positive integers  $T_0$  and  $\tau_1 > \tau_0$  such that, for  $T \geq T_0$  and  $\tau \geq \tau_1$ , the triplet  $(A_{11}, C_1, \bar{C}_1)$ , obtained from (27) by taking  $H := H_\tau$  in (23)–(25), is a minimal realization of a strictly positive real function (28).

We note that, in van Overshee and De Moor (1993), the large Hankel matrix

$$\tilde{H}_\tau = (y_\tau^+)'(E\{y_\tau^+(y_\tau^+)'\})^{-1}E\{y_\tau^+(y_\tau^-)'\} \\ \times (E\{y_\tau^-(y_\tau^-)'\})^{-1}y_\tau^-$$

is used in place of  $\hat{H}_\tau$ . This leads to a procedure which is equivalent to the one described above. Moreover, the computation of a second singular-value decomposition in van Overshee and De Moor (1993), based on  $H_{\tau+1} := E\{y_{\tau+1}^+(y_{\tau+1}^-)'\}$ , together with a subsequent change of bases, is actually redundant, as can be deduced from the following proposition. In fact, a considerable amount of computation is needed in van Overshee and De Moor (1993) to compensate for the fact that taking  $z(\tau + 1)$ , computed from a second singular-value decomposition, as a basis in  $\hat{X}_{(\tau+1)-}$  would lead to a Kalman filter model with time-varying parameters.

*Proposition 7.* To each coherent pair of bases  $\hat{x}(\tau)$  and  $\hat{x}(\tau)$  in the finite-interval predictor spaces  $\hat{X}_{\tau-}$  and  $\hat{X}_{\tau+}$ , there corresponds a minimal factorization

$$H_\tau = \Omega_\tau \bar{\Omega}'_\tau \quad (125)$$

of the block Hankel matrix  $H_\tau$ . Here

$$\Omega_\tau \hat{x}(\tau) = E^{Y_\tau} y_\tau^+ \quad \text{and} \quad \bar{\Omega}'_\tau \hat{x}(\tau) = E^{Y_\tau} y_\tau^- \quad (126)$$

Conversely, given a minimal factorization (125),

$$\hat{x}(\tau) = \bar{\Omega}'_\tau (T_\tau^-)^{-1} y_\tau^- \quad \text{and} \quad \hat{x}(\tau) = \Omega'_\tau (T_\tau^+)^{-1} y_\tau^+ \quad (127)$$

is a coherent pair of bases in  $\hat{X}_{\tau-}$  and  $\hat{X}_{\tau+}$ .



*Proof.* Let  $\hat{x}(\tau)$  and  $\tilde{\hat{x}}(\tau)$  be a coherent choice of bases in  $\hat{X}_{\tau-}$  and  $\hat{X}_{\tau+}$ . Then, for any  $X_{\tau}$  as defined in Theorem 6, there is a unique pair  $(x(\tau), \tilde{x}(\tau))$  of dual bases such that (89) and (90) hold. Let  $\Omega_{\tau}$  and  $\tilde{\Omega}_{\tau}$  be the matrices defined via

$$E^{X_{\tau}} y_{\tau}^{+} = \Omega_{\tau} x(\tau) \quad \text{and} \quad E^{X_{\tau}} y_{\tau}^{-} = \tilde{\Omega}_{\tau} \tilde{x}(\tau). \tag{128}$$

Then, the splitting property (Lindquist and Picci, 1985; Lindquist and Picci, 1991) of  $X_{\tau}$  with respect to  $Y_{\tau}^{-}$  and  $Y_{\tau}^{+}$  yields

$$E\{y_{\tau}^{+}(y_{\tau}^{-})'\} = E\{E^{X_{\tau}} y_{\tau}^{+}(E^{X_{\tau}} y_{\tau}^{-})'\},$$

which, in view of (128), is the same as (125). Applying  $E^{Y_{\tau}^{-}}$  and  $E^{Y_{\tau}^{+}}$  to, respectively, the first and second equations of (128), the splitting property yields (126).

As for the converse statement, equations (127) follow from the construction in the proof of Theorem 6, from which it also follows that the resulting bases  $\hat{x}(\tau)$  and  $\tilde{\hat{x}}(\tau)$  are constructed from the same  $(A, C, \tilde{C})$  and are therefore coherent.

As soon as the parameters  $(A, C, \tilde{C})$  have been fixed by a particular choice of  $x(\tau)$  in the representation (89) in Theorem 6, we must choose  $\hat{x}(\tau + 1)$  as

$$\hat{x}(\tau + 1) = E^{Y_{\tau+1}} U_X(\tau) \tag{129}$$

to stay within the same uniform choice of bases. More specifically Proposition 7 implies that  $\Omega_{\tau}$  and  $\tilde{\Omega}_{\tau}$  are uniquely determined once  $x(\tau)$  has been selected. Hence,  $(A, C, \tilde{C})$  is uniquely determined by the Ho-Kalman algorithm so that

$$\tilde{\Omega}_{\tau+1} = \begin{bmatrix} \tilde{C} \\ \tilde{\Omega}_{\tau} A' \end{bmatrix}$$

is prescribed, as is

$$\hat{x}(\tau + 1) = \tilde{\Omega}_{\tau}' (T_{\tau+1}^{-})^{-1} y_{\tau+1}^{-}. \tag{130}$$

Of course, this analysis is purely conceptual, demonstrating that the step determining  $\hat{x}(\tau + 1)$  by an extra singular-value decomposition, as in van Overschee and De Moor (1993), is actually redundant. If we actually were to determine  $\hat{x}(\tau + 1)$  as described above, we would better compute  $\tilde{\Omega}_{\tau+1}$  from  $\tilde{\Omega}_{\tau+1} = \Omega_{\tau}^{-L} H_{\tau+1}$ , where the left inverse is very easily obtained from the singular-value decomposition of  $H_{\tau}$ .

We stress that Assumption 1, although quite limiting, is absolutely crucial in insuring that the subspace identification algorithms mentioned above will actually work. Note that for *generic* data these algorithms may break down for any fixed  $\tau$ . The same is true for all other subspace methods which deal with identification of covariance models (or equivalent) involving stochastic signals.

On the other hand, Assumption 1 introduces a quite unrealistic condition which, as we have seen in Section 2, is untestable. Moreover, we have absolutely no procedure to estimate  $T_0$  and  $\tau_1$  in Proposition 8, as the proof is based only on continuity arguments.

### 7. STOCHASTIC MODEL REDUCTION

As we have already pointed out, some truncation procedure or stochastic model reduction technique may have to be employed in the partial stochastic realization step in order to keep the dimension of the model at a reasonable level. To justify any such procedure one must either assume that there is an underlying "true" system of sufficiently low order, i.e., invoke Assumption 1, or to perform rational covariance extension (Kalman, 1981; Georgiou, 1987; Kimura, 1987; Byrnes *et al.*, 1995; Byrnes and Lindquist, 1996) to extend the covariance sequence (83) to an infinite one. The latter can be done in many ways, one of which is the maximum entropy extension.

In either case, the truncation problem is equivalent to approximating a positive real matrix function

$$Z(z) = C(zI - A)^{-1} \tilde{C}' + \frac{1}{2} \Lambda_0 \tag{131}$$

of a degree  $n$  which is often too large, by another positive real matrix function  $Z_1$  of lower degree. In this section we shall investigate how this can be done and also how such an approximation affects the canonical correlation structure.

One main question to be addressed is whether the principal subsystem truncation (26) preserves positive realness and balancing, and hence the leading canonical correlation coefficients, as originally claimed by Desai and Pal (1982). As it turns out, the answer is affirmative to the first but not to the second of these questions. This also explains the nature of the subspace-identification approximation obtained by setting some canonical correlation coefficients equal to zero.

It is instructive to first consider the continuous-time counterpart of this problem since the latter is simpler and exhibits more desirable properties. Also, it has been widely believed that the continuous-time results are valid also in the present discrete-time setting, which in general is not true.

It is well-known (see, e.g. Faurre *et al.*, 1979) that an  $m \times m$  matrix function  $Z$  with minimal realization

$$Z(s) = C(sI - A)^{-1} \tilde{C}' + \frac{1}{2} R \tag{132}$$

is positive real with respect to the right half plane if and only if there is a symmetric matrix  $P > 0$  such that

$$M(P) := \begin{bmatrix} -AP - PA' & \bar{C}' - PC' \\ \bar{C} - CP & R \end{bmatrix} \geq 0, \tag{133}$$

where here we assume that  $R$  is positive definite and symmetric. In this case there are two solutions of (133),  $P_-$  and  $P_+$ , with the property that any other solution of (133) satisfies

$$P_- \leq P \leq P_+. \tag{134}$$

These extremal solutions play the same role as  $P_-$  and  $P_+$  in the discrete-time setting, and

$$\text{rank}M(P_-) = m = \text{rank}M(P_+). \tag{135}$$

If the state space coordinates are chosen so that both  $P_-$  and  $\bar{P}_+ := P_+^{-1}$  are diagonal and equal, and thus, by (65), equal to the diagonal matrix  $\Sigma$  of canonical correlation coefficients, we say that  $(A, C, \bar{C})$  is *stochastically balanced*.

Now, suppose that  $\Sigma$  is partitioned as in (21) with  $\sigma_{r+1} < \sigma_r$ , and consider the corresponding principal subsystem truncation (27). Using the stochastic realization framework, Harshavaradhana *et al.* (1984) showed that

$$Z_1(s) = C_1(sI - A_{11})^{-1}\bar{C}'_1 + \frac{1}{2}R \tag{136}$$

is a minimal realization of a positive real function and conjectured that  $(A_{11}, C_1, \bar{C}'_1)$  is stochastically balanced. We shall next show that this conjecture is true, as has already been done by Ober (1991) in a framework of canonical forms.

First, note that positivity is easily proved by inserting (21) into (133) to yield

$$\begin{bmatrix} -A_{11}\Sigma_1 - \Sigma_1 A'_{11} & * & \bar{C}'_1 - \Sigma_1 C'_1 \\ * & * & * \\ \bar{C}_1 - C_1 \Sigma_1 & * & R \end{bmatrix} \geq 0, \tag{137}$$

where blocks which play no role in the analysis are marked by an asterisk. Consequently,

$$M_1(\Sigma_1) = \begin{bmatrix} -A_{11}\Sigma_1 - \Sigma_1 A'_{11} & \bar{C}'_1 - \Sigma_1 C'_1 \\ \bar{C}_1 - C_1 \Sigma_1 & R \end{bmatrix} \geq 0. \tag{138}$$

Since, in addition, it can be shown that  $A_{11}$  is stable (Pernebo and Silverman, 1982; Harshavaradhana *et al.*, 1984), i.e. has all its eigenvalues in the open left half plane, (136) is positive real, but it remains to prove that  $(A_{11}, C_1, \bar{C}'_1)$  is a minimal realization. This was done in Harshavaradhana *et al.* (1984). It is important to observe here that, contrary to the situation in the discrete-time setting,  $\text{rank} M_1(\Sigma_1) = \text{rank} M(\Sigma) = m$  and  $\text{rank} M_1(\Sigma_1^{-1}) = \text{rank} M(\Sigma^{-1}) = m$ , important facts that will be seen to imply that the reduced system is stochastically balanced.

Recall that in the continuous-time setting the spectral density  $\Phi(s) = Z(s) + Z(-s)'$  is *coercive* if,

for some  $\epsilon > 0$ , we have  $\Phi(s) \geq \epsilon I$  for all  $s$  on the imaginary axis. This is equivalent to the condition that  $R > 0$  and  $\Phi$  has no zeros on the imaginary axis (Faurre *et al.*, 1979, Theorem 4.17).

*Theorem 9.* Let (132) be positive real (in the continuous-time sense) with  $\Phi(s) := Z(s) + Z(-s)$  coercive, and let  $(A, C, \bar{C})$  be in stochastically balanced form. Then, if  $\sigma_{r+1} < \sigma_r$ , the reduced system  $(A_{11}, C_1, \bar{C}'_1)$  defines a positive real function (136) for which it is a minimal realization in stochastically balanced form, and  $\Phi_1(s) := Z_1(s) + Z_1(-s)$  is coercive.

*Proof.* We have already shown that  $Z_1$  is positive real, and we refer the reader to Harshavaradhana *et al.* (1984) for the proof that  $(A_{11}, C_1, \bar{C}'_1)$  is a minimal realization of  $Z_1$ . It remains to show that  $\Phi_1$  is coercive and that  $(A_{11}, C_1, \bar{C}'_1)$  is stochastically balanced, i.e. that  $P_{1-} = \Sigma_1 = P_{1+}^{-1}$ , where  $P_{1-}$  and  $P_{1+}$  are solutions to the algebraic Riccati equation  $A_{11}P_1 + P_1A'_{11} + (\bar{C}'_1 - P_1C'_1)R^{-1}(\bar{C}'_1 - P_1C'_1)' = 0,$

$$\tag{139}$$

such that any other solution  $P_1$  of (139) satisfies  $P_{1-} \leq P_1 \leq P_{1+}$ . To this end, note that since  $M_1(\Sigma_1)$  and  $M_1(\Sigma_1^{-1})$  have rank  $m$ , both  $\Sigma_1$  and  $\Sigma_1^{-1}$  satisfy (139). Therefore, as is well known (Molinari, 1977) and easy to show,  $Q := \Sigma_1^{-1} - \Sigma_1$  satisfies

$$\Gamma_1 Q + Q\Gamma_1' + QC'_1R^{-1}C_1Q = 0, \tag{140}$$

where

$$\Gamma_1 = A_{11} - (\bar{C}'_1 - \Sigma_1 C'_1)R^{-1}C_1. \tag{141}$$

Since  $\Phi$  is coercive,  $\Sigma^{-1} - \Sigma = P_+ - P_- > 0$  (Faurre *et al.*, 1979, Theorem 4.17) so that  $\sigma_1 < 1$ . Hence  $Q > 0$ , and therefore (140) is equivalent to

$$\Gamma_1 Q^{-1} + Q^{-1}\Gamma_1 + C'_1R^{-1}C_1 = 0. \tag{142}$$

Now, since  $(C_1, A_{11})$  is observable, then, in view of (141), so is  $(C_1, \Gamma_1)$ . Since, in addition, the Lyapunov equation (142) has a positive definite solution  $Q^{-1}$ ,  $\Gamma_1$  must be a stability matrix. Therefore,  $\Sigma_1$  is the minimal (stabilizing) solution  $P_{1-}$  of (139). In the same way, using the backward setting, we show that  $\bar{P}_{1+} := P_{1+}^{-1} = \Sigma_1$ . Consequently,  $(A_{11}, C_1, \bar{C}'_1)$  is stochastically balanced. Since  $P_{1+} - P_{1-} > 0$ ,  $\Phi_1$  is coercive.

Let us now return to the discrete-time setting. Let us recall that, if  $(A, C, \bar{C}, \frac{1}{2}\Lambda_0)$  is a minimal realization of (131), the matrix function  $Z$  is positive real if and only if the linear matrix inequality (122) has a symmetric solution  $P > 0$ . Conversely, given the positive real rational function (131) with the property that  $\Phi(z) = Z(z) + Z(z^{-1})'$  is the spectral density of the time series  $y$ , the state covariance  $P$  of

any minimal stochastic realization (37) of  $y$  satisfies (122) and the matrices  $B, D$  in (37) satisfy (123). Consequently, as pointed out in Section 5, the matrices  $B$  and  $D$  can be determined via a matrix factorization of  $M(P)$  once  $P$  has been determined.

Now, if  $(A, C, \bar{C})$  is in stochastically balanced form, Theorem 5 implies that  $M(\Sigma) \geq 0$ . In view of (67) and (27),  $M(\Sigma)$  may be written

$$\begin{bmatrix} \Sigma_1 - A_{11}\Sigma_1A'_{11} - A_{12}\Sigma_2A'_{12} & * & \bar{C}'_1 - A_{11}\Sigma_1C'_1 - A_{12}\Sigma_2C'_2 \\ * & * & * \\ \bar{C}_1 - C_1\Sigma_1A'_{11} - C_2\Sigma_2A'_{12} & * & \Lambda_0 - C_1\Sigma_1C'_1 - C_2\Sigma_2C'_2 \end{bmatrix}$$

where, as before, the blocks which do not enter the analysis are marked with an asterisk. Since  $M(\Sigma) \geq 0$ , this implies that

$$M_1(\Sigma_1) - \begin{bmatrix} A_{12} \\ C_2 \end{bmatrix} \Sigma_2 \begin{bmatrix} A_{12} \\ C_2 \end{bmatrix}' \geq 0, \quad (143)$$

where

$$M_1(\Sigma_1) = \begin{bmatrix} \Sigma_1 - A_{11}\Sigma_1A'_{11} & \bar{C}'_1 - A_{11}\Sigma_1C'_1 \\ \bar{C}_1 - C_1\Sigma_1A'_{11} & \Lambda_0 - C_1\Sigma_1C'_1 \end{bmatrix} \quad (144)$$

is the matrix function (122) corresponding to the reduced triplet  $(A_{11}, C_1, \bar{C}_1)$ . Therefore,  $M(\Sigma_1) \geq 0$ , so if we can show that  $A_{11}$  is stable, i.e. has all its eigenvalues strictly inside the unit circle, it follows that

$$Z_1(z) = C_1(zI - A_{11})^{-1}\bar{C}'_1 + \frac{1}{2}\Lambda_0 \quad (145)$$

is positive real. As we shall see below, this is true without the requirement needed in continuous time that  $\sigma_{r+1} < \sigma_r$ .

For  $(A_{11}, C_1, \bar{C}_1)$  also to be balanced,  $\Sigma_1$  would have to be the minimal solution  $P_{1-}$  of  $M_1(P_1) \geq 0$ , which in turn would require that  $\text{rank } M_1(\Sigma_1) = \text{rank } M(\Sigma) = m$ . Due to the extra positive semidefinite term in (143), however, this will in general not be the case and therefore  $\Sigma_1 \geq P_{1-}$  will correspond to an external realization, as will  $\Sigma_1^{-1} \leq P_{1+}$ ; see Lindquist and Picci (1991).

To show that  $(A_{11}, C_1, \bar{C}_1)$  is minimal we need to assume that  $\Phi$  is coercive, or, equivalently, that  $Z$  is strictly positive real. It is well known (Faurre *et al.*, 1979, Theorem A4.4) that this implies that

$$P_+ - P_- > 0. \quad (146)$$

In fact, if  $\Lambda_0 > 0$ , which in particular holds if  $y$  is full rank, (146) is equivalent to coercivity. Coercivity also implies that

$$\Lambda_0 - CP_-C' > 0. \quad (147)$$

*Remark 2.* With  $(A, C, \bar{C})$  in balanced form,  $P_- = \Sigma = \bar{P}_+$  and, in view of (46),  $P_+ = \Sigma^{-1}$ . Hence, (146) becomes  $\Sigma^{-1} > \Sigma$ , which obviously holds if

and only if  $\sigma_1 < 1$ , which in turn is equivalent to  $H^- \cap H^+ = 0$ . Consequently, given the full rank condition  $\Lambda_0 > 0$ , coercivity is equivalent to the past and the future spaces of  $y$  having a trivial intersection.

*Theorem 10.* Let (131) be positive real, and let  $(A, C, \bar{C})$  be in stochastically balanced form. Then, the reduced-degree function (145) obtained via principal subsystem decomposition (28) is positive real. Moreover, if  $Z$  is strictly positive real, then so is  $Z_1$ , and  $(A_{11}, C_1, \bar{C}_1, \frac{1}{2}\Lambda_0)$  is a minimal realization of  $Z_1$ .

For the proof we need the following lemma, the proof of which is given in Appendix D.

*Lemma 1.* Let the matrix function  $Z$  be given by (131), where  $\Lambda_0 > 0$ , but where  $(C, A)$  and  $(\bar{C}, A')$  are not necessarily observable, and suppose that (122) has two positive definite symmetric solutions,  $P_1$  and  $P_2$ , such that

$$P_2 - P_1 > 0. \quad (148)$$

Then,  $Z$  is strictly positive real.

*Proof of Theorem 10.* To prove that  $Z_1$  is positive real it remains to show that  $A_{11}$  is stable. To this end, we note that  $P$  is the reachability gramian of (37). In particular, if  $(A, C, \bar{C})$  is stochastically balanced, the reachability gramian of the system (48) equals  $\Sigma$  so, in view of Theorem 4.2 in Pernebo and Silverman (1982),  $A_{11}$  is stable. By Remark 2, coercivity of  $\Phi$  implies that  $\Sigma^{-1} - \Sigma > 0$ , from which it follows that  $\Sigma_1^{-1} - \Sigma_1 > 0$  and that  $\Lambda_0 > 0$ . Moreover, by construction,  $M_1(\Sigma_1) \geq 0$  and  $M_1(\Sigma_1^{-1}) \geq 0$ . Therefore, by Lemma 1,  $Z_1$  is strictly positive real if  $Z$  is.

To prove minimality, we prove that  $(C_1, A_{11})$  is observable. Then, the rest follows by symmetry. By regularity condition (147)

$$\Lambda_0 - C_1\Sigma_1C'_1 \geq \Lambda_0 - C\Sigma C' > 0,$$

and consequently, since  $M_1(\Sigma_1) \geq 0$ ,  $\Sigma_1$  satisfies the algebraic Riccati inequality

$$A_{11}P_1A'_{11} - P_1 + (\bar{C}'_1 - A_{11}P_1C'_1) (\Lambda_0 - C_1P_1C'_1)^{-1} (\bar{C}'_1 - A_{11}P_1C'_1)' \geq 0,$$

but in general not with equality. Now, since  $A_{11}$  is stable,  $(A'_{11}, C'_1)$  is stabilizable. Moreover, given condition (40), we have proved above that the reduced-degree spectral density  $\Phi_1$  is coercive. Therefore, by Theorem 2 in Molinari (1975), there is a unique symmetric  $P_{1-} > 0$  which satisfies (149) with equality and for which

$$\Gamma_{1-} := A_{11} - (\bar{C}'_1 - A_{11}P_{1-}C'_1)(\Lambda_0 - C_1P_{1-}C'_1)^{-1}C_1$$

is stable. It is well known (Faurre *et al.*, 1979) that  $P_{1-}$  is the minimal symmetric solution of the linear matrix inequality  $M_1(P_1) \geq 0$ , i.e. that any other symmetric solution  $P_1$  satisfies  $P_1 \geq P_{1-}$ . We also know that  $M_1(\Sigma_1^{-1}) \geq 0$ . Next, since  $\Sigma_1^{-1} - \Sigma_1 > 0$ , a fortiori it holds that  $Q := \Sigma_1^{-1} - P_{1-} > 0$ . A tedious but straight-forward calculation shows that  $Q$  satisfies

$$\Gamma_{1-}(Q^{-1} - C_1' R^{-1} C_1)^{-1} \Gamma_{1-}' - Q \geq 0,$$

from which it follows that

$$Q^{-1} - C_1' R^{-1} C_1 - \Gamma_{1-}' Q^{-1} \Gamma_{1-} \leq 0. \quad (149)$$

(cf. Faurre *et al.* (1979), pp. 85 and 95).

Now, suppose that  $(C_1, A_{11})$  is not observable. Then, there is a nonzero  $a \in \mathbb{C}^r$  and a  $\lambda \in \mathbb{C}$ , such that  $[C_1, \lambda I - A_{11}]a = 0$ . and therefore, in view of (149),

$$(1 - |\lambda|^2)a^* Q^{-1} a \leq 0.$$

But,  $\lambda$  is an eigenvalue of the stable matrix  $A_{11}$ , implying that  $|\lambda| < 1$ , so we must have  $a = 0$  contrary to assumption. Consequently,  $(C_1, A_{11})$  is observable.

A remaining question is whether there is some balanced order-reduction procedure in discrete time which preserves both positivity and balancing. That this, is the case in continuous time implies that the answer is affirmative, but the reduced system cannot be a simple principal subsystem truncation.

*Theorem 11.* Let (6) be strictly positive real and let  $(A, C, \bar{C})$  be in stochastically balanced form. Moreover, given a decomposition (27), such that  $\sigma_{r+1} < \sigma_r$ , let

$$\begin{aligned} A_r &= A_{11} - A_{12}(I + A_{22})^{-1} A_{21}, \\ C_r &= C_1 - C_2(I + A_{22})^{-1} A_{21}, \\ \bar{C}_r &= \bar{C}_1 - \bar{C}_2(I + A_{22}')^{-1} A_{12}', \\ \Lambda_{r0} &= \Lambda_0 - C_2(I + A_{22})^{-1} \bar{C}_2' - \bar{C}_2(I + A_{22}')^{-1} C_2'. \end{aligned}$$

Then,  $(A_r, C_r, \bar{C}_r, \Lambda_{r0})$  is a minimal realization of a strictly positive real function

$$Z_r(z) = C_r(zI - A_r)^{-1} \bar{C}_r' + \frac{1}{2} \Lambda_{r0}. \quad (150)$$

Moreover,  $(A_r, C_r, \bar{C}_r, \Lambda_{r0})$  is stochastically balanced with canonical correlation coefficients  $\sigma_1, \sigma_2, \dots, \sigma_r$ .

To understand why this reduced-order system does preserve both positivity and balancing, note that for

$$T = \begin{bmatrix} I - A_{12}(I + A_{22})^{-1} & 0 \\ 0 & I \\ 0 & -C_2(I + A_{22})^{-1} & I \end{bmatrix}$$

we obtain

$$TM(\Sigma)T' = \begin{bmatrix} \Sigma_1 - A_r \Sigma_1 A_r' & * & \bar{C}_r' - A_r \Sigma_1 C_r' \\ * & * & * \\ \bar{C}_r - C_r \Sigma_1 A_r' & * & \Lambda_{r0} - C_r \Sigma_1 C_r' \end{bmatrix},$$

and consequently, if  $M_r(P)$  is the the matrix function (122) corresponding to the reduced-order system,  $M_r(\sigma_1) \geq 0$  and  $\text{rank } M_r(\Sigma_1) \leq \text{rank } M(\Sigma)$ .

To prove Theorem 11 we observe that  $(A_r, C_r, \bar{C}_r, \Lambda_{r0})$  is precisely what one obtains if one transforms  $(A, C, \bar{C}, \Lambda_0)$  by the appropriate linear fractional transform to the continuous-time setting and then, after reduction, back to discrete time again as suggested in Ober (1991). The proof is deferred to Appendix D.

## 8. CONCLUSIONS

The purpose of this paper is to analyze a class of popular subspace identification procedures for state space models in the theoretical framework of rational covariance extension, balanced model reduction, and geometric theory for splitting subspaces. We have pointed out that these methods are based on the hidden Assumption 1 which is not entirely natural and which is in general untestable.

The procedures of Aoki (1990) and van Overshee and De Moor (1993) can be regarded as prototypes for this class of algorithms. We point out that they are essentially equivalent to the Ho-Kalman algorithm in which the basic factorization is performed by singular-value decomposition of a block Hankel matrix of finite covariance data, as in Aoki (1990), or of a normalized version of this matrix, as in van Overshee and De Moor (1993). The latter normalization is natural in that it yields a matrix representation of the abstract Hankel operator of geometric stochastic systems theory in orthonormal coordinates and allows for theoretical verification of the truncation step.

A major problem with these algorithms is that they are based on realization algorithms for *deterministic* systems. Therefore, they require that the positive degree of the data equals the algebraic degree. To achieve this, one must assume that the data are generated exactly by an underlying system *and* that the amount of data is sufficient for constructing an accurate partial covariance sequence the length of which is sufficient in relation to the dimension of the underlying system. Hence, it is absolutely crucial that a reliable upper bound of the dimension of the "true" underlying system is available.

We stress that these stringent assumptions are *not* satisfied for generic data, as was pointed out in

Section 2. In fact, in Byrnes and Lindquist (1996) it is shown that the positive degree has no generic value. In fact, just for the moment considering the single-output case, for each  $p$ , such that  $r \leq p \leq v$ , there is a nonempty open set of partial covariance sequences having positive degree  $p$  in the space of sequences of length  $v$ . Secondly, for any  $r$ , it is possible to construct examples of long partial covariance sequences having algebraic degree  $r$  but having arbitrarily large positive degree (Theorem 3).

In Section 7 we proved an open question concerning the preservation of positivity in the original (discrete-time) model reduction procedure of Desai and Pal (1984). Unlike that of the later paper (Desai *et al.*, 1985), this procedure is equivalent to the principal subsystem truncation used in van Overshee and De Moor (1993), but not to the one in Aoki (1990). We prove that positivity is preserved provided that the original data satisfies Assumption 1, justifying setting the smaller canonical correlation coefficients equal to zero. Unlike the situation in continuous time, this truncation does not preserve balancing. The validity of the corresponding procedure of Aoki (1990) has not been settled.

The contribution of this paper is to provide theoretical understanding of these identification algorithms and to point out possible pitfalls of such procedures. Hence, the primary purpose is not to suggest alternative procedures. Nevertheless, we would like to point out that a two-stage procedure equivalent to covariance extension followed by model reduction would work on any finite string of data, thus eliminating the need for Assumption 1. However, we leave open the question of how such a procedure should be implemented with respect to the data. The approximation would then of course depend on which covariance extension is used, a maximum-entropy extension or some other.

*Acknowledgements*— This research was supported in part by grants from TFR, the Göran Gustafsson Foundation, the SCIENCE project "System Identification" and LADSEB-CNR. We would like to thank the referees and the associate editor for the careful review of our paper and for many useful suggestions, which have led to considerable improvements of this paper.

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## APPENDIX A. PROOF OF THEOREM 3

We first give a proof for the special case  $n = 1$ . Consider a scalar function

$$Z(z) = \frac{1}{2} \frac{z+b}{z+a} \quad (\text{A.1})$$

with a scalar sequence (4), such that  $\Lambda_0 = 1$ . Now it is well known (see, e.g. Schur, 1918; ?) that  $T_v$  is positive definite if and only if

$$|\gamma_t| < 1 \quad t = 0, 1, 2, \dots, v-1, \quad (\text{A.2})$$

where  $\{\gamma_0, \gamma_1, \gamma_2, \dots\}$  are the so-called Schur parameters. There is a bijective relation between partial sequences (1) and partial sequences  $\{\gamma_0, \gamma_1, \dots, \gamma_{v-1}\}$  of the same length (Schur, 1918; Akhiezer, 1965). In Byrnes *et al.* (1991), it was shown that the Schur parameters of (A.1) are generated by the nonlinear dynamical system

$$\begin{cases} \alpha_{t+1} = \frac{\alpha_t}{1 - \gamma_t^2} & \alpha_0 = \frac{1}{2}(a+b), \\ \gamma_{t+1} = \frac{-\gamma_t \alpha_t}{1 - \gamma_t^2} & \gamma_0 = \frac{1}{2}(b-a) \end{cases} \quad (\text{A.3})$$

and that  $T_t$  becomes singular precisely when there is finite escape. It was also shown in Byrnes *et al.* (1991) that  $\{\alpha_t\}$  is generated by a linear system

$$\begin{bmatrix} u_{t+1} \\ v_{t+1} \end{bmatrix} = \begin{bmatrix} 2/\kappa & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_t \\ v_t \end{bmatrix}, \quad (\text{A.4})$$

where  $\alpha_t = v_t/u_t$  and  $\kappa := (a+b)(1+ab)^{-1}$ . If  $\kappa$  is greater than one in modulus, the coefficient matrix of (A.4) has complex eigenvalues and is thus, modulo a constant scalar factor, similar to

$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix},$$

where  $\theta := \arctan \sqrt{\kappa^2 - 1}$ . Hence,  $\alpha_t$  is the slope of a line through the origin in  $\mathbb{R}^2$  which rotates counter-clockwise with the constant angle  $\theta$  in each time step. Consequently

$$\arctan \alpha_{t+1} = \arctan \alpha_t + \theta.$$

Moreover, assuming that  $\alpha_0 > 0$ , the Schur condition  $\gamma_t < 1$  will fail as soon as  $\alpha_{t+1}$  becomes negative or infinite, as can be seen from the first of recursions (A.3). Hence, (A.2) holds if and only if

$$\arctan \alpha_v < \frac{\pi}{2}. \quad (\text{A.5})$$

Therefore, for a small  $\epsilon > 0$ , take  $a = 1 - \epsilon$  and  $b = 1 + \epsilon$ , yielding a stable  $Z$ . Then,  $\kappa = \frac{2}{2-\epsilon^2} > 1$  and  $\theta = \arctan \left( \frac{\epsilon}{2-\epsilon^2} \sqrt{4-\epsilon^2} \right)$ . We may choose  $\epsilon$  so that

$$\frac{\vartheta}{v+1} < \theta < \frac{\vartheta}{v},$$

where  $\vartheta := \frac{\pi}{2} - \arctan \alpha_0$ . Then, (A.5) holds so that  $T_v > 0$ , but we also have

$$\arctan \alpha_{v+1} > \frac{\pi}{2}$$

so that  $T_{v+1} \not> 0$ .

Next, let  $n$  be arbitrary. Consider the scalar function

$$Z(z) = \frac{1}{2} \frac{\psi_n(z) + \frac{1}{2}(a+b)\psi_{n-1}(z)}{\varphi_n(z) + \frac{1}{2}(a+b)\varphi_{n-1}(z)},$$

where  $\{\varphi_t\}$  and  $\{\psi_t\}$  are the Szegő polynomials of the first and second kind, respectively (Akhiezer, 1965).

The function  $Z$  has the property that its first  $n$  Schur parameters,  $\{\gamma_0, \gamma_1, \dots, \gamma_{n-1}\}$ , are precisely the data which uniquely determines  $\varphi_n, \varphi_{n-1}, \psi_n$  and  $\psi_{n-1}$  (Georgiou, 1987; Kimura, 1987; Byrnes *et al.*, 1994). Now, in Byrnes *et al.* (1994), it is shown that the remaining Schur parameters are generated by

$$\begin{cases} \alpha_{r+1} = \frac{\alpha_r}{1 - \gamma_{r+n-1}^2} \\ \gamma_{r+1} = \frac{-\gamma_r \alpha_r}{1 - \gamma_{r+n-1}^2} \end{cases} \quad \alpha_0 = \frac{1}{2}(a + b),$$

Hence, we have reduced the problem to the case  $n = 1$ . If we choose the initial Schur parameters sufficiently small so that  $\varphi_n(z)$  and  $\varphi_{n-1}(z)$  are approximately  $z^n$  and  $z^{n-1}$

$$\varphi_n(z) + \alpha_0 \varphi_{n-1}(z)$$

is stable if we choose  $a := 1 - 2\epsilon$  and  $b := 1 + \epsilon$  for some small  $\epsilon > 0$ . Then  $\kappa > 1$  and the proof for the case  $n = 1$  carries through with a trivial modification.

APPENDIX B. THE HILBERT SPACE OF A SAMPLE FUNCTION

Let  $y = \{y(t)\}_{t \geq 0}$  be a zero-mean wide-sense-stationary stochastic process defined on a probability space  $\{\Omega, \mathcal{A}, P\}$ , such that the limit (11) exists for almost all trajectories  $\{y_t = y(t, \omega); t = 0, 1, \dots\}$ . It is relatively easy to show that whenever the limit exists, the  $m \times m$  matrix function  $k \rightarrow \Lambda_k$  obtained from a particular trajectory is then a bona fide covariance function. (The continuous-time analog of this property was observed already by Wiener (1933).) If, moreover, the sample limit is (almost surely) independent of the particular trajectory and, hence, necessarily coincides with the "ensemble" covariance function, we shall call such a process *second-order stationary*. Conditions for second order stationarity are given, for example, on p. 210 in Hannan (1970). It is obvious from Birkhoff's ergodic theorem that any (zero-mean) strictly stationary ergodic process is also second-order ergodic.

In this Appendix we shall show that the properties of the Hilbert space structure associated to a stationary time series  $y$ , defined on p. 10, are identical to those of the Hilbert space induced by a second-order ergodic process.\*

The two frameworks, i.e. the statistical "time-series" structure and the "probabilistic" structure, are in fact isomorphic. To see this, pick a "representative" trajectory of  $y$ , i.e. one in the subset of  $\Omega$  (of probability one) for which the limit (11) exists. Clearly there will be no loss of generality in assuming that the probability space  $\Omega$  of  $y$  is the "sample space", of all possible trajectories of  $y$ , i.e. the set of all semi-infinite sequences  $\omega = \{\omega_0, \omega_1, \omega_2, \dots\}, \omega_t \in \mathbb{R}^m$ . With this choice,  $\mathcal{A}$  will be the usual  $\sigma$ -algebra of cylinder subsets of  $\Omega$  and the  $t$ th random variable of the process,  $y(t)$ , is just the canonical projection function

$$y(t, \omega) : \omega \rightarrow \omega_t.$$

Let us arrange the tails of the observed sample trajectory of the process in a sequence of  $m \times \infty$  matrices  $y := \{y(k)\}_{k \geq 0}$  as in (33). For  $\omega$  in the subset of  $\Omega$  where the time averages converge, define the map  $T_\omega$ ,

$$T_\omega : a'y(t) \rightarrow a'y(t) \quad t \geq 0 \quad a \in \mathbb{R}^m$$

associating the  $i$ th scalar components of each  $m$ -dimensional random vector  $y(t)$  of the process to the corresponding  $i$ th

\* For a process of this kind the Hilbert space  $H(y)$  is the closure in  $L^2(\Omega, \mathcal{A}, P)$  of the linear vector space generated by the scalar random variables  $\omega \rightarrow y_i(t, \omega)$  (Rozanov, 1963).

(infinite) row of the  $m \times \infty$  matrix  $y(t)$  constructed from the corresponding sample path  $\{y(t, \omega); t \in \mathbb{Z}\}$ . By second-order ergodicity, the set of all such  $\omega \in \Omega$  will have probability measure one and the map  $T_\omega$  will in fact be norm preserving, since by construction we have

$$\Lambda_{t-s} = Ey(t)y(s)' = Ey(t)y(s)',$$

where  $\Lambda_t$  is the covariance matrix of  $y$ . The map  $T_\omega$  can then be extended by linearity and continuity to a *unitary* linear operator  $T_\omega : H(y) \rightarrow H(y)$  which commutes with the action of the natural shift operators (both of which we denote  $U$ ), in these two Hilbert spaces:

$$\begin{array}{ccc} H(y) & \xrightarrow{U} & H(y) \\ T_\omega \downarrow & & \downarrow T_\omega \\ H(y) & \xrightarrow{U} & H(y) \end{array}$$

This isomorphism allows us to employ exactly the same formalism and notations used in the geometric theory of stochastic systems (Lindquist and Picci, 1985; Lindquist and Picci, 1991) in the present statistical setup, where we build estimates of the parameters of models describing the data in terms of an observed time series instead of stochastic processes. This provides a remarkable conceptual unity and admits a straightforward derivation in the style of stochastic realization theory of the formulas in the paper van Overshee and De Moor (1993), there obtained with considerable effort through lengthy and formal manipulations.

APPENDIX C. THE INVARIANT FORM OF THE KALMAN FILTER

Given a stationary stochastic system (37), the Kalman filter is usually determined via the matrix Riccati equation

$$\begin{aligned} Q(t+1) &= AQ(t)A' - [AQ(t)C' + BD'] \\ &[CQ(t)C' + DD']^{-1}[AQ(t)C' + BD']' + BB', \end{aligned} \tag{C.1}$$

where  $Q(0) = P := E\{x(0)x(0)'\}$ . Here

$$Q(t) = E\{[x(t) - \hat{x}(t)][x(t) - \hat{x}(t)]'\}, \tag{C.2}$$

and the Kalman gain is given by

$$K(t) = [AQ(t)C' + BD'] [CQ(t)C' + DD']^{-1}. \tag{C.3}$$

These equations of course depend on  $P, B$  and  $D$ , which vary as the splitting subspace  $X$  varies over  $\mathcal{X}$ , whereas  $(A, C, \bar{C})$  is invariant if a uniform choice of bases is made.

However, as we shall see, the gain  $K$  depends only on the triplet  $(A, C, \bar{C})$  and hence one should be able to replace (C.1) and (C.3) with equations which also only depend on  $(A, C, \bar{C})$ , and hence are invariant over  $\mathcal{X}$ . Clearly, in view of Theorem 6,  $P_-(t)$ , as defined by (93), has this property. Moreover,

$$Q(t) = P - P_-(t)$$

and, consequently, in view of (39), and the Lyapunov equation

$$P = APA' + BB',$$

$P, B$  and  $D$  in (C.1) and (C.3) can be eliminated to yield precisely (94) and (92). A symmetric argument yields the backward equations.

It is easy to see that as  $Q(t) \rightarrow Q_\infty$  monotonously,  $P_-(t) \rightarrow P_-$ , and hence  $P \geq P_-$ , as should be.

APPENDIX D. SOME DEFERRED PROOFS

*Proof of Theorem 6.* Since  $X$  is a splitting subspace for the infinite past  $H^-$  and the infinite future  $H^+$ , by stationarity,  $X_\tau$  splits  $H_\tau^- := U^\tau H^-$  and  $H_\tau^+ := U^\tau H^+$ . But  $Y_\tau^- \subset H_\tau^-$  and  $Y_\tau^+ \subset H_\tau^+$ , and hence  $X_\tau$  splits  $Y_\tau^-$  and  $Y_\tau^+$  also (see, e.g. Lindquist and Picci, 1985; Lindquist and Picci, 1991). Now, using the projection formula in the footnote of page 718, we have for any  $b'y_\tau^+ \in Y_\tau^+$

$$E^{Y_\tau^-} b'y_\tau^+ = b' \begin{bmatrix} \Lambda_1 & \Lambda_2 & \dots & \Lambda_\tau \\ \Lambda_2 & \Lambda_3 & \dots & \Lambda_{\tau+1} \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda_\tau & \Lambda_{\tau+1} & \dots & \Lambda_{2\tau-1} \end{bmatrix} \begin{bmatrix} \Lambda_0 & \Lambda_1 & \dots & \Lambda_\tau \\ \Lambda'_1 & \Lambda_0 & \dots & \Lambda_{\tau-1} \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda'_\tau & \Lambda'_{\tau-1} & \dots & \Lambda_0 \end{bmatrix}^{-1} y_\tau^-$$

$$= b' \Omega_\tau \tilde{\Omega}'_\tau (T_\tau^-)^{-1} y_\tau^-$$

$$= b' \Omega_\tau \xi,$$

where  $\Omega_\tau$  and  $\tilde{\Omega}_\tau$  are appropriate finite-dimensional observability and constructibility matrices (19) of full rank. If  $\tau > \tau_0$ , there is a minimal factorization  $H = \Omega_\tau \tilde{\Omega}'_\tau$ , such that  $\xi := \tilde{\Omega}'_\tau (T_\tau^-)^{-1} y_\tau^-$  has  $n$  components, and

$$E\{\xi\xi'\} = \tilde{\Omega}'_\tau (T_\tau^-)^{-1} \tilde{\Omega}_\tau > 0.$$

Therefore, since the components of  $\xi$  belong to  $\hat{X}_{\tau-}$ ,  $\dim \hat{X}_{\tau-} \geq n = \dim X_\tau$  so, since  $\hat{X}_{\tau-}$  is minimal,  $X_\tau$  must also be minimal and  $\hat{X}_{\tau-}$  be spanned by the components of  $\xi$ .

Next, from the backward system (44) we see that

$$y_\tau^- = \tilde{\Omega}_\tau \tilde{x}(\tau) + \text{terms orthogonal to } X_\tau,$$

and therefore, by the same projection formula,

$$E^{Y_\tau^-} a'x(\tau) = a'E\{x(\tau)\tilde{x}(\tau)'\} \tilde{\Omega}'_\tau (T_\tau^-)^{-1} y_\tau^- = a'\xi.$$

Consequently,  $E^{Y_\tau^-} X_\tau = \{a'\xi \mid a \in \mathbb{R}^n\} = \hat{X}_{\tau-}$ , establishing the first of identities (88). The second follows from a symmetric argument.

The representation formula (89) follows from the minimality of  $X_\tau$  as a splitting subspace for  $Y_\tau^+$  and  $Y_\tau^-$ , which, in particular, implies that the constructibility operator,

$$C_\tau := E^{Y_\tau^-}_{|X_\tau} : X_\tau \rightarrow \hat{X}_{\tau-}$$

is *injective* (Lindquist and Picci, 1985; Lindquist and Picci, 1991). In other words, for each  $k = 1, 2, \dots, n$ , there is a unique random variable  $x_k(\tau) \in X_\tau$  whose projection onto  $Y_\tau^-$  is  $\hat{x}_k(\tau)$ . To show that  $x(0)$  form a uniform choice of bases as  $X$  varies over  $\mathcal{X}$ , first take  $X$  to be the stationary backward predictor space  $X_+$  and let  $x_+(\tau)$  be the unique basis in  $U^\tau X_+$ , such that  $\hat{x}(\tau) = E^{Y_\tau^-} x_+(\tau)$ . Now, let  $X \in \mathcal{X}$  be arbitrary. Then, since  $X_\tau$  is a splitting subspace for  $Y_\tau^-$  and  $U^\tau X_+ \subset U^\tau H^+$  (Lindquist and Picci, 1991, Proposition 2.1(vi)), we have

$$\hat{x}(\tau) = E^{Y_\tau^-} x_+(\tau) = E^{Y_\tau^-} E^{X_\tau} x_+(\tau),$$

and therefore, by the uniqueness of the representation (89),  $x(0) = E^X x_+(0)$  for all  $X \in \mathcal{X}$ , which is a well-known characterization of uniform choice of bases (see Section 6 in Lindquist and Picci (1991)). A symmetric argument in the backward setting yields the corresponding statement for (90).

*Proof of Proposition 6.* Suppose that the underlying system prescribed by Assumption 1 has a positive real function  $Z$  of

MacMillan degree  $n$ , and let (1) be a corresponding partial covariance sequence, where  $\nu$  is large enough for the Hankel matrix  $H$ , defined by (5), to have rank  $n$ . Let  $(A, C, \tilde{C})$  be the triplet determined from  $H$  via (16)–(18). Likewise, let  $H_T$  be the Hankel matrix obtained by exchanging the covariance data by estimates

$$\{\Lambda_{0T}, \Lambda_{1T}, \dots, \Lambda_{\nu T}\}$$

of type (119), and let  $(A_T, C_T, \tilde{C}_T)$  be the corresponding triplet obtained via (16)–(18). We want to prove that

$$Z_T(z) := C_T(zI - A_T)^{-1} \tilde{C}_T' + \frac{1}{2} \Lambda_{0T}$$

is strictly positive real for a sufficiently large  $T$ . Now, if  $\deg Z_T \neq \deg Z$ , replace  $\Sigma$  by  $\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}$ ,  $U$  by  $\begin{bmatrix} U & 0 \end{bmatrix}$ ,  $V$  by  $\begin{bmatrix} V & 0 \end{bmatrix}$ , and  $\Sigma^{-1}$  by  $\begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix}$  in (16)–(18) in the appropriate calculation so that  $(A, C, \tilde{C})$  and  $(A_T, C_T, \tilde{C}_T)$  have the same dimensions. This will not affect  $Z$  and  $Z_T$ . By continuity,  $(A_T, C_T, \tilde{C}_T, \Lambda_{0T})$  can be made arbitrarily close to  $(A, C, \tilde{C}, \Lambda_0)$  in any norm by choosing  $T$  sufficiently large. Thus, the same holds for

$$\max_{\theta \in [0, 2\pi]} \|Z(e^{i\theta}) - Z_T(e^{i\theta})\|$$

and hence, since  $\Phi(z) := Z(z) + Z(z^{-1})'$  satisfies (40), so will  $\Phi_T(z) := Z_T(z) + Z_T(z^{-1})'$  for sufficiently large  $T$ . Moreover, since  $|\lambda(A)| < 1$ , we have  $|\lambda(A_T)| < 1$  by continuity for sufficiently large  $T$ . Consequently, there is a  $T_0$  such that  $Z_T$  is strictly positive real for  $T \geq T_0$ .

*Proof of Theorem 7.* Let  $Z$ , defined by (6), be strictly positive real, and let  $(A, C, \tilde{C})$  be chosen in stochastically balanced form. Then, by Theorem 10,  $Z_1$ , defined by (145) in terms of the principal subsystem truncation  $(A_{11}, C_1, \tilde{C}_1)$ , is also strictly positive real. We want to prove that this property is carried over to rational matrix function

$$Z_{\tau 1}(z) = (C_\tau)_1(zI - (A_\tau)_{11})^{-1} (\tilde{C}_\tau)_1 + \frac{1}{2} \Lambda_0$$

for  $\tau$  sufficiently large.

To this end, let  $Q_\tau$  be defined by (115). Since the canonical correlation coefficients (108) tend to the canonical correlation coefficients (63) as  $\tau \rightarrow \infty$ ,  $\Sigma_\tau \rightarrow \Sigma$ . Moreover, as explained in the text preceding Theorem 7, the Riccati solution  $P_-(t)$  tends to  $Q_\tau \Sigma Q_\tau$  as  $t \rightarrow \infty$  if the initial condition is taken to be  $P_-(\tau) = \Sigma_\tau$ . Consequently, for any  $\epsilon > 0$ , there is a sufficiently large  $\tau$ , such that  $\|\Sigma_\tau - \Sigma\| < \frac{\epsilon}{2}$  and  $\|\Sigma_\tau - Q_\tau \Sigma Q_\tau'\| < \frac{\epsilon}{2}$  so that  $\|\Sigma - Q_\tau \Sigma Q_\tau'\| < \epsilon$ . Hence,  $Q_\tau$  tends to a limit  $Q_\infty$  with the property  $\Sigma = Q_\infty \Sigma Q_\infty'$ . Using the same argument in the backward direction, the second of relations (116) shows that  $Q_\infty$  also satisfies  $\Sigma = Q_\infty^{-T} \Sigma Q_\infty^{-1}$ . Consequently, by the same argument as in the proof of Theorem 5,  $Q_\infty$  is a signature matrix, and hence in particular diagonal. Therefore

$$\begin{aligned} & ((A_\tau)_{11}, (C_\tau)_1, (\tilde{C}_\tau)_1) \\ & \rightarrow ((Q_\infty)_{11} A (Q_\infty)_{11}^{-1}, C (Q_\infty)_{11}^{-1}, \tilde{C} (Q_\infty)_{11}^{-1}) \\ & \text{as } \tau \rightarrow \infty, \end{aligned}$$

where  $(Q_\infty)_{11}$  is the corresponding truncation of the signature matrix, and consequently, by continuity,  $Z_{\tau 1} \rightarrow Z_1$ . Hence, since  $Z_1$  is positive real, then so is  $Z_{\tau 1}$  for  $\tau$  sufficiently large.

*Proof of Lemma 1.* Let us first consider the case when  $(A, C, \tilde{C})$  is a minimal triplet. Then,  $Z$  is positive real by the Positive Real Lemma, and the linear matrix inequality (122) has a minimal and a maximal solution,  $P_-$  and  $P_+$ , respectively, which, in



particular, have the property that  $P_- \leq P_1$  and  $P_2 \leq P_+$ . Then, in view of (148),  $P_+ - P_- > 0$ , and therefore  $Z$  is strictly positive real (Faurre *et al.*, 1979, Theorem A4.4).

Next, let us reduce the general case to the case considered above. If  $(C, A)$  is not observable, change the coordinates in state space, through a transformation  $(A, C, \tilde{C}) \rightarrow (QAQ^{-1}, CQ^{-1}, Q\tilde{C}')$ , so that

$$C = [\hat{C} \ 0] \quad A = \begin{bmatrix} \hat{A} & 0 \\ * & * \end{bmatrix} \quad \tilde{C} = [\hat{C} \ *],$$

where  $(\hat{C}, \hat{A})$  is observable. Then, if  $P_1$  and  $P_2$  have the corresponding representations

$$P_1 = \begin{bmatrix} \hat{P}_1 & * \\ * & * \end{bmatrix} \quad P_2 = \begin{bmatrix} \hat{P}_2 & * \\ * & * \end{bmatrix},$$

it is easy to see that  $\hat{P}_1$  and  $\hat{P}_2$  satisfy the reduced version of the linear matrix inequality (122) obtained by exchanging  $(A, C, \tilde{C})$  for  $(\hat{A}, \hat{C}, \hat{C})$  and that, in this new setting, (148) holds, i.e.  $\hat{P}_2 - \hat{P}_1 > 0$ . If  $(\tilde{C}, \hat{A}')$  is not observable, we proceed by removing these unobservable modes. First note that  $\hat{P}_1^{-1}$  and  $\hat{P}_2^{-1}$  satisfy the dual linear matrix inequality obtained by exchanging  $(\hat{A}, \hat{C}, \hat{C})$  by  $(\hat{A}', \hat{C}, \hat{C})$ . Then, changing coordinates in state space so that

$$\hat{C} = [\tilde{C} \ *] \quad \hat{A}' = \begin{bmatrix} \tilde{A}' & 0 \\ * & * \end{bmatrix} \quad \hat{C} = [\tilde{C} \ 0],$$

with  $(\tilde{C}, \tilde{A}')$  observable, and defining

$$\hat{P}_1^{-1} = \begin{bmatrix} \tilde{P}_1^{-1} & * \\ * & * \end{bmatrix} \quad \hat{P}_2^{-1} = \begin{bmatrix} \tilde{P}_2^{-1} & * \\ * & * \end{bmatrix},$$

we see that  $(\tilde{A}, \tilde{C}, \tilde{C}, \frac{1}{2}\Lambda_0)$  is a minimal realization of  $Z$ . Moreover,  $\tilde{P}_1$  and  $\tilde{P}_2$  satisfy the corresponding linear matrix inequality (122) and have the property (148) in this setting. Hence, the problem is reduced to the case already studied above.

*Proof of Theorem 11.* It is well-known that the discrete-time setting can be transformed to the continuous-time setting via a bilinear transformation  $s = \frac{z-1}{z+1}$ , mapping the unit disc onto the left half plane so that

$$Z_c(s) = Z_d \left( \frac{1+s}{1-s} \right) \tag{D.1}$$

is positive real in the continuous-time sense if and only if  $Z_d$  is positive real in the discrete-time sense. It is not hard to show (see, e.g. Glover, 1984; Faurre *et al.*, 1979) that, if  $(A_d, C_d, \tilde{C}_d, \frac{1}{2}\Lambda_0)$  and  $(A_c, C_c, \tilde{C}_c, \frac{1}{2}R)$  are realizations of  $Z_d$  and  $Z_c$ , respectively, we have

$$\begin{cases} A_c = (A_d + I)^{-1}(A_d - I), \\ C_c = \sqrt{2}C_d(A_d + I)^{-1}, \\ \tilde{C}_c = \sqrt{2}\tilde{C}_d(A_d + I)^{-1}, \\ R = \Lambda_0 - C_d(A_d + I)^{-1}\tilde{C}_d' - \tilde{C}_d(A_d + I)^{-1}C_d' \end{cases} \tag{D.2}$$

and inversely

$$\begin{cases} A_d = (I - A_c)^{-1}(I + A_c), \\ C_d = \sqrt{2}C_c(I - A_c)^{-1}, \\ \tilde{C}_d = \sqrt{2}\tilde{C}_c(I - A_c)^{-1}, \\ \Lambda_0 = R + C_c(I - A_c)^{-1}\tilde{C}_c' + \tilde{C}_c(I - A_c)^{-1}C_c'. \end{cases} \tag{D.3}$$

Under this transformation the observability gramian and the constructibility gramian (i.e. the observability gramian of  $(\tilde{C}, A')$ ) are preserved so that  $(A_d, C_d, \tilde{C}_d, \frac{1}{2}\Lambda_0)$  is a minimal realization if and only if  $(A_c, C_c, \tilde{C}_c, \frac{1}{2}R)$  is (see, e.g. Glover, 1984, p. 1119). Moreover, coercivity is preserved, and the solution sets of the corresponding linear matrix inequalities (133) and (122) are identical. (This is because  $P$  is the reachability gramian of a spectral factor and this gramian is also preserved.)

Therefore, Theorem 11 is a straight-forward consequence of Theorem 9. In fact, transforming the problem of Theorem 11 via (D.2) to the continuous-time setting, all the requirements of Theorem 131 are satisfied. Then, performing principal subsystem decomposition in the continuous-time setting and transforming the reduced-order positive real function thus obtained via (D.3) back to discrete time, the desired result is obtained.