state observers has been developed, which can guarantee the asymptotic convergence of the observation error between the observer state estimate and the true state. The adaptive robust state observer proposed in this technical note is continuous, and can be easily implemented in practical control problems.

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On Degree-Constrained Analytic Interpolation With Interpolation Points Close to the Boundary

Johan Karlsson and Anders Lindquist

***Abstract*—In the recent article [4], a theory for complexity-constrained interpolation of contractive functions is developed. In particular, it is shown that any such interpolant may be obtained as the unique minimizer of a (convex) weighted entropy gain. In this technical note we study this optimization problem in detail and describe how the minimizer depends on weight selection and on interpolation conditions. We first show that, if, for a sequence of interpolants, the values of the entropy gain of the interpolants converge to the optimum, then the interpolants converge in H_2, but not in H_∞. This result is then used to describe the asymptotic behavior of the interpolant as an interpolation point approaches the boundary of the domain of analyticity. For loop shaping to specifications in control design, it might at first seem natural to place strategically additional interpolation points close to the boundary. However, our results indicate that such a strategy will have little effect on the shape. Another consequence of our results relates to model reduction based on minimum-entropy principles, where one should avoid placing interpolation points too close to the boundary.**

***Index Terms*—Analytic interpolation, generalized entropy rate, sensitivity shaping.**

I. INTRODUCTION

Many important engineering problems lead to analytic interpolation, where the interpolant represents a transfer function of, for example, a feedback control system or a filter and therefore is required to be a rational function of bounded degree. In recent years, a complete theory of analytic interpolation with degree constraint has been developed, which provides complete smooth parameterizations of whole classes of such interpolants in terms of a weighting function belonging to a finite-dimensional space, as well as convex optimization problems for determining them; see [3] and [4] and references therein.

This theory provides a framework for tuning an engineering design based on analytic interpolation to satisfy additional design specification without increasing the degree of the transfer function. Occasionally, the number of tuning parameters is too small to satisfy the design specifications, and then the parameter space needs to be enlarged by increasing the degree bound. In [12], this was done by adding new interpolation conditions, often close to the boundary.

In this technical note, we present some negative results concerning this strategy and explain why, after all, the solution in [12] is satisfactory. We show that unless the weighting function is changed, adding new interpolation points close to the boundary will have little effect on the interpolant. We illustrate this by analyzing a simple example from robust control.

We also show that interpolation conditions close to the unit disc have little effect on the minimum-entropy solution and can thus be discarded (Remark 2). Recently, some procedures for model reduction based on the minimum-entropy solution have been proposed [1], [13], which amount to interpolating in the mirror images of selected spectral zeros.

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Our results suggest (at least for bounded-real interpolants) that dominant spectral zeros close to the boundary should not be selected in this procedure. However, by choosing more general weights [8], this situation can be avoided.

In Section II, we begin by reviewing some pertinent results from [4] amplified with a generalization from the more recent paper [10]. Then, in Section III, we provide a motivation example from robust control, which is then revisited in Section V after having presented the main results in Section IV. Some proofs are deferred to Section VI.

II. Background

Consider the classical Pick problem of finding a function f in the Schur class

$$\mathcal{S} := \{f \in H_\infty(\mathbb{D}) : \|f\|_\infty \leq 1\}$$

that satisfies the interpolation condition

$$f(z_k) = w_k, \quad k = 0, 1, \ldots, n \tag{1}$$

where (z_k, w_k), $k = 0, 1, \ldots, n$, are given pairs of points in the open unit disc $\mathbb{D} := \{z : |z| < 1\}$. It is well known that such an $f \in \mathcal{S}$ exists if and only if the Pick matrix

$$P := \left[\frac{1 - w_k \bar{w}_\ell}{1 - z_k \bar{z}_\ell}\right]_{k,\ell=0}^{n} \tag{2}$$

is positive semi-definite, and that the function f is uniquely determined if and only if the matrix P is singular. In the latter case, f is a Blaschke product of degree equal to the rank of P. Here we shall take P to be positive definite, in which case there are infinitely many solutions to the Pick problem. A complete parameterization of the solutions of this so called Nevanlinna-Pick interpolation problem was given by Nevanlinna (see, e.g., [2]) in 1929. The parameterization is in terms of a linear fractional transformation centered around a rational solution of degree n, known as the *central solution*.

In a research program leading to [3] and [4], the subset of all solutions of the Nevanlinna-Pick problem that are rational of degree at most n were parameterized. Most engineering problems require such degree constraints, which completely alter the basic mathematical problem. More precisely, let $\mathcal{K}$ be the space of all functions

$$f(z) = \frac{\rho(z)}{\tau(z)}$$

where $\rho(z)$ is an arbitrary polynomial of degree at most n and

$$\tau(z) = \prod_{k=0}^{n}(1 - \bar{z}_k z).$$

Clearly, $\mathcal{K}$ is a subspace of the Hardy space $H_2(\mathbb{D})$. Moreover, let $\mathcal{K}_0$ be the the subset of all $f \in \mathcal{K}$ such that $\rho(z)$ has all its roots in the complement of $\mathbb{D}$ and $\rho(0) > 0$. In this (rational) context, Theorem 1 in [4] can be stated in the following way.

Theorem 1: Suppose that the Pick matrix (2) is positive definite. Let σ be an arbitrary function in $\mathcal{K}_0$. Then there exists a unique pair $(a, b) \in \mathcal{K}_0 \times \mathcal{K}$ such that:

1) $f = b/a \in \mathcal{S}$;
2) $f(z_k) = w_k$, $k = 0, 1, \ldots, n$;
3) $|a|^2 - |b|^2 = |\sigma|^2$ a.e. on $\mathbb{T} := \{z : |z| = 1\}$.

Conversely, any pair $(a, b) \in \mathcal{K}_0 \times \mathcal{K}$ satisfying 1 and 2 determines, via 3, a unique $\sigma \in \mathcal{K}_0$.

Consequently, the solutions (a, b) corresponding to interpolants of degree at most n are completely parameterized by the zeros of $\sigma \in \mathcal{K}_0$; i.e., the n-tuples $\{\lambda_1, \ldots, \lambda_n\}$ of complex number in the complement of $\mathbb{D}$; these are called the *spectral zeros*. For each such choice of spectral zeros, the corresponding interpolant $f \in \mathcal{S}$ can be determined by minimizing the strictly convex functional $\mathbb{K}_\Psi : \mathcal{S} \to \mathbb{R} \cup \{\infty\}$, given by

$$\mathbb{K}_\Psi(f) = -\int_{\mathbb{T}} \Psi \log(1 - |f|^2) dm(z)$$

over the class of interpolants, where $\Psi := |\sigma|^2$ and m is the normalized Lebesgue measure on $\mathbb{T}$. In fact, in the present context, Theorem 5 in [4] can be stated as follows.

Theorem 2: Suppose that the Pick matrix (2) is positive definite. Let σ be an arbitrary function in $\mathcal{K}_0$, and set $\Psi := |\sigma|^2$. Then the functional $\mathbb{K}_\Psi$ has a unique minimizer in the class of functions that satisfy the interpolation conditions (1), and this minimizer is precisely the unique function $f \in \mathcal{S}$ satisfying conditions 1, 2, and 3 in Theorem 1.

Remark 1: When $z_0 = 0$, then the central solution corresponds to $\Psi \equiv 1$. The corresponding functional $\mathbb{K}_1$ is the usual entropy gain, and the central solution is therefore equal to the *minimum entropy solution* (see, e.g., [11]). Then $\sigma \equiv 1 \in \mathcal{K}_0$, and hence the generic degree of the minimum entropy solution is n, and the corresponding spectral zeros are located at the conjugate inverses (mirror images in unit circle) of $\{z_k\}_{k=1}^n$; all in harmony with Theorems 1 and 2. If zero is not an interpolation point, then $1 \notin \mathcal{K}_0$ and the generic degree of the minimum entropy solution is instead $n + 1$.

By varying Ψ we can tune the interpolant without increasing its degree. In the context of our motivating example in Section III, this amounts shaping the sensitivity function without increasing the McMillan degree of the closed-loop system. By adding interpolation conditions we can increase the number of tuning parameters at the cost of increased degree of the interpolant.

Theorems 1 and 2 can be generalized to the case that Ψ is an arbitrary log-integrable function on $\mathbb{T}$. This was done in the following way in [10].

Theorem 3: Suppose that the Pick matrix (2) is positive definite and that $\Psi = |\sigma|^2$ is a log-integrable nonnegative function on the unit circle, where σ is analytic but need not belong to $\mathcal{K}_0$. Then f is the minimizer of $\mathbb{K}_\Psi$ in the class of functions that satisfy the interpolation conditions (1) if and only if the following three conditions hold:

i) $f = b/a \in \mathcal{S}$ where $b \in \mathcal{K}$ and a is outer;
ii) $f(z_k) = w_k$ for $k = 0, \ldots, n$;
iii) $|a|^2 - |b|^2 = |\sigma|^2$.

Any such minimizer is necessarily unique.

This allows for shaping the interpolant without the constraint that σ belong to $\mathcal{K}_0$, but at the expense of increased degree; for a precise statement, see [10]. An interesting special case of this is when $\Psi \in C(\mathbb{T})_+$, i.e., Ψ is positive and continuous on the unit circle.

The theory described above allows us to choose an interpolant that best satisfies additional design specifications. In fact, the map from σ to (a, b) defined by Theorem 1 is smooth [5], [6], and hence a given design can be tuned via Ψ to smoothly change the interpolant. An obvious first choice of Ψ is to make it large in frequency bands where $|f|$ needs to be small. This technical note is an attempt to gain understanding of the underlying function theory involved in tuning the interpolant. In the subsequent paper [10], we have derived a *systematic* procedure for shaping interpolants based on design specifications.

III. A Motivating Example

The purpose of this technical note is to show how the interpolant changes as the weight Ψ is changed and as additional interpolation points are introduced, especially close to the boundary of $\mathbb{D}$. To illustrate this point, we consider an example on sensitivity shaping in robust

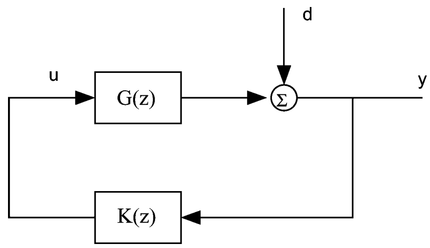

Fig. 1. Feedback system.

control from [12]. Fig. 1 depicts a feedback system with u denoting the control input to the plant

$$G(z) = \frac{1}{z - 1.05}$$

to be controlled, d represents a disturbance, and y is the resulting output, which is fed back through a compensator $K(z)$ to be designed. The goal is to determine a controller $K(z)$ so that the feedback system in Fig. 1 satisfies the design specifications

$$|S(e^{i\theta})| < 2.0 \sim 6.02 \text{ dB} \quad 0 \le \theta \le \pi \tag{3}$$
$$|S(e^{i\theta})| < 0.1 \sim -20.1 \text{ dB} \quad 0 \le \theta \le 0.3 \tag{4}$$
$$|T(e^{i\theta})| < 0.5 \sim -6.02 \text{ dB} \quad 2.5 \le \theta \le \pi \tag{5}$$

in terms of the sensitivity function $S = (1 - GK)^{-1}$ and the complementary sensitivity function $T = 1 - S$. The plant $G(z)$ has one unstable pole at $z = 1.05$ and one non-minimum phase zero at $z = \infty$. It follows from H_∞ control theory (see, e.g., [7]) that the feedback system is internally stable if and only if the sensitivity function $S(z)$, the transfer function from d to y, is analytic in $\mathbb{D}^C := \{z : |z| > 1\}$, the complement of the closed unit disc, and satisfies the interpolation conditions

$$S(1.05) = 0, \quad S(\infty) = 1.$$

By the design specification (3), the interpolants are required to satisfy $\|S\|_\infty \le \gamma := 2$. Setting

$$f(z) = \frac{1}{2} S(z^{-1}), \tag{6}$$

f fits into the framework of Theorem 1 with interpolation conditions

$$f(0.9524) = 0, \quad f(0) = \frac{1}{2}. \tag{7}$$

Since $n = 1$, there exists a one-parameter family of degree-one interpolants satisfying $\|f\|_\infty \le 1$ that may be parametrized by its corresponding spectral zero λ. Fig. 2 shows the solutions $S(z) = 2f(z^{-1})$ as $1/\lambda$ varies from -1 to 1 with the grid 0.2.

Clearly, none of these designs satisfies the specifications. Therefore, following Nagamune [12], we add the interpolation conditions

$$f(-0.9901) = \frac{1}{2} \tag{8}$$

and

$$f(0.9901e^{\pm 0.3i}) = 0. \tag{9}$$

Here (8), motivated by the design specification (5), ensures, via (6), that $T(1.01) = 0$, while (9), motivated by the design specification (4), ensures that $S(1.01e^{\pm 0.3i}) = 0$. The number of interpolation conditions adds up to $n + 1 = 5$, and therefore Theorem 1 allows for parameterizing all solutions of degree $n = 4$ by choosing n spectral zeros. As in [12], we choose the spectral zeros in $0.97e^{\pm 0.55i}$ and $0.9e^{\pm 1.55i}$, which corresponds to the weight

$$\Psi_{\mathrm{N}} = \left| \frac{(1 - 0.9e^{1.55i}z)(1 - 0.9e^{-1.55i}z)}{(z - 1.05)(z + 1.01)} \right|^2 \times \left| \frac{(1 - 0.97e^{0.55i}z)(1 - 0.97e^{-0.55i}z)}{(z - 1.01e^{0.3i})(z - 1.01e^{-0.3i})} \right|^2. \tag{10}$$

The corresponding sensitivity function S_{N}, depicted in Fig. 3 together with the weight Ψ_{N}, satisfies the design specifications (3)–(5). The interpolation points and spectral zeros for this design are depicted in Fig. 4.

From the plots in Fig. 3, one first notices that in the example where a large weight in the low-frequency region is used, the magnitude of the sensitivity is low. This seems to be intuitive since the high weight in the entropy functional penalizes the sensitivity more in that region than in others. However, the weight is also large in the high-frequency area, and in this case there is no significant change in the sensitivity.

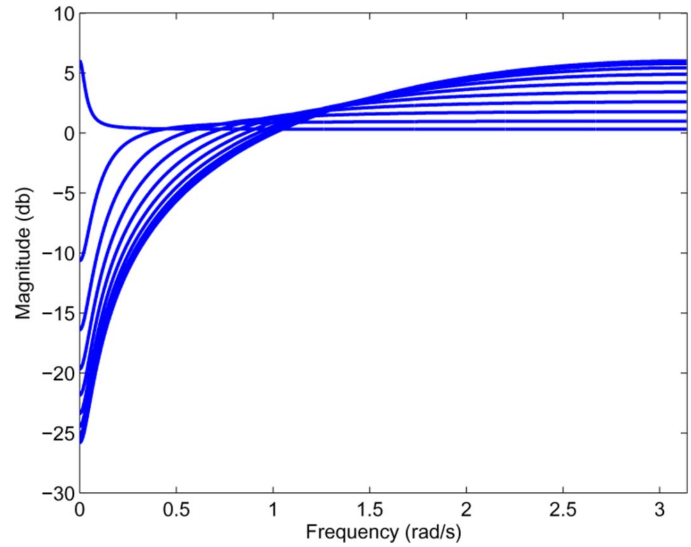

Fig. 2. Degree-one sensitivity functions corresponding to spectral-zero selections with $1/\lambda$ between -1 to 1 with grid 0.2.

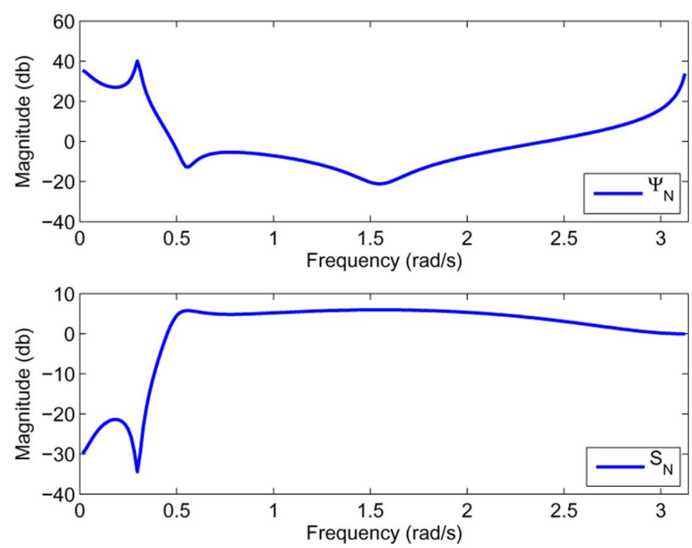

Fig. 3. Weight Ψ_{N} (above) and the magnitude of the sensitivity function S_{N} (below).

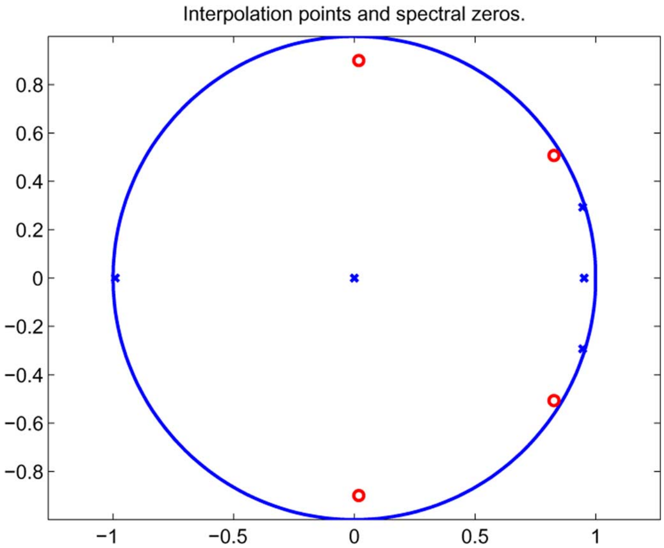

Fig. 4. Intepolation points ($\times$) and the mirror images ($\circ$) of the spectral zeros corresponding to S_{N}.

In order to understand the effects of interpolation points and spectral zeros in this design, in Section IV we develop results for interpolation points close to the unit circle. Then, in Section V we revisit the example above.

IV. Main Results

As the example of Section III suggests, we need to investigate how the interpolant changes as additional interpolation points are introduced close to the unit circle. The following theorem is one of our main results.

Theorem 4: Let $\Psi = |\sigma|^2 \in C(\mathbb{T})_+$, where $\sigma \in \mathcal{K}_0$, and let $\hat{f}$ be the minimizer of $\mathbb{K}_\Psi(f)$ subject to the interpolation conditions

$$f(z_k) = w_k, \quad k = 0, 1, \ldots, n.$$

Moreover, given $|w| < 1$, let f_λ be the minimizer of $\mathbb{K}_\Psi(f)$ subject to

$$f(z_k) = w_k, \quad k = 0, 1, \ldots, n, \quad f(\lambda) = w.$$

Then $f_\lambda \to \hat{f}$ in H_2 as $|\lambda| \to 1$.

This theorem indicates that adding interpolation conditions close to the unit circle will not affect the design in any important way unless we also change the weighting function Ψ. For the proof we first need to show that if the generalized entropy of interpolants converge to the optimum, then the interpolants converge to the optimal interpolant in H_2. The following theorem is proven in Section VI.

Theorem 5: Let $\Psi \in C(\mathbb{T})_+$, and let $\hat{f}$ be the minimizer of $\mathbb{K}_\Psi(f)$ subject to $f(z_k) = w_k$, $k = 0, 1, \ldots, n$. If f_ℓ satisfies $f_\ell(z_k) = w_k$, $k = 0, 1, \ldots, n$, and $\mathbb{K}_\Psi(f_\ell) \to \mathbb{K}_\Psi(\hat{f})$, then $f_\ell \to \hat{f}$ in H_2.

It should be noted that this result could not be strengthened to H_∞ convergence. A counterexample could be constructed by noting that $\mathbb{K}_\Psi(f + \alpha\chi_{E_\ell}) \to \mathbb{K}_\Psi(f)$ if $m(E_\ell) \to 0$. Here, χ denotes the characteristic function and α is a scalar such that $0 < |\alpha| < 1 - \|f\|_\infty$. But $\|\alpha\chi_{E_\ell}\|_\infty = \alpha$ for all ℓ. This argument works equally well for $f + g_\ell$, where $g_\ell \in \phi H_2$ and $|g_\ell|$ is an appropriate approximation of χ_{E_ℓ}.

A second step in proving Theorem 4 is to investigate how the interpolant changes as the data is transformed under a Möbius transformation. For $\lambda \in \mathbb{D}$, let b_λ be the Blaschke factor

$$b_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z}.$$

Then the following proposition tells us how the entropy is changed as the range is transformed by a Möbius transformation.

Proposition 6: The map $\rho(\cdot, \lambda, \Psi) : \mathcal{S} \to \mathbb{R}$ defined by

$$\rho(f, \lambda, \Psi) = \int_{\mathbb{T}} \Psi \log \frac{|1 - \bar{\lambda} f|^2}{1 - \bar{\lambda}\lambda} dm(z)$$

is continuous, and

$$\mathbb{K}_\Psi(b_\lambda(f)) = \mathbb{K}_\Psi(f) + \rho(f, \lambda, \Psi).$$

Moreover, if $\Psi = |\sigma|^2$ where $\sigma \in \mathcal{K}_0$, then $\rho(f_1, \lambda, \Psi) = \rho(f_2, \lambda, \Psi)$, whenever $f_1(z_k) = f_2(z_k)$ for $k = 0, 1, \ldots, n$.

Proof: First part is trivial, second part follows from [4, p. 8, Lemma 10]. ■

As a corollary we have the following proposition, which tells us that the solution obtained from the transformed data is the solution transformed with the same transformation.

Proposition 7: Let $\sigma \in \mathcal{K}_0$, and let f be the the corresponding solution to the analytic interpolation problem $f(z_k) = w_k$, $k = 0, 1, \ldots, n$, $\|f\|_\infty \leq 1$ prescribed by Theorem 1. Then $g = b_\lambda(f)$ is the interpolant corresponding to the same σ of the analytic interpolation problem $g(z_k) = b_\lambda(w_k)$, $k = 0, 1, \ldots, n$, $\|g\|_\infty \leq 1$.

A simple proof of Proposition 7, derived directly from Theorem 1 using basic principles, is given in Section VI.

To conclude the proof of Theorem 4, we first prove a version in which the interpolation value w equals zero.

Theorem 8: Let $\Psi \in C(\mathbb{T})_+$, and let $\hat{f}$ be the minimizer of $\mathbb{K}_\Psi(f)$ subject to the interpolation conditions $f(z_k) = w_k$, $k = 0, 1, \ldots, n$. Moreover, let f_λ be the minimizer of $\mathbb{K}_\Psi(f)$ subject to $f(z_k) = w_k$, $k = 0, 1, \ldots, n$ and $f(\lambda) = 0$. Then $f_\lambda \to \hat{f}$ in H_2 as $|\lambda| \to 1$.

In Theorems 4 and 8, f_λ may not exist for certain $\lambda \in \mathbb{D}$, since the corresponding Pick matrix may not be positive definite. However, there is always an $\epsilon > 0$ such that f_λ exists whenever $1 - \epsilon < |\lambda| < 1$, hence the limit results are still valid.

Proof: Let $M_\lambda = \{g : g \in \mathcal{S}, g(z_k) = |\lambda| w_k / \bar{\lambda} b_\lambda(z_k)\}$. First note that if $g \in M_\lambda$, then $g b_\lambda \bar{\lambda} / |\lambda|$ satisfies the interpolation conditions. Furthermore

$$\mathbb{K}_\Psi(g) = \mathbb{K}_\Psi\left(\frac{g b_\lambda \bar{\lambda}}{|\lambda|}\right) \geq \mathbb{K}_\Psi(f_\lambda) \geq \mathbb{K}_\Psi(\hat{f})$$

by the definitions of f_λ and $\hat{f}$. If we could prove that

$$\min_{g \in M_\lambda} \mathbb{K}_\Psi(g) \to \mathbb{K}_\Psi(\hat{f}) \text{ as } |\lambda| \to 1 \tag{11}$$

then $\mathbb{K}_\Psi(f_\lambda) \to \mathbb{K}_\Psi(\hat{f})$, and by Theorem 5 it follows that $f_\lambda \to \hat{f}$ in H_2. However, since $|\lambda| w_k / \bar{\lambda} b_\lambda(z_k) \to w_k$ as $|\lambda| \to 1$, there is a sequence of functions $g_\lambda \in M_\lambda$ such that $g_\lambda \to \hat{f}$ in H_∞. By H_∞ continuity of $\mathbb{K}_\Psi$, (11) holds. ■

Note that Theorem 8 holds for any positive and continuous Ψ, whereas Theorem 4 only holds if $\Psi = |\sigma|^2$ and σ belong to $\mathcal{K}_0$. This is because the proof of Theorem 4 requires the use of Proposition 7, where $\sigma \in \mathcal{K}_0$ is a condition.

We are now in a position to prove Theorem 4. To this end, let $g_\lambda = b_w(f_\lambda)$ and $g = b_w(\hat{f})$. By Proposition 7 and Theorem 1, g_λ is the unique minimizer of $\mathbb{K}_\Psi(f)$ such that $f(z_k) = b_w(w_k)$, $k = 1, \ldots, n$, and $f(\lambda) = 0$. Furthermore, g is the unique minimizer of $\mathbb{K}_\Psi(f)$ such that $f(z_k) = b_w(w_k)$, $k = 1, \ldots, n$. By Theorem 8, $g_\lambda \to g$ in H_2. Since b_w is Lipschitz continuous, $f_\lambda \to \hat{f}$ in H_2. This concludes the proof of Theorem 4.

Remark 2: Consider a Pick problem with $z_0 = 0$ and several interpolation points close to the unit circle. Since $1 \in \mathcal{K}_0$ for all $n \geq 1$, it follows from Theorem 4 that the corresponding interpolation conditions have little effect on the minimum-entropy interpolant and could

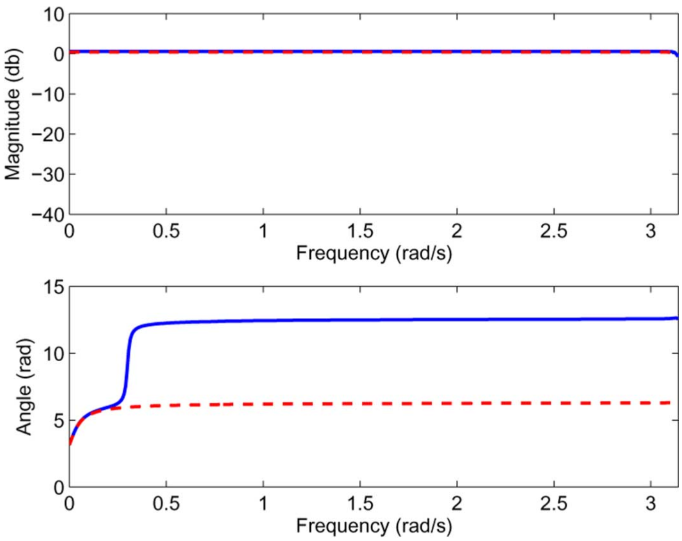

Fig. 5. Sensitivity functions corresponding to the maximum entropy solution, i.e., $\Psi \equiv 1$. The dashed lines correspond to interpolation conditions (7) and (8) and the solid lines (7)–(9).

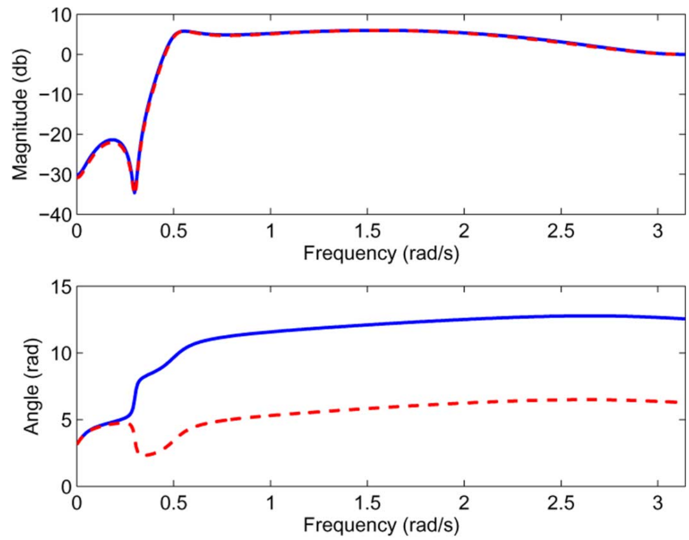

Fig. 6. Sensitivity functions corresponding to the weight Ψ_{N}. The dashed lines correspond to interpolation conditions (7) and (8) and the solid lines to (7)–(9).

be removed. Unless the interpolation value equals zero (Theorem 8), the situation is more complicated for a more general choice of Ψ, since removing an interpolation condition generally produces a $\sigma \notin \mathcal{K}_0$.

V. Revisiting the Example

We now return to the example of Section III. For determining the sensitivity function, two design tools were used, namely adding interpolation conditions and changing the weight Ψ by adding spectral zeros. We shall now investigate have these strategies have affected the design. As a starting point we choose the maximum entropy interpolant corresponding to the interpolation conditions (7) and (8). This sensitivity function, obtained by using the weight $\Psi \equiv 1$, is depicted in Fig. 5 with a dashed line. Next we observe what happens to the maximum entropy solution when we add the interpolation conditions (9), which requires the interpolant to be zero at the points $0.9901e^{\pm 0.3i}$. This interpolation point is close to the unit circle. The corresponding sensitivity function is depicted by the solid line in Fig. 5. As is seen, the added interpolation points have neglible effect on the modulus of the interpolant and only a local effect on the phase around 0.3 rad/sec, where there is a sharp shift of 2π in the phase. This is in harmony with Theorem 4 which states that the effect of adding additional interpolation points close to the unit circle is small in the H_2-norm.

However, the theory allows for shaping the interpolant by specifying spectral zeros or, equivalently, the weight Ψ. In the motivating example of Section III, the spectral zeros were chosen to be in $0.97e^{\pm 0.55i}$ and $0.9e^{\pm 1.55i}$, which corresponds to the weight Ψ_{N} given by (10). The sensitivity function S_{N} obtained, via (6), by using this weight and the interpolation constraints (7)–(9) is depicted in Fig. 6 with a solid line. If we remove the interpolation condition (9) we obtain the sensitivity function depicted by a dashed line in Fig. 6. As can be seen, also in this case, the only significant change resulting from the additional interpolation condition is the change of 2π in the phase around $\theta = 0.3$. As before this change is very local close to the added interpolation condition.

Therefore, solely adding the interpolation condition (9) does not change the solution significantly. The change in the magnitude is negligible, as is the change in the phase, except for the region close to the added interpolation point, where there is a sharp shift of 2π in the phase. Since the shift occurs over a short interval, the change in H_2 norm is minor, and as the interpolation point approaches the boundary this shift will have negligible effect on the H_2 norm. This example shows why the same convergence result could not hold for the H_∞ norm.

Let us return to Fig. 3 and the fact that there is no significant change in the sensitivity in the high frequency area, despite the large weight in this region. This could be due to the interpolation condition (8), which lies very close to the boundary in the high frequency region. Note that any effect (8) has on the interpolant is not in conflict with Theorem 4. This is because $\sigma \notin \mathcal{K}_0$ if (8) is removed, and hence Theorem 4 is not applicable. What can be concluded is that the weight has a large effect on the design. This is further exploited in [10] where a systematic procedure for finding appropriate weights is developed.

VI. Proofs

For the proof of Theorem 5 we will use concepts from convex analysis. Let $\Lambda : \mathbb{X} \to \mathbb{R}$ be a strictly convex functional, where $\mathbb{X}$ is compact and convex. Then the minimum

$$\beta = \min_{x \in \mathbb{X}} \Lambda(x)$$

exists and is attained at a unique $x \in \mathbb{X}$. Consider the set K_ϵ of ϵ-suboptimal solutions

$$K_\epsilon = \{x \in \mathbb{X} : \Lambda(x) < \beta + \epsilon\}, \quad \epsilon > 0.$$

It seems reasonable that the "size" of K_ϵ tends to zero as $\epsilon \to 0$. However, to state and prove this properly, we need topological considerations and the concept of strong convexity.

Definition 1: A functional Λ is *strongly convex with respect to the norm* $\|\cdot\|$ if there exists an $\alpha : [0, \infty) \to [0, \infty)$ that is continuous, strictly increasing and satisfies $\alpha(0) = 0$, for which

$$\frac{1}{2}(\Lambda(x) + \Lambda(y)) \geq \Lambda\left(\frac{x+y}{2}\right) + \alpha(\|x - y\|)$$

holds for all $x, y \in \mathbb{X}$.

Lemma 9: Let $\mathbb{X}$ be a convex set, and let Λ be a strongly convex functional on $\mathbb{X}$ with respect to the norm $\|\cdot\|$. Moreover, let $\hat{x}$ be the minimum of $\Lambda(x)$ such that $x \in \mathbb{X}$. Then $\Lambda(x_k) \to \Lambda(\hat{x}), x_k \in \mathbb{X}$, implies $\|x_k - \hat{x}\| \to 0$.

An equivalent statement is that, if Λ is strongly convex with respect to the norm $\|\cdot\|$, then

$$\sup\{\|x-y\| : x, y \in K_\epsilon\} \to 0$$

as $\epsilon \to 0$, or, equivalently, K_ϵ is a neighborhood basis for the optimal point $\hat{x}$ in the topology induced by the norm $\|\cdot\|$.

Proof: Assume that the statement of Lemma 9 does not hold, i.e., that there exists an $\epsilon > 0$ so that for any $\delta > 0$ it is possible to find an $x \in \mathbb{X}$ so that $|\Lambda(x) - \Lambda(\hat{x})| < \delta$ and $\|x - \hat{x}\| > \epsilon$. Let $\delta < \alpha(\epsilon)$. Then $\Lambda(\hat{x}) + \delta \geq 1/2(\Lambda(x) + \Lambda(\hat{x})) \geq \Lambda(x + \hat{x}/2) + \alpha(\|x - \hat{x}\|) \geq \Lambda(x + \hat{x}/2) + \alpha(\epsilon)$. This contradicts that $\hat{x}$ is the minimizer, and hence the validity of Lemma 9 is proved by contradiction. ∎

In order to apply this result to the entropy functional, we need to show that $\mathbb{K}_\Psi$ is strongly convex with respect to the H_2 norm.

Proposition 10: Let $\Psi \in C(\mathbb{T})_+$. Then the entropy functional $\mathbb{K}_\Psi$ is strongly convex with respect to the H_2 norm.

Proof: For $|f| < 1$ and $|g| < 1$, we have the following inequality:

$$\begin{aligned}\frac{1}{2}(-\log(1-|f|^2) - \log(1-|g|^2)) \\ \geq -\log\left(1 - \left|\frac{f+g}{2}\right|^2\right) + \frac{1}{2}\log\left(1 + \frac{|f-g|^2}{2}\right). \quad (12)\end{aligned}$$

To see this, use the parallelogram law

$$|f+g|^2 + |f-g|^2 = 2|f|^2 + 2|g|^2$$

to obtain

$$\begin{aligned}&\left(1 - \frac{|f+g|^2}{4}\right)^2 \\ &= (1-|f|^2)(1-|g|^2) - |f|^2|g|^2 \\ &\quad + \frac{|f-g|^2}{2} + \frac{1}{16}\left(2|f|^2 + 2|g|^2 - |f-g|^2\right)^2 \\ &= (1-|f|^2)(1-|g|^2) + \frac{|f-g|^2}{4}\left(2 - |f|^2 - |g|^2\right) \\ &\quad + \frac{1}{4}\left(|f|^2 - |g|^2\right)^2 + \frac{1}{16}|f-g|^4 \\ &\geq (1-|f|^2)(1-|g|^2) + \frac{|f-g|^2}{4}\left(2 - |f|^2 - |g|^2\right).\end{aligned}$$

Consequently, since

$$\frac{2 - |f|^2 - |g|^2}{(1-|f|^2)(1-|g|^2)} \geq 2$$

we have

$$\frac{\left(1 - \frac{|f+g|^2}{4}\right)^2}{(1-|f|^2)(1-|g|^2)} \geq 1 + \frac{|f-g|^2}{2},$$

from which (12) follows. Then, multiplying (12) by Ψ and integrating, we obtain

$$\begin{aligned}\frac{1}{2}(\mathbb{K}_\Psi(f) + \mathbb{K}_\Psi(g)) \geq \mathbb{K}_\Psi\left(\frac{f+g}{2}\right) \\ + \frac{1}{2}\int_{\mathbb{T}} \Psi \log\left(1 + \frac{|f-g|^2}{2}\right) dm. \quad (13)\end{aligned}$$

Since $\log(1+t) \geq t/2$ for $t \in [0, 2]$, the last term in (13) is bounded from below by

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{T}} \Psi \log\left(1 + \frac{|f-g|^2}{2}\right) dm \geq \int_{\mathbb{T}} \Psi \frac{|f-g|^2}{4} dm \\ \geq \frac{\min \Psi}{4}\|f-g\|_2^2\end{aligned}$$

establishing the strong continuity of $\mathbb{K}_\Psi$ ∎

Theorem 5 then follows from Lemma 9 and Proposition 10.

We also provide a more direct proof of Proposition 7. In fact, $b_\lambda(f)$ clearly satisfies the interpolation conditions $b_\lambda(f(z_k)) = b_\lambda(w_k)$ and $\|b_\lambda(f)\|_\infty < 1$. Let $f = b/a$. Then

$$\frac{\beta}{\alpha} = b_\lambda(f) = \frac{\lambda - f}{1 - \bar{\lambda} f} = \frac{a\lambda - b}{a - \bar{\lambda} b}$$

and hence

$$\begin{aligned}\alpha\alpha^* - \beta\beta^* &= (a - \bar{\lambda}b)(a - \bar{\lambda}b)^* - (a\lambda - b)(a\lambda - b)^* \\ &= (1 - \lambda\bar{\lambda})(aa^* - bb^*).\end{aligned}$$

This shows that $g = b_\lambda(f)$ is the interpolant corresponding to σ.

VII. Conclusion

In this technical note, we have studied the (generalized) entropy functional $\mathbb{K}_\Psi$ of [4] and the interpolants solving the optimization problem

$$\min \mathbb{K}_\Psi(f) \text{ s.t. } f(z_k) = w_k, \quad k = 0, 1, \ldots, n.$$

It is shown that, if the entropies of a sequence of interpolants converge to the minimum, then the corresponding interpolants converge in H_2, but not necessarily in H_∞. Furthermore, if the interpolation values are transformed by a Möbius transform, so is the minimizing interpolant. Next we show that the introduction of an additional interpolation point close to the boundary produces an insignificant increase in the entropy gain. Taken together with the results above, this implies that the change in the interpolant is small in H_2 norm.

We have analyzed a design example from robust control, studied by Nagamune [12], in the context of our results. The effect of an added interpolation condition close to the boundary turns out to be small, in harmony with Theorems 4 and 8. In the solution of Nagamune [12], a main objective of the additional interpolation conditions was to increase the dimension of $\mathcal{K}$, thereby allowing for more design parameters. However, by instead applying the parameterization of Theorem 3, we could solve the optimization problem for a larger class of Ψ without the additional interpolation conditions. Adding interpolation points restricts the admissible set, and, if they have negligible effect, one would expect better solutions without them. A complete theory for this is developed in [10].

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Robust H^∞ Control of an Uncertain System Via a Stable Output Feedback Controller

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***Abstract*—This technical note presents a new approach to the robust control of an uncertain system via a stable output feedback controller. The uncertain systems under consideration contain structured uncertainty described by integral quadratic constraints. The controller is designed to achieve absolute stabilization with a specified level of disturbance attenuation. The main result involves solving a state feedback version of the problem by solving an algebraic Riccati equation dependent on a set of scaling parameters. Then two further algebraic Riccati equations are solved, which depend on a further set of scaling parameters.**

***Index Terms*—Absolute stabilization, H^∞ control, integral quadratic constraints, strong stabilization.**

I. Introduction

This technical note considers the problem of robust H^∞ control via a stable output feedback controller. It is well known that the use of stable controllers is preferable to the use of unstable feedback controllers in many practical control problems; e.g., see [1]–[3]. Indeed, the use of unstable controllers can lead to problems with actuator and sensor failure, sensitivity to plant uncertainties and implementation problems. This has motivated a number researchers to consider problems of H^∞ control via the use of stable controllers; e.g., see [1]–[4].

In this technical note, we propose a new approach to the problem of robust H^∞ control via a stable output feedback controller. We consider a class of uncertain systems with structured uncertainty described by integral quadratic constraints (IQCs); e.g., see [5] and [6]. Indeed, our results build on the results of [5] which provide necessary and sufficient conditions for the absolute stabilization of such uncertain systems with a specified level of disturbance attenuation (but with no requirement that the output feedback controller is stable). The key idea behind our approach is to begin with an uncertain system of the type considered in [5] and then add an additional uncertainty to form a new uncertain system. This additional uncertainty has the property that for one specific value of the uncertainty, the new uncertain system reduces to the original uncertain system and thus any suitable controller for the new uncertain system will also solve the problem of absolute stabilization with a specified level of disturbance attenuation for the original system. Also, for a different value of the new uncertainty, the new uncertain system reduces to a certain open-loop system in such a way that the controller is forced to be stable. Because our approach involves the addition of new uncertainties, our results provide only sufficient conditions rather than necessary and sufficient conditions for absolute stabilization with a specified level of disturbance attenuation. However, because the new uncertainty is explicitly constructed, this can give some indication about the degree of conservatism introduced.

Our main result is obtained applying the results of [5] to the new uncertain system. This gives a stable output feedback controller solving a problem of absolute stabilization with a specified level of disturbance attenuation. This is achieved by solving a pair of algebraic Riccati equations dependent on a set of scaling parameters. The controller obtained is of the same order of the plant.

The remainder of the technical note proceeds as follows: In Section II of the technical note, we set up the problem of absolute stabilization with a specified level of disturbance attenuation via a stable output feedback controller. Section III introduces the new uncertain system for which we will apply the results of [5] in order to obtain a stable controller which guarantees absolute stabilization with a specified level of disturbance attenuation. The construction of this new uncertain system involves solving a state feedback version of the approach of [5] applied to the original uncertain system. This involves solving an algebraic Riccati equation of the H^∞ type which is dependent on a set of scaling parameters. This leads to our main result which is a procedure for constructing the required stable controller. This procedure involves solving a pair algebraic Riccati equations of the H^∞ type which are dependent on an additional set of scaling parameters. The final controller is constructed from the solutions to these Riccati equations. Section IV presents an example which illustrates the theory presented in the technical note. This example, which involves an H^∞ control problem for a linear time-invariant (LTI) system without uncertainty, is taken from [2]. We show that for this example, our approach is slightly less conservative than the approach of [2].

II. Problem Statement

We consider an output feedback H^∞ control problem for an uncertain system of the form

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) + \sum_{s=1}^{k} D_s \xi_s(t)\\
z(t) &= C_1 x(t) + D_{12} u(t)\\
\zeta_1(t) &= K_1 x(t) + G_1 u(t)\\
&\vdots\\
\zeta_k(t) &= K_k x(t) + G_k u(t)\\
y(t) &= C_2 x(t) + D_{21} w(t)
\end{aligned} \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the *state*, $w(t) \in \mathbf{R}^p$ is the *disturbance input*, $u(t) \in \mathbf{R}^m$ is the *control input*, $z(t) \in \mathbf{R}^q$ is the *error output*, $\zeta_1(t) \in \mathbf{R}^{h_1}, \ldots, \zeta_k(t) \in \mathbf{R}^{h_k}$ are the *uncertainty outputs*, $\xi_1(t) \in \mathbf{R}^{r_1}, \ldots, \xi_k(t) \in \mathbf{R}^{r_k}$ are the *uncertainty inputs* , and $y(t) \in \mathbf{R}^l$ is the *measured output*. The uncertainty in this system is described by a set of equations of the form

$$\begin{aligned}
\xi_1(t) &= \phi_1\left(t, \zeta_1(\cdot)|_0^t\right)\\
&\vdots\\
\xi_k(t) &= \phi_k\left(t, \zeta_k(\cdot)|_0^t\right)
\end{aligned} \tag{2}$$

where the following IQC is satisfied.

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