On degree-constrained analytic interpolation with interpolation points close to the boundary

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***Abstract*— In the recent article [2] a paradigm for complexity constrained interpolation of contractive functions is developed. In particular, it is shown that any such interpolant may be obtained from a convex optimization problem, minimizing a generalized entropy gain. With this as a background, we study the optimization problem in detail and derive certain properties of it. One of the main results is that, if, for a sequence of interpolants, the values of the generalized entropy gain of the interpolants converge to the optimum, then the interpolants converge in H_2. This result is used in order to get the asymptotic behavior of the interpolant as an interpolation point approach the boundary of the domain of analyticity. Finally, a control design example which has been considered by Nagamune [8] is studied, and the results are reexamined in this framework.**

I. Introduction

Many important engineering problems lead to analytic interpolation, where the interpolant represents a transfer function of, for example, a feedback control system or a filter and therefore is required to be a rational function of bounded degree. In recent years, a complete theory of analytic interpolation with degree constraint has been developed; see [1], [2] and references therein. The theory provides complete smooth parameterizations of whole classes of such interpolants in terms of a weighting function belonging to a finite-dimensional space, as well as convex optimization problems for determining them.

This theory provides a framework for tuning an engineering design based on analytic interpolation to satisfy additional design specification without increasing the degree of the transfer function. How to do this in a systematic way remains partly an open question. Occasionally the number of tuning parameters is too small to achieve the specifications, and then the parameter space needs to be increased by increasing the degree bound. Nagamune [8] has suggested that this be done by adding new interpolation condition, often close to the boundary.

In this paper we study how the interpolant behaves as new interpolation points are added close to the boundary. From the analysis we show that unless the weighting function is changed, adding new interpolation points close to the boundary will have little effect on the interpolant. We illustrate this by analyzing a simple example from robust control.

In Section II we begin by reviewing some pertinent results from [2]. Then, in Section III, we provide a motivation example from robust control, which is then revisited in Section V after having presented the main results in Section IV. Some proofs are deferred to Section VI.

II. Background

Consider the classical Pick problem of finding a function f in the Schur class

$$\mathcal{S} = \{f \in H_\infty(\mathbb{D}) \, : \, \|f\|_\infty \le 1\}$$

that satisfies the interpolation condition

$$f(z_k) = w_k, \quad k = 0, 1, \dots, n, \tag{1}$$

where $(z_k, w_k), k = 0, 1, \dots, n$, are given pairs of points in the open unit disc $\mathbb{D} = \{z \, : \, |z| < 1\}$. It is well-known that such an $f \in \mathcal{S}$ exists if and only if the Pick matrix

$$P(z, w) = \left[\frac{1 - w_k \bar{w}_\ell}{1 - z_k \bar{z}_\ell}\right]_{k,\ell=0}^n \tag{2}$$

is positive semi-definite, and that the function f is unique if and only if the matrix $P(z, w)$ is singular. In the latter case f is a Blaschke product of degree equal to the rank of $P(z, w)$. Here we shall take $P(z, w)$ to be positive definite, in which case there are infinitely many solutions to the Pick problem. A complete parameterization of the solutions of this so called Nevanlinna-Pick interpolation problem was given by Nevanlinna [9] in 1929. The parameterization is in terms of a linear fractional transformation centered around a rational solution of degree n, known as the *central solution*.

In a research program culminating in [1], [2], the subset of all solutions of the Nevanlinna-Pick problem that are rational of degree at most n were parameterized. Most engineering problems require such degree constraints, which completely alter the basic mathematical problem. More precisely, let $\mathcal{K}$ be the space of all functions

$$f(z) = \frac{\rho(z)}{\tau(z)},$$

where $\rho(z)$ is an arbitrary polynomial of degree at most n and

$$\tau(z) = \prod_{k=1}^{n} (1 - \bar{z}_k z).$$

Clearly $\mathcal{K}$ is a subspace of the hardy space H^2. Moreover, let $\mathcal{K}_0$ be the the subset of all $f \in \mathcal{K}$ such that $\rho(z)$ has all its roots in the complement of $\mathbb{D}$ and $\rho(0) > 0$. In this context, Theorem 1 in [2] can be stated in the following way.

Theorem 1: Suppose that the Pick matrix $P(z, w)$ is positive definite. Let σ be an arbitrary function in $\mathcal{K}_0$. Then there exists a unique pair $(a, b) \in \mathcal{K}_0 \times \mathcal{K}$ such that

1) $f = b/a \in \mathcal{S}$
2) $f(z_k) = w_k, \; k = 0, 1, \dots, n$, and,

3) $|a|^2 - |b|^2 = |\sigma|^2$ a.e. on $\mathbb{T} := \{z : |z| = 1\}$.

Conversely, any pair $(a, b) \in \mathcal{K}_0 \times \mathcal{K}$ satisfying 1 and 2 determines, via 3, a unique $\sigma \in \mathcal{K}_0$.

Consequently, the solutions (a, b) corresponding to interpolants of degree at most n are completely parameterized by the zeros of $\sigma \in \mathcal{K}_0$; i.e., the n-tuples $\{\lambda_1, \dots, \lambda_n\}$ of complex number in the complement of $\mathbb{D}$; these are called the *spectral zeros*. For each such choice of spectral zeros, the corresponding interpolant $f \in \mathcal{S}$ can be determined by minimizing the strictly convex functional

$$\mathbb{K}_\Psi : \mathcal{S} \to \mathbb{R}, \quad \mathbb{K}_\Psi(f) = -\int_{\mathbb{T}} \Psi \log(1 - |f|^2) dm(z),$$

over the class of interpolants, where $\Psi := |\sigma|^2$ and m is the normalized Lebesgue measure on $\mathbb{T}$. In fact, in the present context, Theorem 5 in [2] can can be stated as follows.

Theorem 2: Suppose that the Pick matrix $P(z, w)$ is positive definite. Let σ be an arbitrary function in $\mathcal{K}_0$, and set $\Phi := |\sigma|^2$. Then the functional $\mathbb{K}_\Psi$ has a unique minimizer in the class of functions that satisfy the interpolation conditions (1), and this minimizer is precisely the unique function $f \in \mathcal{S}$ satisfying conditions 1, 2 and 3 in Theorem 1.

The central solution corresponds to $\Psi \equiv 1$. The corresponding functional $\mathbb{K}_1$ is the usual entropy gain, and therefore the central solution is often called the *minimum entropy solution* (see, e.g. [7]). If $z_0, w_0) = (0, 0)$, then $\sigma \equiv 1 \in \mathbb{K}_0$, and the corresponding spectral zeros are at the conjugate inverse (mirror image in unit circle) of $\{z_k\}_{k=1}^n$.

The theory described by Theorems 1 and 2 allows us to choose an interpolant that best satisfies additional design specifications. In fact, the map from σ to (a, b) defined by Theorem 1 is smooth [3], [4], and hence a given design can be tuned via Ψ to smoothly change the interpolant. An obvious first choice of Ψ is to make it large in frequency bands where low sensitivity $|S|$ is required. However, how to design systematic tuning strategies remains a partly open problem, and to settle this question is an important task. This paper is an attempt to gain understanding of the underlying function theory involved in tuning the interpolant.

III. A MOTIVATING EXAMPLE

The purpose of this paper is to show how the interpolants change as additional interpolation points are introduced, especially close to the boundary of $\mathbb{D}$. To illustrate this point, we consider an example on sensitivity shaping in robust control from [8], which we shall return to later in this paper. Figure 1 depicts a feedback system with u denoting the control input to the plant

$$G(z) = \frac{z}{1 - 1.05z}$$

to be controlled, d represents a disturbance, and y is the resulting output, which is fed back through a compensator $K(z)$ to be designed. The goal is to determine a controller $K(z)$ so that the feedback system in Figure 1 satisfies the

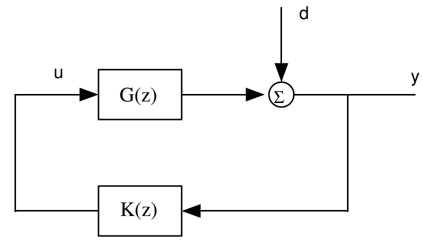

Fig. 1. A feedback system.

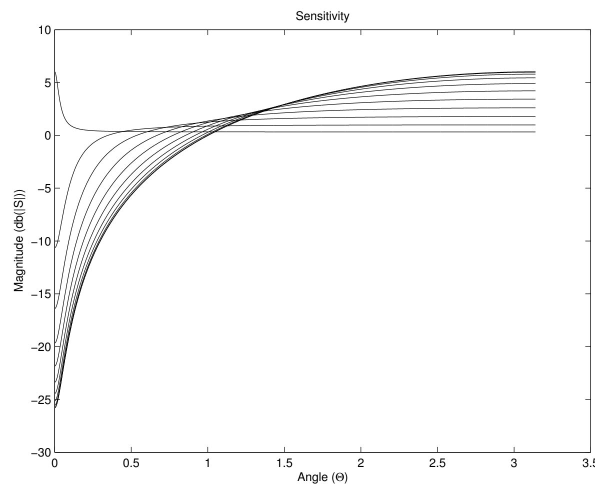

Fig. 2. *The degree* 1 *interpolants corresponding to* $1/\lambda$ *from* -1 *to* 1 *with grid* 0.2.

design specifications

$$\begin{aligned} |S(e^{i\theta})| &< 2 \sim 6,02dB && 0 \le \theta \le \pi \\ |S(e^{i\theta})| &< 0,1 \sim -20dB && 0 \le \theta \le 0,3 \\ |T(e^{i\theta})| &< 0.5 \sim -6,02dB && 2,5 \le \theta \le \pi \end{aligned} \tag{3}$$

in terms of the sensitivity function $S = (1 - GK)^{-1}$ and the complementary sensitivity function $T = 1 - S$.

The plant $G(z)$ has one unstable pole at $z = 0.9524$ and one non-minimum phase zero at $z = 0$. It follows from H^∞ control theory (see, e.g.,[5]) that the feedback system is internally stable if and only if the sensitivity function $S(z)$, the transfer function from d to y, is analytic in $\mathbb{D}$ and satisfies the interpolation conditions

$$S(0.9524) = 0, \quad S(0) = 1. \tag{4}$$

A simple calculation shows that there exist analytic interpolants such that $\|S\|_\infty < \gamma := 2$. Therefore, since $n = 1$, by Theorem 1, there exists a family of degree-one interpolants satisfying $\|S/\gamma\|_\infty \le 1$ that may be parametrized by one spectral zero, λ. Figure 2 shows the solutions S as $1/\lambda$ goes from -1 to 1 with the grid 0.2.

Clearly none of these designs satisfies the specifications. Therefore, following Nagamune [8], we add the interpolation condition

$$S(-0,9901) = 1 \tag{5}$$

and consider the corresponding family of interpolants of degree two, described by Theorem 1. Choosing the two

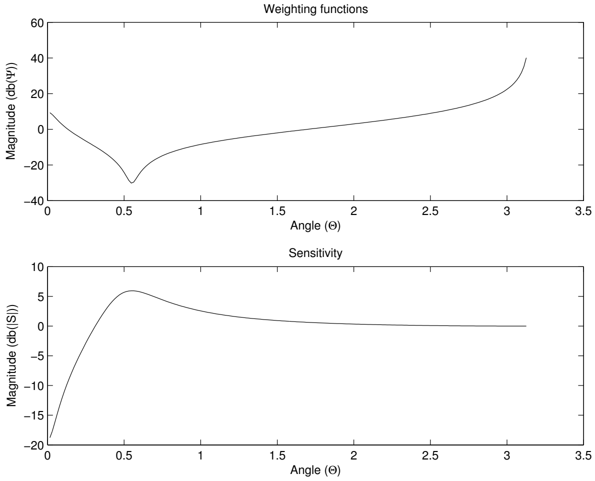

Fig. 3. *Ψ and $|S|$ associated with the interpolation conditions* (4) *and* (5) *and spectral zeros in the mirror image of* $0.97e^{\pm 0.55i}$.

spectral zeros to be the mirror image (in the unit circle) of $0.97e^{\pm 0.55i}$, we obtain the sensitivity function depicted in Figure 3. This design clearly does not satisfiy the specifications either, and it can be shown that nor does any other S of degree two [8]. Below in Section V, we shall see what happens when further interpolation conditions are added close to the unit circle.

IV. Main results

As the example of Section III suggests, we need to investigate how the interpolant changes as additional interpolation points are introduced close to the unit circle. The following theorem is one of our main results.

Theorem 3: Let $\Psi = \sigma\sigma^*$, where $\sigma \in \mathcal{K}_0$, and let $\hat{f}$ be the minimizer of $\mathbb{K}_\Psi(f)$ subject to the interpolation conditions

$$f(z_k) = w_k, \quad k = 0, 1, \ldots, n.$$

Moreover, given $|w| \leq 1$, let f_λ be the minimizer of $\mathbb{K}_\Psi(f)$ subject to

$$f(z_k) = w_k, \quad k = 0, 1, \ldots, n, \quad f(\lambda) = w.$$

Then $f_\lambda \to \hat{f}$ in H_2 as $|\lambda| \to 1$.

This theorem indicates that adding interpolation conditions close to the unit circle will not affect the design in any important way unless we also change the weighting function Ψ. For the proof we first need to show that if the generalized entropy of interpolants converge to the optimum, then the interpolants converge to the optimal interpolant in H_2. The following theorem is proved in Section VI.

Theorem 4: Let $\hat{f}$ be the minimizer of $\mathbb{K}_\Psi(f)$ subject to $f(z_k) = w_k$, $k = 0, 1, \ldots, n$. If f_ℓ satisfies $f_\ell(z_k) = w_k$, $k = 0, 1, \ldots, n$, and $\mathbb{K}_\Psi(f_\ell) \to \mathbb{K}_\Psi(\hat{f})$, then $f_\ell \to \hat{f}$ in H_2.

It should be noted that this result could not be strengthened to H_∞ convergence. An idea for a counterexample could be formed from noting that $\mathbb{K}_\Psi(f + \alpha\chi_{E_\ell}) \to \mathbb{K}_\Psi(f)$ if $m(E_\ell) \to 0$. Here χ denotes the characteristic function and α is a scalar such that $0 < |\alpha| < 1 - \|f\|_\infty$. But $\|\alpha\chi_{E_\ell}\|_\infty = \alpha$ for all ℓ. This argument works equally well for $f + g_\ell$, where $g_\ell \in \phi H_2$ and $|g_\ell|$ is an appropriate approximation of χ_{E_ℓ}.

A second step in proving Theorem 3 is to investigate how the interpolant changes as the data is transformed under a Möbius transformation. For $\lambda\mathbb{D}$, let b_λ be the Blaschke factor

$$b_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda}z}.$$

Then the following proposition tells us how the entropy is changed as the range is transformed by a Möbius transformation.

Proposition 5: The map $\rho(\cdot, \lambda, \Psi) : \mathbb{S} \to \mathbb{R}$ defined by

$$\rho(f, \lambda, \Psi) = \int_{\mathbb{T}} \Psi \log \frac{|1 - \bar{\lambda}f|^2}{1 - \bar{\lambda}\lambda} dm(z)$$

is continuous, and

$$\mathbb{K}_\Psi(b_\lambda(f)) = \mathbb{K}_\Psi(f) + \rho(f, \lambda, \Psi).$$

Moreover, if $\Psi = |\sigma|^2$ where $\sigma \in \mathcal{K}_0$, then $\rho(f_1, \lambda, \Psi) = \rho(f_2, \lambda, \Psi)$, whenever $f_1(z_k) = f_2(z_k)$ for $k = 0, 1, \ldots, n$.

Proof: First part is trivial, second part follows from [2, p. 8 and Lemma 10]. ■

As a corollary we have the following proposition, which tells us that the solution obtained from the transformed data is the solution transformed with the same transformation.

Proposition 6: Let $\sigma \in \mathcal{K}_0$, and let f be the the corresponding solution to the analytic interpolation problem $f(z_k) = w_k, k = 0, 1, \ldots, n$, $\|f\|_\infty < 1$ prescribed by Theorem 1. Then $g = b_\lambda(f)$ is the interpolant corresponding to the same σ of the analytic interpolation problem $g(z_k) = b_\lambda(w_k), k = 0, 1, \ldots, n$, $\|g\|_\infty < 1$.

A simple proof from basic principles of Proposition 6 is given in Section VI.

To conclude the proof of Theorem 3, we first prove a version in which the interpolation value is 0.

Theorem 7: Let $\hat{f}$ be the minimizer of $\mathbb{K}_\Psi(f)$ subject to the interpolation conditions $f(z_k) = w_k$, $k = 0, 1, \ldots, n$. Moreover, let f_λ be the minimizer of $\mathbb{K}_\Psi(f)$ subject to $f(z_k) = w_k$, $k = 0, 1, \ldots, n$ and $f(\lambda) = 0$. Then $f_\lambda \to \hat{f}$ in H_2 as $|\lambda| \to 1$.

Proof: Let $M_\lambda = \{g : g \in \mathbb{S}, g(z_k) = \frac{w_k}{b_\lambda(z_k)}\}$. First note that if $g \in M_\lambda$, then gb_λ satisfies the interpolation conditions. Furthermore

$$\mathbb{K}_\Psi(g) = \mathbb{K}_\Psi(gb_\lambda) \geq \mathbb{K}_\Psi(f_\lambda) \geq \mathbb{K}_\Psi(\hat{f})$$

by definition of f_λ and $\hat{f}$. If we can prove that

$$\min_{g \in M_\lambda} \mathbb{K}_\Psi(g) \to \mathbb{K}_\Psi(\hat{f}) \text{ as } |\lambda| \to 1, \tag{6}$$

then $\mathbb{K}_\Psi(f_\lambda) \to \mathbb{K}_\Psi(\hat{f})$ and by Theorem 4 it follows that $f_\lambda \to \hat{f}$ in H_2. However, since $\frac{w_k}{b_\lambda(z_k)} \to w_k$ as $|\lambda| \to 1$, there is a sequence of functions $g_\lambda \in M_\lambda$ such that $g_\lambda \to \hat{f}$ in H_∞. By H_∞ continuity of $\mathbb{K}_\Psi$, (6) holds. ■

Note that Theorem 7 holds for any positive and continuous Ψ, whereas Theorem 3 only holds if $\Psi = \sigma\sigma^*$ and σ belong to $\mathcal{K}_0$. This is because the proof of Theorem 3 require the use of Proposition 6.

In fact, we are now in a position to prove Theorem 3. To this end, let $g_\lambda = b_w(f_\lambda)$ and $g = b_w(\hat{f})$. By Proposition 6 and Theorem 1, g_λ is the unique minimizer of $\mathbb{K}_\Psi(f)$ such that $f(z_k) = b_w(w_k)$, $k = 1, \dots, n$, and $f(\lambda) = 0$. Furthermore g is the unique minimizer of $\mathbb{K}_\Psi(f)$ such that $f(z_k) = b_w(w_k)$, $k = 1, \dots, n$. By Theorem 7, $g_\lambda \to g$ in H_2. Since b_λ is Lipschitz continuous, $f_\lambda \to \hat{f}$ in H_2.

V. Revisiting the example

We now return to the example of Section III. Our starting point is the interpolant corresponding to the interpolation conditions (4) and (5) and to the spectral zeros in the mirror image of $0.97e^{\pm 0.55i}$; i.e.,

$$\Psi = \frac{|z - 0.97e^{0.55i}|^2 |z - 0.97e^{-0.55i}|^2}{|z + 0.9901|^2 |z + 1.05|^2}. \qquad (7)$$

As shown in Figure 2, this design does not satisfy the specifications (3).

Following Nagamune in [8], we now add an additional pair of complex interpolation points close to the unit circle, thus adding the interpolation conditions

$$S(0.9459 \pm 0.2926i) = 0 \qquad (8)$$

to (4) and (5). This allows for sensitivity functions of degree four. Indeed, there still exist interpolants such that $\|S\|_\infty < 2$, so we may, as before, consider consider interpolants such that $\|S/\gamma\|_\infty \le 1$, where $\gamma = 2$.

In Figure 4 the old (degree two) design of Figure 2 is depicted (solid line) together with a new (degree four) design (dashed-dotted line) corresponding to the extended set of five interpolation points but retaining the same Ψ, namely (7). We se that the modulus of the sensitivity function has not changed much in harmony with Theorem 3.

It is also interesting to see how the phase of the solution is changed as additional interpolation conditions are introduced. Figure 5 show phases of the degree two (solid line) and degree four (dashed-dotted line) corresponding to the Psi given by (7). The phases correspond well, except for the region close to the added interpolation point. In that region there is a sharp shift of 2π in the phase. Since the shift occurs over a short interval, the H_2 norm is not affected much, and as the interpolation condition approaches the boundary this shift will have negligle effect on the H_2 norm. This example shows clearly why the same could not be true for the H_∞-norm.

However, adding the two interpolation conditions (8), allows us to choose arbitrarily an additional two spectral zeros. Following Nagamune [8], we choose these to be in the mirror image of $z = 0.9e^{\pm 1.55i}$. This yields the sensitivity function

$$\begin{aligned} S_N &= \frac{-0.0045(z + 1.7392)(z - 0.9524)}{(z + 0.4166 - 17.5386i)(z + 0.4166 + 17.5386i)} \\ &\times \frac{(z - 0.9459 - 0.2926i)(z - 0.9459 + 0.2926i)}{(z + 0.9544 - 0.5181i)(z - 0.9544 + 0.5181i)}, \end{aligned}$$

which indeed satisfies the design specifications (3); see Figure 6.

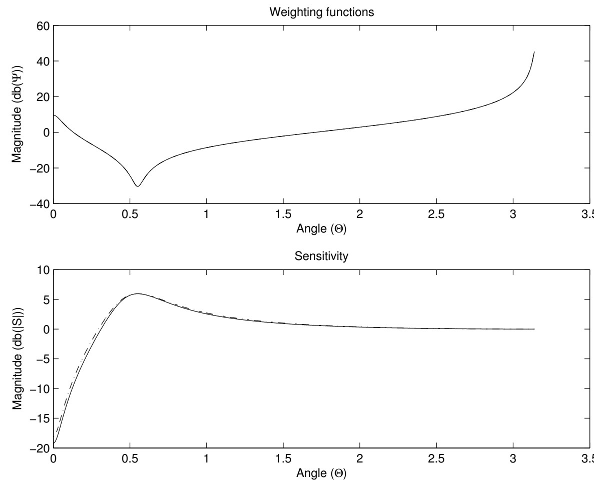

Fig. 4. *The solid lines is the design of Figure 2 and the dash-doted lines is the design with the additional interpolation conditions* (8) *but with the same* Ψ.

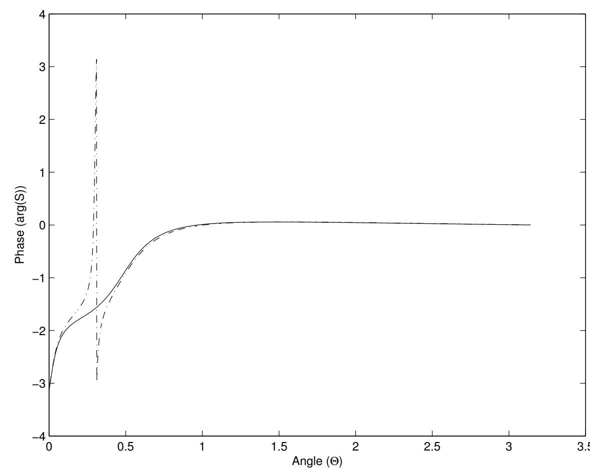

Fig. 5. *The solid line shows the phase of the design of Figure 2 and the dash-doted line the phase of the design with the additional interpolation conditions* (8) *but with the same* Ψ.

As a comparason, the spectral zeros of the maximum entropy solution is the mirror image of $(0.9524, 0.9459 \pm 0.2926i, -0.9901)$ (which is the mirror image of the interpolation points, except for 0), and this yields the interpolant

$$\begin{aligned} S_{ME} &= 0.8705 \frac{(z + 1.0082)(z - 0.9524)}{(z + 1.0095)(z - 1.0500)} \\ &\times \frac{(z - 0.9459 - 0.2926i)(z - 0.9459 + 0.2926i)}{(z - 0.9649 - 0.2985i)(z - 0.9649 + 0.2985i)}. \end{aligned}$$

A plot of the weights and of the modules of the interpolants are shown in Figure 6 and the corresponding interpolation points and spectral zeros in Figure 7.

From Figure 6 it is interesting to note that the magnitude of the sensitivity function corresponding to the ME solution is almost flat. This is expected in view of Theorem 3, since the only interpolation condition that is not close to the boundary is $S(0) = 1$. The minimizer of $\mathbb{K}_1(S)$ such that $S(0) = 1$ is $S \equiv 1$, which is close to S_{ME}.

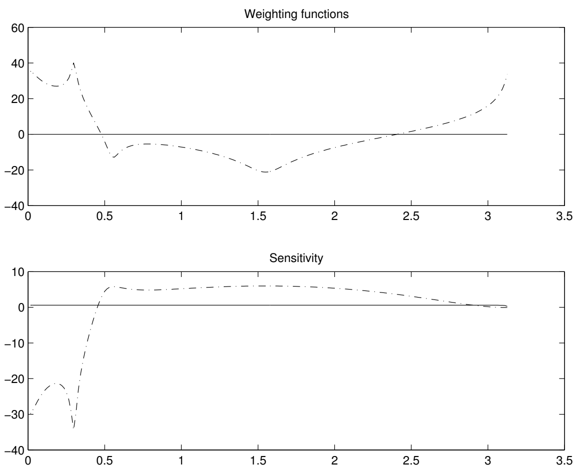

Fig. 6. *The solid line depicts $|S_{ME}|$ corresponding to the ME-solution. The dash-dot depicts $|S_N|$ correponding to the solution by Nagamune,*

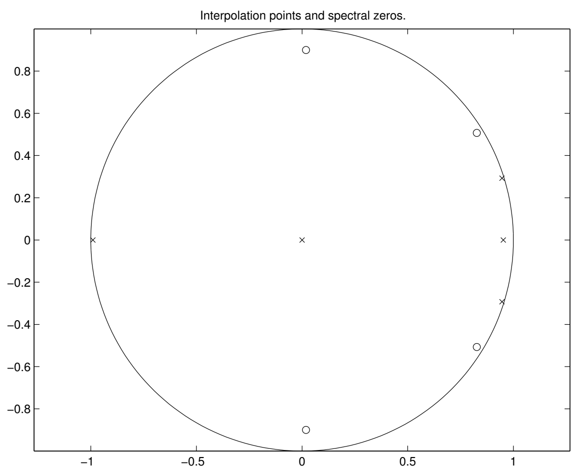

Fig. 7. $\times$*- the intepolation points (and the mirror image of the spectral zeros corresponding to S_{ME}, except 0).* $\circ$*- the mirror image of the spectral zeros corresponding to S_N.*

From the plots in Figure 6 one first notices that in the example where a large weight in the low frequency region is used, the magnitude of the sensitivity is low. This seems to be intuitive since the high weight in the entropy functional penalizes the sensitivity more in that region than in others. However, the weight is also large in the high frequency area, and in this case there is no significant change in the sensitivity. One reason could be that this is due to the interpolation condition $S(-0.9901) = 1$, which lie very close to the boundary in the high frequency region.

VI. PROOFS

For the proof of Theorem 4 we will use arguments from convex analysis. To this end we need some lemmas.

Let $\Lambda : K \to \mathbb{R}$ be a stictly convex functional, where K is compact and convex. Then the minimum

$$\beta = \min_{x \in K} \Lambda(x)$$

exists and is obtained at a unique $x \in K$. Consider the set K_ϵ of $\epsilon-$suboptimal solutions

$$K_\epsilon = \{x \in K : \Lambda(x) \leq \beta + \epsilon\}, \epsilon \geq 0.$$

It seams reasonable that the "size" of K_ϵ converges to zero as $\epsilon \to 0$. However, to guarantee this, we need to consider topological aspects and the concept of strong convexity.

Definition 1: A functional Λ is *strongly convex with respect to the norm* $\|\cdot\|$ if $\frac{1}{2}(\Lambda(x) + \Lambda(y)) \geq \Lambda(\frac{x+y}{2}) + \alpha(\|x-y\|)$, where $\alpha : [0,\infty) \to [0,\infty)$ is continuous, strictly increasing and such that $\alpha(0) = 0$.

Lemma 8: Let $\mathbb{X}$ be a convex set. Let Λ be a strongly convex functional on $\mathbb{X}$ with respect to $\|\cdot\|$. Let x^* be a minimum of $\Lambda(x)$ such that $x \in \mathbb{X}$. Then $\Lambda(x_k) \to \Lambda(x^*), x_k \in \mathbb{X}$ implies $\|x_k - x^*\| \to 0$.

An equivalent statment is that, if Λ is strongly convex with respect to $\|\cdot\|$. Then $\sup\{\|x-y\| : x, y \in K_\epsilon\} \to 0$ as $\epsilon \to 0$, or equivalently, K_ϵ is a neighboorhood basis for the optimal point in the topology induced by $\|\cdot\|$.

Proof: Assume not, i.e. $\exists\epsilon\forall\delta\exists x$ s. t. $|\Lambda(x) - \Lambda(x^*)| < \delta$ and $\|x - x^*\| > \epsilon$. Let $\delta < \alpha(\epsilon)$. Then $\Lambda(x^*) + \delta \geq \frac{1}{2}(\Lambda(x) + \Lambda(x^*)) \geq \Lambda(\frac{x+x^*}{2}) + \alpha(\|x - x^*\|) \geq \Lambda(\frac{x+x^*}{2}) + \alpha(\epsilon)$, which contradicts that x^* is the minimizer. ■

In order to apply this theory to the entropy functional, we will need to show that $\mathbb{K}_\Psi$ is strictly convex with respect to the H_2-norm.

Proposition 9: $\mathbb{K}_\Psi$ is strongly convex with respect to the H_2-norm.

Proof: For $|f| < 1$ and $|g| < 1$, we have the following inequality

$$\begin{aligned} &\tfrac{1}{2}(-\log(1-|f|^2) - \log(1-|g|^2)) \\ &\geq -\log\left(1 - \left|\tfrac{f+g}{2}\right|^2\right) + \tfrac{1}{2}\log\left(1 + \tfrac{|f-g|^2}{8}\right). \end{aligned} \tag{9}$$

Multiply by Ψ and integrate, this leads to

$$\begin{aligned} &\tfrac{1}{2}(\mathbb{K}_\Psi(f) + \mathbb{K}_\Psi(g)) \\ &\geq \mathbb{K}_\Psi\left(\tfrac{f+g}{2}\right) + \tfrac{1}{2}\int_{\mathbb{T}} \Psi \log\left(1 + \tfrac{|f-g|^2}{8}\right) dm. \end{aligned} \tag{10}$$

Since $\log(1+t) \geq t/2$ for $t \in [0, \frac{1}{2}]$, the last term in (10) is bounded from below by

$$\begin{aligned} &\tfrac{1}{2}\int_{\mathbb{T}} \Psi \log\left(1 + \tfrac{|f-g|^2}{8}\right) dm \\ &\geq \int_{\mathbb{T}} \Psi \tfrac{|f-g|^2}{32} dm \geq \tfrac{\min \Psi}{32}\|f-g\|_2^2. \end{aligned}$$

■

Then Theorem 4 follows from Lemma 8 and Proposition 9.

We also provide a more direct proof of Proposition 6. In fact, $b_\lambda(f)$ clearly satisfies the interpolation conditions $b_\lambda(f(z_k)) = b_\lambda(w_k)$ and $\|b_\lambda(f)\|_\infty < 1$. Let $f = \frac{b}{a}$, then $\frac{\beta}{\alpha} = b_\lambda(f) = \frac{\lambda - f}{1 - \bar{\lambda}f} = \frac{a\lambda - b}{a - \bar{\lambda}b}$, hence

$$\begin{aligned} \alpha\alpha^* - \beta\beta^* &= (a - \bar{\lambda}b)(a - \bar{\lambda}b)^* - (a\lambda - b)(a\lambda - b)^* \\ &= (1 - \lambda\bar{\lambda})(aa^* - bb^*). \end{aligned}$$

This shows that $g = b_\lambda(f)$ is the interpolant corresponding to σ.

VII. Conclusions

In this article we have studied the entropy functional from [2] and the interpolants solving the optimization problem

$$\min \mathbb{K}_\Psi(f) \ \text{ s.t. } \ f(z_k) = w_k, \ k = 0, 1, \ldots, n.$$

It is shown that, if the entropy of a sequence of interpolants converge to the minimum, then the corresponding interpolants converge in H_2. Furthermore, if the interpolation values are transformed by a Möbius transform, so is the minimizing interpolant. These results are used for obtaining an asymptotic result in the case that an interpolation point approaches the boundary.

In a few simple design examples, we have examined the effect interpolation conditions close to the boundary have on the interpolant. In these examples, the effect of adding interpolation points close to the boundary is small in harmony with Theorem 7 and Theorem 3. In the solution of Nagamune, a main objective of the additional interpolation conditions is to increase the dimension of $\mathcal{K}$, thereby allowing for more design parameters.

In view of this, one may ask if we could solve the optimization problem for a larger class of Ψ *without* the additional interpolation conditions. Adding interpolation points restrict the admissible set, and if they have negligle effect one would expect better solutions without them. A theory for this is developed in [6].

VIII. Acknowledgements

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