Stability-Preserving Rational Approximation Subject to Interpolation Constraints

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***Abstract*—A quite comprehensive theory of analytic interpolation with degree constraint, dealing with rational analytic interpolants with an *a priori* bound, has been developed in recent years. In this paper, we consider the limit case when this bound is removed, and only stable interpolants with a prescribed maximum degree are sought. This leads to weighted H_2 minimization, where the interpolants are parameterized by the weights. The inverse problem of determining the weight given a desired interpolant profile is considered, and a rational approximation procedure based on the theory is proposed. This provides a tool for tuning the solution to specifications. The basic idea could also be applied to the case with bounded analytic interpolants.**

***Index Terms*—Interpolation, model reduction, quasi-convex optimization, rational approximation, stability.**

I. Introduction

Stability-preserving model reduction is a topic of major importance in systems and control, and over the last decades numerous such approximation procedures have been developed; see, e.g., [3], [15], [18], [1] and references therein. In this paper we introduce a novel approach to stability-preserving model reduction that also accommodates interpolation contraints, a requirement not uncommon in systems and control. By choosing the weights appropriately in a family of weighted H_2 minimization problems, the minimizer will both have low degree and match the original system.

As we shall see in this paper, stable interpolation with degree constraint can be regarded as a limit case of bounded analytic interpolation under the same degree constraint—a topic that has been thoroughly researched in recent years; see [5] and [9].

More precisely, let f be a function in $H(\mathbb{D})$, the space of functions analytic in the unit disc $\mathbb{D} = \{z : |z| < 1\}$, satisfying:

i) the interpolation condition

$$f(z_k) = w_k, \quad k = 0, \ldots, n \tag{1}$$

ii) the *a priori* bound $\|f\|_\infty \leq \gamma$;

iii) the condition that f be rational of degree at most n;

where $z_0, z_1, \ldots, z_n \in \mathbb{D}$ are taken to be distinct (for simplicity) and $w_0, w_1, \ldots, w_n \in \mathbb{C}$. It was shown in [5] that, for each such f, there is a rational function $\sigma(z)$ of the form

$$\sigma(z) = \frac{p(z)}{\tau(z)}, \quad \tau(z) := \prod_{k=0}^{n}(1 - \bar{z}_k z)$$

where $p(z)$ is a polynomial of degree n with $p(0) > 0$ and $p(z) \neq 0$ for $z \in \mathbb{D}$ such that f is the unique minimizer of the generalized entropy functional

$$-\int_{-\pi}^{\pi} |\sigma(e^{i\theta})|^2 \gamma^2 \log(1 - \gamma^{-2}|f(e^{i\theta})|^2)\frac{d\theta}{2\pi}$$

subject to the interpolation conditions (1). In fact, there is a complete parameterization of the class of all interpolants satisfying (i)–(iii) in terms of the zeros of σ, which also are *spectral zeros* of f; i.e., zeros of $\gamma^2 - f(z)f^*(z)$ located in the complement of the unit disc. It can also be shown that this parameterization is smooth, in fact a diffeomorphism [6].

This smooth parameterization in terms of spectral zeros is the center piece in the theory of analytic interpolation with degree constraints; see [4] and [5] and references therein. By tuning the spectral zeros one can obtain an interpolant that better fulfills additional design specifications. However, one of the stumbling-blocks in the application of this theory has been the lack of a systematic procedure for achieving this tuning. In fact, the relation between the spectral zeros of f and f itself is nontrivial, and how to choose the spectral zeros in order to obtain an interpolant which satisfy the given design specifications is a partly open problem.

In order to understand this problem better, in this paper we will focus on the limit case as $\gamma \to \infty$; i.e., the case when condition (ii) is removed. We shall refer to this problem—which is of considerable interest in its own right—as *stable interpolation with degree constraint*. Note that, as $\gamma \to \infty$,

$$-\gamma^2 \log(1 - \gamma^{-2}|f|^2) \to |f|^2$$

and hence (see Proposition 2)

$$-\int_{-\pi}^{\pi} |\sigma|^2 \gamma^2 \log(1 - \gamma^{-2}|f|^2)\frac{d\theta}{2\pi} \to \int_{-\pi}^{\pi} |\sigma f|^2 \frac{d\theta}{2\pi}.$$

For the case $\sigma \equiv 1$, this connection between the H_2 norm and the corresponding entropy functional have been studied in [14]. Consequently, the stable interpolants with degree constraint turn out to be minimizers of weighted H_2 norms. Indeed, the H_2 norm plays the same role in stable interpolation as the entropy functional does in bounded interpolation. Stable interpolation and H_2 norms are considerably easier to work with than bounded analytic interpolation and entropy functionals, but many of the concepts and ideas are similar.

The purpose of this paper is twofold. First, we want to provide a stability-preserving model reduction procedure that admits interpolation constraints and error bounds. Secondly, this theory is the simplest and most transparent gateway for understanding the full power of bounded analytic interpolation with degree constraint. In fact, our paper provides, together with the results in [12], the key to the problem of how to settle an important open question in the theory of bounded analytic interpolation with degree constraint, namely how to choose spectral zeros. In the present setting, the spectral zeros are actually the poles.

In many applications, no interpolation conditions (or only a few) are given *a priori*. This allows us to use the interpolation points as additional tuning variables, available for satisfying design specifications. Such an approach for passivity-preserving model reduction was taken in [8]. However, a problem left open in [8] was how to actually select spectral zeros and interpolation points in a systematic way in order to obtain the best approximation. This problem, here in the context of stability-preserving model reduction, is one of the topics of this paper.

The paper is outlined as follows. In Section II, we show that the problem of stable interpolation is the limit, as the bound tend to infinity, of the bounded analytic interpolation problem stated above. In Section III, we derive the basic theory for how all stable interpolants with a degree bound may be obtained as weighted H_2-norm minimizers. In Section IV, we consider the inverse problem of H_2 minimization, and in Section V, the inverse problem is used for model reduction of interpolants. The inverse problem and the model reduction

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procedure are closely related to the theory in [12]. A model reduction procedure where no *a priori* interpolation conditions are required are derived in Section VI. This is motivated by a weighed relative error bound of the approximant and gives a systematic way to choose the interpolation points. This approximation procedure is also tunable so as to give small error in selected regions. In the Appendix, we describe how the corresponding quasi-convex optimization problems can be solved. Finally, in Section VII, we illustrate our new approximation procedures by applying them to a simple example and conclude with a simple control design example.

II. Bounded Interpolation and Stable Interpolation

In this section, we show that the H_2 norm is the limit of a sequence of entropy functionals. From this limit, the relation between stable interpolation and bounded interpolation is established, and it is shown that some of the important concepts in the two different frameworks match.

First consider one of the main results of bounded interpolation: a complete parameterization of all interpolants with a degree bound [5]. For this, we will need two key concepts in that theory; the entropy functional

$$\mathbb{K}^{\gamma}_{|\sigma|^2}(f) = -\int_{-\pi}^{\pi} \gamma^2 |\sigma(e^{i\theta})|^2 \log(1-\gamma^{-2}|f(e^{i\theta})|^2)\frac{d\theta}{2\pi}$$

where we take $\mathbb{K}^{\gamma}_{|\sigma|^2}(f) := \infty$ whenever the H_∞ norm $\|f\|_\infty > \gamma$, and the co-invariant subspace

$$\mathcal{K} = \left\{ \frac{p(z)}{\tau(z)} : \tau(z) = \prod_{k=0}^{n} (1-\bar{z}_k z), p \in \mathrm{Pol}(n) \right\}. \quad (2)$$

Here, $\mathrm{Pol}(n)$ denotes the set of polynomials of degree at most n, and $\{z_k\}_{k=0}^n$ are the interpolation points.

In fact, any interpolant f of degree at most n with $\|f\|_\infty \le \gamma$ is a minimizer of $\mathbb{K}^{\gamma}_{|\sigma|^2}(f)$ subject to (1) for some $\sigma \in \mathcal{K}_0$, where

$$\mathcal{K}_0 = \{\sigma \in \mathcal{K} : \sigma(0) > 0, \sigma \text{ outer}\}.$$

Furthermore, all such interpolants are parameterized by $\sigma \in \mathcal{K}_0$. This is one of the main results for bounded interpolation in [5] and is stated more precisely as follows.

Theorem 1: Let $\{z_k\}_{k=0}^n \subset \mathbb{D}$, $\{w_k\}_{k=0}^n \subset \mathbb{C}$, and $\gamma \in \mathbb{R}_+$. Suppose that the Pick matrix

$$P = \left[\frac{\gamma^2 - w_k\bar{w}_\ell}{1-z_k\bar{z}_\ell}\right]_{k,\ell=0}^{n} \quad (3)$$

is positive definite, and let σ be an arbitrary function in $\mathcal{K}_0$. Then there exists a unique pair of elements $(a,b) \in \mathcal{K}_0 \times \mathcal{K}$ such that

i) $f(z) = b(z)/a(z) \in H^\infty$ with $\|f\|_\infty \le \gamma$;
ii) $f(z_k) = w_k, \quad k = 0,1,\ldots,n$;
iii) $|a(z)|^2 - \gamma^{-2}|b(z)|^2 = |\sigma(z)|^2$ for $z \in \mathbb{T}$.

where $\mathbb{T} := \{z : |z| = 1\}$. Conversely, any pair $(a,b) \in \mathcal{K}_0 \times \mathcal{K}$ satisfying (i) and (ii) determines, via (iii), a unique $\sigma \in \mathcal{K}_0$. Moreover, the optimization problem

$$\min \mathbb{K}^{\gamma}_{|\sigma|^2}(f) \text{ s.t. } f(z_k) = w_k, k = 0,\ldots,n$$

has a unique solution f that is precisely the unique f satisfying conditions (i), (ii), and (iii).

The essential content of this theorem is that the class of interpolants satisfying $\|f\|_\infty \le \gamma$ may be parameterized in terms of the zeros of σ, and that these zeros are the same as the *spectral zeros* of f; i.e., the zeros of the spectral outer factor $w(z)$ of $w(z)w^*(z) = \gamma^2 - f(z)f^*(z)$, where $f^*(z) = \overline{f(\bar{z}^{-1})}$.

Let $\|f\| = \sqrt{\langle f, f\rangle}$ denote the norm in the Hilbert space $H_2(\mathbb{D})$ with inner product $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\theta$. As the bound γ tends to infinity

$$-\gamma^2 \log(1-\gamma^{-2}|f|) \to |f|^2.$$

Therefore, the entropy functional $\mathbb{K}^{\gamma}_{|\sigma|^2}(f)$ converge to the weighted H_2 norm $\|\sigma f\|^2$.

Proposition 2: Let $f \in H_\infty(\mathbb{D})$ and σ be rational functions with σ outer. Then:

i) $\mathbb{K}^{\gamma}_{|\sigma|^2}(f)$ is a nonincreasing function of γ;
ii) $\mathbb{K}^{\gamma}_{|\sigma|^2}(f) \to \|\sigma f\|^2$ as $\gamma \to \infty$.

Proof: It clearly suffices to consider only $\gamma \ge \|f\|_\infty$. Then the derivative of $-\gamma^2\log(1-\gamma^{-2}|f|^2)$ with respect to γ is nonpositive for $|f| \le \gamma$, and hence $\mathbb{K}^{\gamma}_{|\sigma|^2}(f)$ is nonincreasing. To establish (ii), note that

$$-\gamma^2\log(1-\gamma^{-2}|f|^2) = |f|^2 + O(\gamma^{-2}|f|^2)$$

and therefore $-|\sigma|^2\gamma^2\log(1-\gamma^{-2}|f|^2) \to |\sigma f|^2$ pointwise in $\mathbb{T}$ except for σ with poles in $\mathbb{T}$. There are two cases of importance. First, if σ has no poles in $\mathbb{T}$, or if a pole of σ coincided with a zero of f of at least the same multiplicity, then $-|\sigma|^2\gamma^2\log(1-\gamma^{-2}|f|^2)$ is bounded, and (ii) follows from bounded convergence. Secondly, if σ has a pole in $\mathbb{T}$ at a point in which f does not have a zero, then both $\mathbb{K}^{\gamma}_{|\sigma|^2}(f)$, and $\|\sigma f\|^2$ are infinite for any γ. ■

The condition $\|f\|_\infty < \infty$ is needed in Proposition 2. Otherwise, if $\|f\|_\infty = \infty$, then $\mathbb{K}^{\gamma}_{|\sigma|^2}(f)$ is infinite for any γ, while $\|\sigma f\|^2$ may be finite if σ has zeros in the poles of f on $\mathbb{T}$. The next proposition shows that stable interpolation may be seen as the limit case of bounded interpolation when the bound γ tend to infinity.

Proposition 3: Let σ be any outer function such that the minimizer f of

$$\begin{aligned}&\min \|\sigma f\| \\ &\text{such that } f(z_k) = w_k, k = 0,\ldots,n\end{aligned} \quad (4)$$

satisfies $\|f\|_\infty < \infty$. Let f_γ be the minimizer of

$$\begin{aligned}&\min \mathbb{K}^{\gamma}_{|\sigma|^2}(f_\gamma) \\ &\text{such that } f_\gamma(z_k) = w_k, k = 0,\ldots,n\end{aligned}$$

for $\gamma \in \mathbb{R}_+$ large enough so that the Pick matrix (3) is positive definite. Then $\|\sigma(f - f_\gamma)\| \to 0$ as $\gamma \to \infty$.

Proof: By Proposition 2, and since f and f_γ are minimizers of the respective functional, we have

$$\mathbb{K}^{\gamma}_{|\sigma|^2}(f) \ge \mathbb{K}^{\gamma}_{|\sigma|^2}(f_\gamma) \ge \|\sigma f_\gamma\|^2 \ge \|\sigma f\|^2.$$

Moreover, since $\mathbb{K}^{\gamma}_{|\sigma|^2}(f) \to \|\sigma f\|^2$ as $\gamma \to \infty$ it follows that $\|\sigma f_\gamma\|^2 \to \|\sigma f\|^2$, and hence, by Lemma 8, we have $\|\sigma(f - f_\gamma)\| \to 0$ as $\gamma \to \infty$, as claimed.

Note that Proposition 3 holds for any σ which is outer and not only for $\sigma \in \mathcal{K}_0$. However, if $\sigma \in \mathcal{K}_0$, then $\deg f_\gamma \le n$ for any γ. Therefore, since $\|\sigma(f - f_\gamma)\| \to 0$ as $\gamma \to \infty$, for $\sigma \in \mathcal{K}_0$ the minimizer f of (4) will be a stable interpolant of degree at most n. We will return to this in the next section.

It is interesting to note how concepts in the two types of interpolation are related. First of all, the weighted H_2 norm plays the same role in stable interpolation as the entropy functional does in bounded interpolation. Secondly, the spectral zeros, which play a major role in degree constrained bounded interpolation, simply correspond to the poles in stable interpolation. This may be seen from (iii) in Theorem 1.

III. Rational Interpolation and H_2 Minimization

In the previous section, we have seen that minimizers of a specific class of H_2 norms are stable interpolants of degree at most n. This, and also the fact that this class may be parameterized by $\sigma \in \mathcal{K}_0$ can be proved using basic Hilbert space concepts. This will be done in this section.

To this end, first consider the minimization problem

$$\min \|f\| \text{ s.t. } f(z_k) = w_k, \quad k = 0, \dots, n \tag{5}$$

without any weight σ. Let $f_0 \in H_2(\mathbb{D})$ satisfy the interpolation condition (1). Then any $f \in H_2(\mathbb{D})$ satisfying (1) can be written as $f = f_0 + v$, where $B(z) = \prod_{k=0}^{n}(z_k - z)/(1 - \bar{z}_k z)$ and $v \in BH_2$. Therefore, (5) is equivalent to

$$\min_{v \in BH_2} \|f_0 + v\|.$$

By the Projection Theorem (see, e.g., [13]), there exists a unique solution $f = f_0 + v$ to this optimization problem, which is orthogonal to BH_2, i.e., $f \in \mathcal{K} := H_2 \ominus BH_2$. Conversely, if $f \in \mathcal{K}$ and $f(z_k) = w_k$, for $k = 0, \dots, n$, then f is the unique solution of (5). To see this, note that any interpolant in $H_2(\mathbb{D})$ may be written as $f + v$ where $v \in BH_2$. However, since $v \in BH_2 \perp \mathcal{K} \ni f$, we have $\|f + v\|^2 = \|f\|^2 + \|v\|^2$, and hence the minimizer is f, obtained by setting $v = 0$.

We summarize this in the following proposition.

Proposition 4: The unique minimizer of (5) belongs to $\mathcal{K}$. Conversely, if $f \in \mathcal{K}$ and $f(z_k) = w_k$, for $k = 0, \dots, n$, then f is the minimizer of (5).

Consequently, in view of (2), f is a rational function with its poles fixed in the mirror images (with respect to the unit circle) of the interpolation points. By introducing weighted norms, any interpolant with poles in prespecified points may be constructed in a similar way. In fact, the set of interpolants f of degree $\leq n$ may be parameterized in this way. One way to see this is by considering

$$\min \|\sigma f\| \text{ s.t. } f(z_k) = w_k, \quad k = 0, \dots, n \tag{6}$$

where $\sigma \in \mathcal{K}_0$. Since σ is invertible in $H(\mathbb{D})$, (6) is equivalent to

$$\min \|\sigma f\| \text{ s.t. } (\sigma f)(z_k) = \sigma(z_k) w_k, k = 0, \dots, n.$$

According to Proposition 4, this has the optimal solution $\sigma f = b \in \mathcal{K}$, and hence the solution of (6), $f = b/\sigma$, is rational of degree at most n. To see that any solution of degree at most n can be obtained in this way, note that any such interpolant f is of the form $f = b/\sigma, b \in \mathcal{K}, \sigma \in \mathcal{K}_0$. Since $\sigma f = b \in \mathcal{K}$ holds together with the interpolation condition (1) if and only if $\sigma(z_k) f(z_k) = \sigma(z_k) w_k$ for $k = 0, \dots, n$, f is the unique solution of (6), by Proposition 4. This proves the following theorem.

Theorem 5: Let $\sigma \in \mathcal{K}_0$. Then the unique minimizer of (6) belongs to $H(\mathbb{D})$ and is rational of a degree at most n. More precisely, $f = b\sigma$, where $b \in \mathcal{K}$ is the unique solution of the linear system of equations

$$b(z_k) = \sigma(z_k) w_k, \quad k = 0, 1, \dots, n. \tag{7}$$

Conversely, if $f = b\sigma$ for some $b \in \mathcal{K}$ and the interpolation condition (1) holds, then f is the unique minimizer of (6).

In other words, the set of interpolants in $H(\mathbb{D})$ of degree at most n may be parameterized in terms of weights $\sigma \in \mathcal{K}_0$. Another way to look at this is that the poles of the minimizer $f = b\sigma$ are specified by the zeros of σ and that the numerator $b = \beta/\tau$ is determined from the interpolation condition by solving the linear system of equations

$$\beta(z_k) = \tau(z_k) \sigma(z_k) w_k, \quad k = 0, 1, \dots, n \tag{8}$$

for the $n + 1$ coefficients $\beta_0, \beta_1, \dots, \beta_n$ of the polynomial $\beta(z)$. This is a Vandermonde system that is known to have a unique solution (as long as the interpolation point $z_o, z_1, \dots, z_n$ are distinct as here).

Note that this parameterization is not necessarily injective. If, for example, $w_k = 1$ for $k = 0, \dots, n$, then there is a unique function f of degree at most n that satisfies $f(z_k) = w_k, k = 0, \dots, n$. No matter how $\sigma \in \mathcal{K}_0$ is chosen, $b = \sigma$, and hence the minimizer of (6) will be $f \equiv 1$.

IV. The Inverse Problem

In [12], we considered the *inverse problem of analytic interpolation*; i.e., the problem of choosing an entropy functional whose unique minimizer is a prespecified interpolant. In this section, we will consider the counterpart of this problem for stable interpolation. To this end, let us first introduce the subclass $H_{\mathbb{Q}}(\mathbb{D})$ of log-integrable analytic functions in $H(\mathbb{D})$ for which the inner part is rational. In particular, the class of rational analytic (i.e., stable) functions belong to $H_{\mathbb{Q}}(\mathbb{D})$.

Suppose $f \in H_{\mathbb{Q}}(\mathbb{D})$ satisfies the interpolation condition (1). Then, when does there exist σ which is outer such that f is the minimizer of (6)? We refer to this as the *inverse problem of H_2 minimization*. Its solution is given in the following theorem.

Theorem 6: Let $f \in H_{\mathbb{Q}}(\mathbb{D})$ satisfy the interpolation condition $f(z_k) = w_k, k = 0, \dots, n$. Then f is the minimizer of (6), where σ is outer if and only if $\sigma f \in \mathcal{K}$, in which case the minimizer is unique. Such a σ exists if and only if f has no more than n zeros in $\mathbb{D}$.

Proof: The function f is the minimizer of (6) if and only if $b = \sigma f$ is the (unique) minimizer of

$$\min \|b\| \text{ s.t. } b(z_k) := w_k \sigma(z_k), \quad k = 0, \dots, n$$

which, by Proposition 4, holds if and only if $\sigma f = b \in \mathcal{K}$. Such a σ only exists if f has at most n zeros inside $\mathbb{D}$. To see this, first note that, if f has more than n zeros in $\mathbb{D}$, then σf has more than n zeros in $\mathbb{D}$ and can therefore not be of the form p/τ with $p \in \mathrm{Pol}(n)$. On the other hand, if f has less or equal to n zeros in $\mathbb{D}$, then let $p = \prod(z - p_k)$ where p_k are the zeros of f, and set $\sigma := p/(f\tau)$. Then σ is outer and satisfies $\sigma f \in \mathcal{K}$. ■

Theorem 6 defines a map F that sends σ to the unique minimizer f of the optimization problem (6); i.e.,

$$\sigma \mapsto f = F(\sigma). \tag{9}$$

Let W_f be the set of weights σ that give f as a minimizer of (6); i.e., the inverse image $F^{-1}(f)$ of f. By Theorem 6

$$\begin{aligned} W_f &:= F^{-1}(f) = \{\sigma \text{ outer } : \sigma f \in \mathcal{K}\} \\ &= \left\{\sigma = \frac{p}{f\tau} : p \in \mathrm{Pol}(n) \setminus \{0\}, \frac{p}{f} \text{ outer}\right\} \end{aligned} \tag{10}$$

i.e., W_f may be parameterized in terms of the polynomials $p \in \mathrm{Pol}(n)$. For the condition that pf^{-1} is outer to hold for some $p \in \mathrm{Pol}(n)$, it is necessary that f has at most n zeros in $\mathbb{D}$. This is in accordance with Theorem 6. It is interesting to note that the dimension of W_f depends on the number of zeros of f inside $\mathbb{D}$. The more zeros f has inside $\mathbb{D}$, the more restricted is the class W_f. One extreme case is when f has no zeros inside $\mathbb{D}$. Then p could be any stable polynomial of degree n. The other extreme is when f has n zeros in $\mathbb{D}$, in which case p is uniquely determined up to a multiplicative constant.

V. Rational Approximation With Interpolation Constraints

Next, we use the solution of the inverse problem (Theorem 6) to develop an approximation procedure for interpolants. Let $f \in H_{\mathbb{Q}}(\mathbb{D})$ be a function satisfying the interpolation condition (1). We want to construct another function $g \in H_{\mathbb{Q}}(\mathbb{D})$ of degree at most n satisfying the same interpolation condition such that g is as close as possible to f.

Let $\sigma \in W_f$; i.e., let σ be a weight and such that f is the minimizer of (6), and let ρ be close to σ. Then it seems reasonable that the minimizer g of the optimization problem

$$\min \|\rho g\| \quad \text{s.t. } g(z_k) = w_k, \quad k = 0, \dots, n, \tag{11}$$

is close to f. This is the statement of the following theorem.

Theorem 7: Let $f \in H_{\mathbb{Q}}(\mathbb{D})$ satisfy the interpolation condition $f(z_k) = w_k, k = 0, \dots, n$, and let $\sigma \in W_f$. Moreover, let ρ be an outer function such that

$$\left\| 1 - \left| \frac{\rho}{\sigma} \right|^2 \right\|_\infty = \epsilon \tag{12}$$

and let g be the corresponding minimizer of (11). Then

$$\|\sigma(f-g)\|^2 \leq \frac{4\epsilon}{1-\epsilon} \|\sigma f\|^2. \tag{13}$$

For the proof, we need the following useful lemma.

Lemma 8: Let let $g \in H_{\mathbb{Q}}(\mathbb{D})$ satisfy $g(z_k) = w_k$ for $k = 0, \dots, n$, and let f be the minimizer of (6). Then, if $\|\sigma g\|^2 \leq (1+\delta)\|\sigma f\|^2$, we have $\|\sigma(f-g)\|^2 \leq 2\delta \|\sigma f\|^2$.

Proof: From the parallelogram law we have

$$\frac{1}{2}(\|\sigma f\|^2 + \|\sigma g\|^2) = \left\| \sigma \frac{f+g}{2} \right\|^2 + \left\| \sigma \frac{f-g}{2} \right\|^2.$$

Therefore, since f is the minimizer of (6), and hence $\|\sigma f\| \leq \|\sigma(f+g)/2\|$, it follows that

$$\|\sigma(f-g)\|^2 \leq 2(\|\sigma g\|^2 - \|\sigma f\|^2) \leq 2\delta \|\sigma f\|^2$$

which concludes the proof of the lemma. ■

Proof of Theorem 7: In view of (12), we have

$$(1-\epsilon)|\sigma(e^{i\theta})|^2 \leq |\rho(e^{i\theta})|^2 \leq (1+\epsilon)|\sigma(e^{i\theta})|^2$$

for all $\theta \in [-\pi, \pi]$. Therefore, since g is the minimizer of (11), by (12), we have

$$\begin{aligned}\|\sigma g\|^2 &\leq \frac{1}{1-\epsilon}\|\rho g\|^2 \leq \frac{1}{1-\epsilon}\|\rho f\|^2 \\ &\leq \frac{1+\epsilon}{1-\epsilon}\|\sigma f\|^2 = (1+\delta)\|\sigma f\|^2\end{aligned}$$

where $\delta := 2\epsilon/(1-\epsilon)$. Consequently (13) follows from Lemma 8. ■

We have thus shown that if $|\rho(z)/\sigma(z)|$ is close to 1 for $z \in \mathbb{T}$, then $\|\sigma(f-g)\|$ is small. This suggests the following approximation procedure, illustrated in Fig. 1. By Theorem 5, the function F, defined by (9), maps the subset $\mathcal{K}_0$ into the space of interpolants of degree at most n. In Fig. 1 these subsets are depicted by fat lines. The basic idea is to replace the hard problem of approximating f by a function g of degree at most n by the simpler problem of approximating an outer function σ by a function $\rho \in \mathcal{K}_0$.

Theorem 7 suggests various strategies for choosing the functions $\rho \in \mathcal{K}_0$ and $\sigma \in W_f$ depending on the design preferences. If a small error bound for $\|\sigma(f-g)\|$ is desired for a particular $\sigma \in W_f$, this σ should be used together with the $\rho \in \mathcal{K}_0$ that minimizes (12).

However, obtaining a small value of (12) is often more important than the choice of σ. Therefore, in general it is more natural to choose the pair $(\sigma, \rho) \in (W_f, \mathcal{K}_0)$ that minimizes ϵ. For such a pair, setting $q := \tau\rho$, we can be see from (2) and (10) that

$$\epsilon = \left\| 1 - \left| \frac{\rho}{\sigma} \right|^2 \right\|_\infty = \left\| 1 - \left| \frac{qf}{p} \right|^2 \right\|_\infty \tag{14}$$

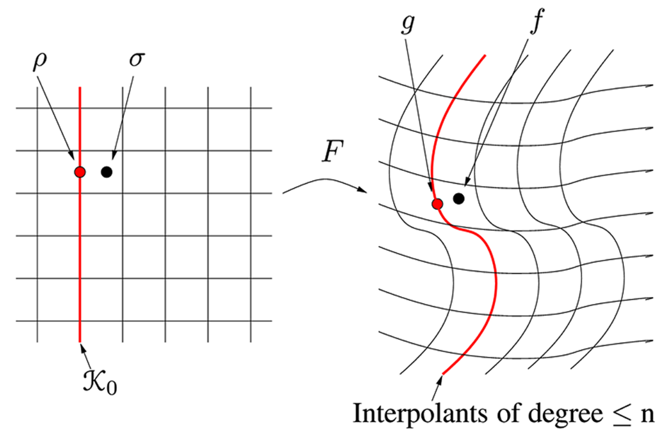

Fig. 1. Map F sending weighting functions to interpolants.

where $q \in \text{Pol}(n)$ and $p \in \text{Pol}(n) \setminus \{0\}$ needs to be chosen so that p/f is outer. It is interesting to note that (14) is independent of $\tau(z) := \prod_{k=0}^n (1 - \bar{z}_k z)$ and hence of the interpolation points $z_0, z_1, \dots, z_n$.

Now suppose that f has ν zeros in $\mathbb{D}$; i.e., ν nonminimum-phase zeros. Then $f = \pi f_0$, where f_0 is outer (minimum phase) and π is an unstable polynomial of degree $\nu \leq n$. Setting $p = \pi p_0$, our optimization problem to minimize ϵ reduces to the problem to find a pair $(p_0, q) \in \text{Pol}(n-\nu) \times \text{Pol}(n)$ that minimizes

$$\epsilon = \left\| 1 - \left| \frac{qf_0}{p_0} \right|^2 \right\|_\infty \tag{15}$$

for a given nonminimum-phase f_0. This is a quasi-convex optimization problem, which can be solved as described in the Appendix (see also [16], [17]). The optimal q yields the optimal $\rho = q/\tau$. The approximant g is then obtained by solving the optimization problem (11) as described in Theorem 5.

One should note that, the more zeros f has inside $\mathbb{D}$, the smaller is the choice of p. Therefore, one expects approximations of non-minimum phase plants to be worse than approximations of plants without unstable zeros.

VI. Rational Approximation

In applications where there are no *a priori* interpolation constraints, the choice of interpolation points serve as additional design parameters. It is then important to choose them so that a good approximation is obtained. The main strategy previously used is to chose interpolation points close to the regions of the unit circle where good fit is desired. The closer to the unit circle the points are placed, the better fit, but the smaller is the region where good fit is ensured; see [8] for further discussions on this. However, in this paper we shall provide a systematic procedure for choosing the interpolation points, based on quasi-convex optimization.

As we have seen in the previous section the choice of interpolation points does not affect ϵ given by (14). However, since $\sigma = p/(f\tau)$, the weighted H_2 error bound (13) in Theorem 7 becomes

$$\left\| \frac{p}{\tau} \frac{f-g}{f} \right\|^2 \leq \frac{4\epsilon}{1-\epsilon} \left\| \frac{p}{\tau} \right\|^2$$

which depends on τ and hence on the choice of interpolation points. In fact, this is a weighed H_2 bound on the relative error $(f-g)/f$. If a specific part of the unit circle is of particular interest, interpolation points may be placed close to that part, which gives a bound on the weighted relative error with high emphasis on that specific region. (For a method to do this by convex optimization, see Remark 3 in the next section.) If no particular part is more important than the rest, we suggest to select τ as the outer part of p; i.e., $|\tau(z)| = |p(z)|$ for $z \in \mathbb{T}$. This gives a natural choice of interpolation points that are the mirror images of the roots of τ. Furthermore, this choice gives the relative error bound $\|(f-g)/f\| \leq 4\epsilon/(1-\epsilon)$. This is summarized in the following theorem.

Theorem 9: Let p and q be polynomials of degrees at most n such that pf^{-1} is outer, and set

$$\epsilon := \left\| 1 - \left| \frac{qf}{p} \right|^2 \right\|_\infty . \tag{16}$$

Let $z_0, z_1, \ldots, z_n \in \mathbb{D}$ and let

$$g = \arg\min \|\rho g\| \text{ s.t. } g(z_k) = f(z_k), \quad k = 0, \ldots, n$$

where $\rho = q/\tau$ and $\tau = \prod_{k=0}^{n}(1 - \bar{z}_k z)$. Then $\deg g \leq n$ and

$$\left\| \frac{p}{\tau} \frac{f-g}{f} \right\|^2 \leq \frac{4\epsilon}{1-\epsilon} \left\| \frac{p}{\tau} \right\|^2 . \tag{17}$$

In particular, if the interpolation points $z_0, z_1, \ldots, z_n$ are chosen so that $|\tau(z)| = |p(z)|$ for $z \in \mathbb{T}$, then

$$\left\| \frac{f-g}{f} \right\|^2 \leq \frac{4\epsilon}{1-\epsilon}. \tag{18}$$

Remark 1: Note that the choice $|\tau| = |p|$ in Theorem 9 implies that the unstable zeros of f become interpolation points. Therefore, for $\epsilon < 1, (f-g)/f$ belongs to H_2.

Remark 2: Our method requires that we choose n to be greater than or equal to the number of unstable zeros. This is a natural design restriction, since the approximation problem becomes more difficult the larger is the number of unstable zeros. It should be noted that other methods for which there is a bound for the relative error, such as balanced stochastic truncation or Glover's relative error method (see, e.g., [10]), will not work either if the number of unstable zeros exceeds n. In fact, in such a case the corresponding phase function will have more than n Hankel singular values equal to 1, and therefore the bound will be infinite, and the problem to minimize $\|(f-g)/f\|_\infty$ over all g of degree at most n will have the optimal solution $g \equiv 0$. Also note that, unlike these methods, our method does not require f to be rational.

VII. The Computational Procedure and Some Illustrative Examples

We summarize the computational procedure suggested by the theory presented above and apply it to some examples.

Given a function $f \in H_{\mathbb{Q}}(\mathbb{D})$ with at most n zeros in $\mathbb{D}$, we want to construct a function $g \in H_{\mathbb{Q}}(\mathbb{D})$ of degree at most n that approximates f as closely as possible. We consider two versions of this problem. First, we assume that f satisfies the interpolation condition (1), and we require g to satisfy the same interpolation conditions. Secondly, we relax the problem by removing the interpolation constraints.

Suppose that f has $\nu \leq n$ zeros in $\mathbb{D}$. Then $f = \pi f_0$, where f_0 is minimum-phase, and π is a polynomial of degree ν with zeros in $\mathbb{D}$. The approximant g can then be determined in two steps.

i) Solve the quasi-convex optimization problem to find a pair $(p_0, q) \in \mathrm{Pol}(n-\nu) \times \mathrm{Pol}(n)$ that minimizes (15), as outlined in the Appendix. This yields optimal ϵ, p_0 and q. Set $p := \pi p_0$.

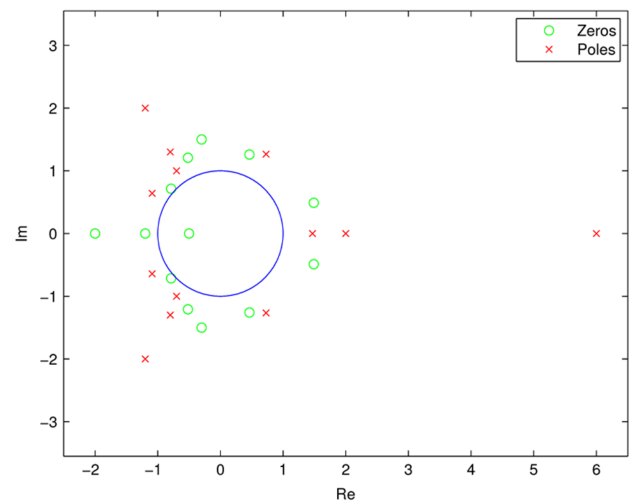

Fig. 2. Poles and zeros of f in Example 1.

ii) Solve the optimization problem (11) with $\rho = q/\tau$, as described in Theorem 5. Exchanging σ for ρ in (8), we solve the Vandermonde system

$$\beta(z_k) = q(z_k) w_k, \quad k = 0, 1, \ldots, n$$

for the $\beta \in \mathrm{Pol}(n)$, which yields

$$g = \frac{\beta}{q} \tag{19}$$

and the bound (17), where $\tau(z) := \prod_{k=0}^{n}(1 - \bar{z}_k z)$.

For the problem without interpolation condition, we replace step (ii) by one of the following steps.

(ii)$'$ Choose $z_0, z_1, \ldots, z_n$ arbitrarily, or as in Remark 3 below. This yields a solution (19) and a bound (17).

(ii)$''$ Choose $z_0, z_1, \ldots, z_n$ so that τ is the outer (minimum-phase) factor of p. This yields a solution (19) and the bound (18) for the relative H_2 error.

Remark 3: If a bound on the weighted error $\|w(f-g)\|$ is desired in Step (ii)$'$, it is natural to choose τ so that $p/(\tau f)$ is as close to w as possible. This may be done by solving the convex optimization problem to find a $\tau \in \mathrm{Pol}(n)$ that minimizes

$$\left\| 1 - \left| \frac{\tau f w}{p} \right|^2 \right\|_\infty$$

as in the Appendix. If instead we need a bound on the weighted relative error $\|w(f-g)/f\|$, we modify the optimization problem accordingly.

Next, we apply these procedures to some numerical examples.

Example 1: Let $f(z) = b(z)/a(z)$ be the stable system of order 13 with $f(\infty) = 7.5$ and whose poles and zeros are given in Fig. 2. This system has one minimum-phase zero. Consider the problem to approximate f by a function g of degree six while preserving the values in the points $(z_0, z_1, \ldots, z_n) = (0, 0.3, 0.5, -0.1, -0.7, -0.3 \pm 0.3i)$. Such an interpolation condition occurs in many applications. Now, suppose we want to find a rational sensitivity function S_n

Step (i) to solve the quasi-convex optimization problem to minimize (15) yields optimal ϵ, p and q, and Step (ii) the approximant g, the Bode plot of which is depicted in Fig. 3 together with that of f. The third

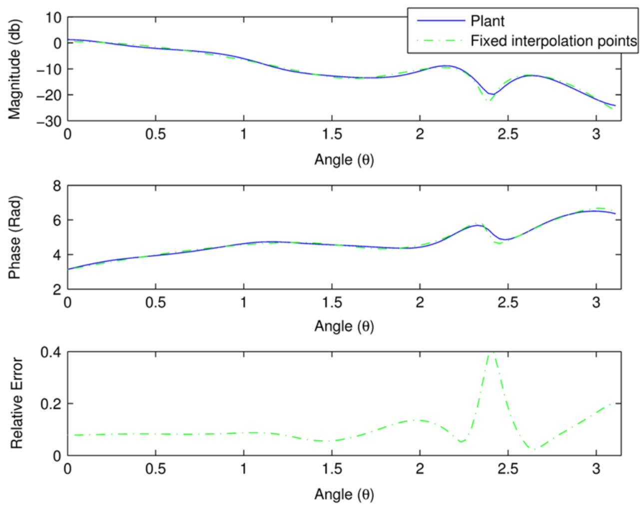

Fig. 3. Bode plots of f and g together with the relative error.

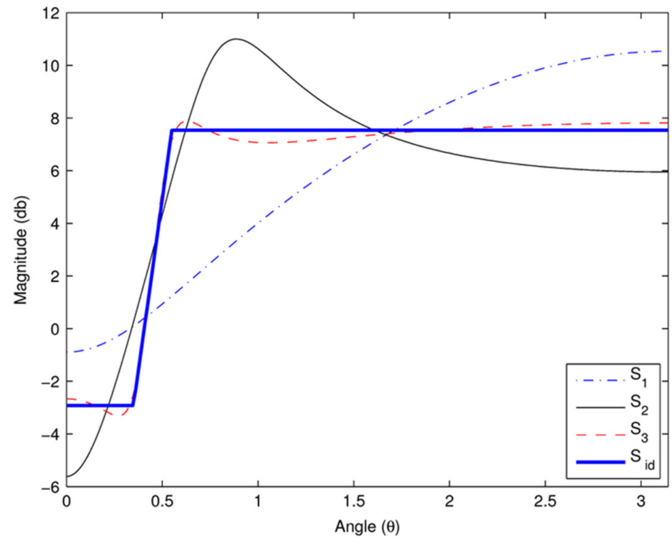

Fig. 4. Approximations of degree 1, 2, and 3.

subplot in the picture shows the relative error $(f-g)/f$. It is important to note that the function g, which is guaranteed to be stable, satisfies the prespecified interpolation conditions and the error bound (17). Fig. 3 shows that g matches f quite well.

Example 2: Finally, we apply our approximation procedure to loop-shaping by low-degree controllers in robust control, where interpolation conditions are needed to ensure internal stability [7]. Given a plant P, a controller is often designed by shaping the sensitivity function $S = 1/(1-PC)$, where P and C are the transfer functions of the plant and the controller respectively. In fact, the design specifications may often be translated into conditions on the sensitivity function. For internal stability of the closed-loop system, the sensitivity function S needs to satisfy the following properties: (i) S is analytic in $\mathbb{D}^c := \{z \mid |z| \geq 1\}$, (discrete time); (ii) $S(z_k) = 1$ whenever z_k is an unstable zero of P; (iii) $S(p_k) = 0$ whenever p_k is an unstable pole of P; Furthermore, in general we require that (iv) S has low degree, and (v) S satisfies additional design specifications. Conditions (i)-(iv) do not, in general, uniquely specify S, so the additional freedom can be utilized to satisfy additional design specifications (v).

As a simple example, also illustrating rational approximation of a nonrational function, consider sensitivity shaping of a feedback system with the plant $P(z) = (z-2)^{-1}$. Since P has an unstable pole at $z = 2$ and an unstable zero at $z = \infty$, the sensitivity function must satisfy $S(\infty) = 1$ and $S(2) = 0$. Then the function $f(z) := S(z^{-1})$ is analytic in $\mathbb{D}$, and satisfies

$$f(0) = 1 \quad \text{and} \quad f(1/2) = 0. \tag{20}$$

of degree n that preserves internal stability and that approximates an ideal sensitivity function $S_{\rm id}$ with the spline-formed shape in bold in Fig. 4. The shape is originally given as a positive function W on the unit circle, and a normalizing factor $\rho > 0$ needs to be chosen so that $|S_{\rm id}(e^{i\theta})| = \rho W(e^{i\theta})$ for $\theta \in [-\pi, \pi]$. An outer function h having the prescribed shape is given by

$$h(z) = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log W(e^{-i\theta})d\theta\right]$$

(see, e.g., [11, p. 63]). Now, define the function $f(z) = \rho h(z)(z-1/2)/(1-z/2)$, where ρ is selected so that $f(0) = 1$. Then f is analytic in $\mathbb{D}$ and satisfies the interpolation conditions (20), and $S_{\rm id}(z) = f(z^{-1})$. Clearly, f is nonrational and $S_{\rm id}$ represents a infinite-dimensional system.[1].

By using the computational procedure in the beginning of the section, we determine the approximants g_n of f of degrees $n = 1, 2$ and 3 which satisfy the interpolation conditions (20). More precisely, g_1 is determined via steps (i) and (ii), whereas, for g_2 and g_3, we need to add one or two extra interpolation points and use (i) and (ii)$'$. That is, $z_0 = 0$ and $z_1 = 1/2$, and z_2 and z_2, z_3, respectively, are determined as in Remark 3 with $w := f^{-1}$.

The magnitudes of the corresponding sensitivity functions S_1, S_2 and S_3, obtained from $S_n(z) = g_n(z^{-1})$, are depicted in Fig. 4. The degree of the controller corresponding to the approximant S_3 is two.

VIII. Concluding Remarks

This paper presents a new theory for stability-preserving model reduction (for plants that need not be minimum-phase) that can also handle prespecified interpolation conditions and comes with error bounds. We have presented a systematic optimization procedure for choosing appropriate weight (and, if desired, interpolation points) so that the minimizer of a corresponding weighted H_2 minimization problem both matches the original system and has low degree.

The study of the H_2 minimization problem is motivated by the relation between the H_2 norm and the entropy functional used in bounded interpolation. Therefore, new concepts derived in this framework are useful for understanding entropy minimization. In fact, the degree reduction methods proposed in this paper easily generalize to the bounded case; see [12] for the method which preserves interpolation conditions. We are currently working on similar bounds for the positive real case; also, see [8].

Appendix

The optimization problem to minimize (16), where p and q are polynomials of fixed degree is *quasi-convex*; i.e., each sublevel set is

[1]For a systematic procedure to determine $S_{\rm id}$ from W for general interpolation constraints, see [12]

convex. For simplicity, we assume that f is real and hence that p and q are real as well.

As a first step, consider the *feasibility problem* of finding a pair (p, q) of polynomials satisfying

$$\left\| 1 - \left| \frac{qf}{p} \right|^2 \right\|_\infty \leq \epsilon \tag{21}$$

for a given ϵ, or, equivalently

$$-\epsilon |p(e^{i\theta})|^2 \leq |p(e^{i\theta})|^2 - |q(e^{i\theta}) f(e^{i\theta})|^2 \leq \epsilon |p(e^{i\theta})|^2$$

for all $\theta \in [-\pi, \pi]$. Since $|p|^2$ and $|q|^2$ are pseudo-polynomials, they have representations $|p(e^{i\theta})|^2 = 1 + \sum_{k=1}^{n_p} p_k \cos(k\theta)$ and $|q(e^{i\theta})|^2 = \sum_{k=0}^{n_q} q_k \cos(k\theta)$, where n_p and n_q are the degree bounds on p and q, respectively, and the first coefficient in $|p|^2$ is chosen to be one without loss of generality. Hence, (21) is equivalent to

$$\begin{aligned} -1 - \epsilon \leq (1+\epsilon) \sum_{k=1}^{n_p} p_k \cos k\theta & \\ - |f(e^{i\theta})|^2 \sum_{k=0}^{n_q} q_k \cos k\theta & \\ 1 - \epsilon \leq (\epsilon - 1) \sum_{k=1}^{n_p} p_k \cos k\theta & \\ + |f(e^{i\theta})|^2 \sum_{k=0}^{n_q} q_k \cos k\theta & \end{aligned}$$

for all $\theta \in [-\pi, \pi]$. There is also a requirement on $1 + \sum_{k=1}^{n_p} p_k \cos(k\theta)$ and $\sum_{k=0}^{n_q} q_k \cos(k\theta)$ to be positive. However, if $\epsilon \in (0, 1)$, then the above constraints will imply positivity. The set of $p_1, p_2, \ldots, p_{n_p}, q_0, q_1, \ldots, q_{n_q}$ satisfying this infinite number of linear constraints is convex.

The most straightforward way to solve this feasibility problem is to relax the infinite number of constraints to a finite grid, which is dense enough to yield an appropriate solution. Here, one must be careful to check the positivity of $1 + \sum_{k=1}^{n_p} p_k \cos(k\theta)$ and $\sum_{k=0}^{n_q} q_k \cos(k\theta)$ in the regions between the grid points. Another method is the Ellipsoid Algorithm, described in detail in [2].

Minimizing (16) then amounts to finding the smallest ϵ for which the feasibility problem has a solution. This can be done by the bisection algorithm, as described in [2]. Note that for $\epsilon = 1$, the trivial solution $q = 0$ is always feasible.

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