INT. J. CONTROL, 1989, VOL. 50, NO. 6, 2301–2312

Geometry of the Kimura–Georgiou parametrization of modelling filters

CHRISTOPHER I. BYRNES† and ANDERS LINDQUIST‡

We discuss questions concerning the geometry of the Kimura–Georgiou parametrization of the set $\mathscr{A}_+(n)$ of degree n positive real transfer functions with the first n coefficients in the Laurent expansion about infinity prescribed. For example, one interesting question which has been raised is whether this set is star-shaped about the maximum entropy solution. This, of course, would be implied by convexity and would imply that $\mathscr{A}_+(n)$ is diffeomorphic to euclidean n-space. All three of these geometric properties would be of interest, for example, when using geometric or optimization techniques to construct an n-dimensional modelling filter with variations about the maximum entropy filter. Our first main result, which also lends support to conjectures concerning convexity and star-shapedness, is that $\mathscr{A}_+(n)$ is in fact diffeomorphic to euclidean n-space. Our proof makes use of certain results from differential topology. There are of course several intimate relations between positive reality and stability properties of both real and complex polynomials. On this basis we observe that the convexity of the parametrization implies the convexity of the set of real Schur polynomials of degree n and is implied by the convexity of the set of complex Schur polynomials of degree n. From this it immediately follows that $\mathscr{A}_+(n)$ is not convex for $n \geqslant 3$ and that $\mathscr{A}_+(1)$ is convex and hence star-shaped. The case $n = 2$ is especially interesting since the real Schur region is convex whereas the complex is not. Another of our main results is to refine the above observations to settle this question in general, showing that for $n \geqslant 3$ there is an open set of Schur parameters for which $\mathscr{A}_+(n)$ fails to be star-shaped about the maximum entropy filter.

1. Introduction

A parametrization of all n-dimensional modelling filters (or, equivalently, all n-dimensional shaping filters) with a given partial sequence

$$c_0, c_1, \ldots, c_n; \quad c_i = E\{y(t+i)y(t)\} \tag{1}$$

of correlation coefficients for a stationary stochastic process $\{y_t\}$ is a long sought-after goal with important applications in signal processing and in speech processing. It is well known (Grenander and Szegö 1958) that to the sequence $c_0, \ldots, c_n$ one can assign a sequence $\gamma_0, \ldots, \gamma_{n-1}$ of Schur parameters satisfying

$$|\gamma_i| < 1 \tag{2}$$

if and only if the Toeplitz matrices

$$T_i = \begin{bmatrix} c_0 & c_1 & \cdots & c_i \\ c_1 & c_0 & \cdots & c_i \\ \vdots & \vdots & & \vdots \\ c_i & c_{i-1} & \cdots & c_0 \end{bmatrix} \quad i = 0, 1, 2, \ldots, n \tag{3}$$

Received 12 August 1988.

† Department of Systems Science and Mathematics, Washington University, St. Louis, MO 63130, U.S.A.

‡ Department of Mathematics, Division of Optimization and Systems Theory, Royal Institute of Technology, 100 44 Stockholm, Sweden.

are positive definite. By extending the sequence $\gamma_0, \ldots, \gamma_{n-1}$ with parameters satisfying (2), Schur obtained a parametrization of all positive real, analytic functions $v(z)$ satisfying

$$v(z) = \hat{c}_0/2 + \sum_{i=1}^{\infty} \hat{c}_i z^i, \quad \hat{c}_i = c_i \quad i = 0, \ldots, n \tag{4}$$

While (4) is an elegant mathematical solution of the *Caratheodory extension problem*, in practice modelling filters $v(z)$ should be realizable by finite dimensional linear systems and it seems quite difficult to characterize rationality of $v(z)$ in terms of the Schur parameters. In addition, the problem statement asks more, viz. that $\deg v(z) \leqslant n$.

In another direction, as we recall in more detail in § 2, the sequence $\gamma_0, \ldots, \gamma_{n-1}$ determines two bases for the vector space of polynomials of degree less than or equal to n: the Szegö polynomials of the first kind, denoted by

$$\phi_n, \phi_{n-1}, \ldots, \phi_0 \equiv 1$$

and of the second kind, denoted by

$$\psi_n, \psi_{n-1}, \ldots, \psi_0 \equiv Z$$

Kimura (1983) and Georgiou (1983) showed independently that any n-dimensional modelling filter $v(z)$ has a representation

$$v(z) = \frac{c_0}{2} \cdot \frac{\psi_n(z) + \alpha_1 \psi_{n-1}(z) + \ldots + \alpha_n}{\phi_n(z) + \alpha_1 \phi_{n-1}(z) + \ldots + \alpha_n} \tag{5}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are real numbers.

Thus, in contrast to the Schur parametrization, for $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ the representation (5) guarantees that $v(z)$ will be rational of degree less than or equal to n. However, it remains to be checked that a given $(\alpha_1, \ldots, \alpha_n)$ satisfies the positive reality conditions. As emphasized by Kimura, an important open problem is therefore to obtain a parametrization of those values of $\alpha = (\alpha_1, \ldots, \alpha_n)$ in (5) for which $v(z)$ is positive real and stable (Kimura 1986, see also Georgiou 1987). Even qualitative information, such as convexity of the set of positive real functions in the Kimura–Georgiou parametrization, would be useful and is suggested by low dimensional analysis and by a large number of simulations in higher dimensions. In this context, Kimura asked whether some Kharitonov-like property would hold with respect to this parametrization. Explicitly, one observes first that the choice

$$\alpha_1 = \ldots = \alpha_n = 0$$

corresponds to the maximum entropy solution, with modelling filter

$$v_0(z) = \frac{c_0}{2} \cdot \frac{\psi_n(z)}{\phi_n(z)} \tag{6}$$

which is, of course, always positive real.

If $v(z)$ defined by (5)) is positive real, the question is whether the interval

$$v_\lambda(z) = \frac{c_0}{2} \cdot \frac{\psi_n(z) + \lambda(\alpha_1 \psi_{n-1}(z) + \ldots + \alpha_n)}{\phi_n(z) + \lambda(\alpha_1 \phi_{n-1}(z) + \ldots + \alpha_n)} \quad 0 \leqslant \lambda \leqslant 1 \tag{7}$$

consists entirely of positive real functions. Thus, in the Kimura–Georgiou parametrization, the rational n-dimensional solutions to this covariance extension problem would be *star-shaped* about the maximal entropy solution. This property would follow from convexity, which seems extremely plausible on the basis of extensive numerical simulation by several groups of researchers. This is further supported by our proof (given in § 3) that the rational degree n solutions to the covariance extension problem admit a parametrization as euclidean space. Part of the importance of establishing either convexity, star-shapedness or the existence of euclidean parametrizations would be in facilitating the construction, using geometric or optimization techniques, of variations from the maximum entropy solution. Such variations would of course have shaping filters with a richer zero structure.

Turning to linear properties, we use the fact that the set of positive real functions, and its geometry, is related both to the set of real Schur polynomials and to the set of complex Schur polynomials in such a way that the star-shapedness of the degree n positive real functions, in the Kimura–Georgiou parametrization, implies star-shapedness of the degree n real monic Schur polynomials and is implied by the star-shapedness of the degree n complex monic Schur polynomials. Thus, for $n = 1$, convexity is immediate.

Another of the main results of this paper is that, in general, the Kimura–Georgiou parametrization is not convex, nor even star-shaped about the maximum entropy solution. In fact, for $n \geqslant 3$ there is an open set of γ and α values for which this Kharitonov-like property fails. However, our numerical results suggest that the Schur parameters may need to be extremely small, suggesting that this property may hold for a reasonable range of Schur parameters. For reasons concerning the real and complex Schur regions, the case $n = 2$ is decidedly non-trivial. While convexity is suggested by extensive simulation, it may turn out that the region is star-shaped about the maximum entropy solution but non-convex.

2. Kimura–Georgiou parametrization

In what follows, without loss of generality, we set $c_0 = 1$. Therefore, given a sequence of real numbers $\{1, c_1, c_2, \ldots, c_n\}$ such that the Toeplitz matrix T_n is positive definite, we consider the class $\mathscr{C}_n$ of rational functions

$$v(z) = \frac{1}{2} \cdot \frac{b(z)}{a(z)} \tag{8 a}$$

of degree at most n such that

$$v(z) = \frac{1}{2} + c_1 z^{-1} + c_2 z^{-2} + \ldots + c_n z^{-n} + O(z^{-n-1}) \tag{8 b}$$

and

$$\left.\begin{aligned} v(z) \text{ is analytic for } |z| \geqslant 1 \\ v(z) + v(1/z) > 0 \text{ for } |z| = 1 \end{aligned}\right\} \tag{9}$$

This is precisely the class of functions (8) such that

$$a(z) \text{ and } b(z) \text{ are monic of degree } n \tag{10 a}$$

$$a(z)b(1/z) + a(1/z)b(z) > 0 \quad \text{for } |z| = 1 \tag{10 b}$$

$$a(z) \text{ is stable, i.e. all zeros on } |z| < 1 \tag{10 c}$$

where the functions $v(z)$ of degree less than n are represented by $a(z)$ and $b(z)$ with common (stable) factors.

In this representation $b(z)$ will also be stable, because $v(z)$ is positive real if and only if $1/v(z)$ is. Kimura (1983) and Georgiou (1983) have shown independently that any $v(z) \in \mathscr{C}_n$ can be written

$$v(z) = \frac{1}{2} \cdot \frac{\psi_n(z) + \alpha_1 \psi_{n-1}(z) + \ldots + \alpha_n \psi_0(z)}{\phi_n(z) + \alpha_1 \phi_{n-1}(z) + \ldots + \alpha_n \phi_0(z)} \tag{11}$$

where $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ are real numbers and $\{\phi_k(z)\}$ and $\{\psi_k(z)\}$ are the Szegö polynomials of the first and second kind respectively. These are polynomial orthogonal on the unit circle, and they are defined by the recursions

$$\left.\begin{aligned} \phi_{t+1}(z) &= z\phi_t(z) - \gamma_t \phi_t^*(z) \quad \phi_0(z) = 1 \\ \phi_{t+1}^*(z) &= \phi_t^*(z) - \gamma_t z \phi_t(z) \quad \phi_0^*(z) = 1 \end{aligned}\right\} \tag{12}$$

and

$$\left.\begin{aligned} \psi_{t+1}(z) &= z\psi_t(z) + \gamma_t \psi_t^*(z) \quad \psi_0(z) = 1 \\ \psi_{t+1}^*(z) &= \psi_t^*(z) + \gamma_t z \psi_t(z) \quad \psi_t^*(z) = 1 \end{aligned}\right\} \tag{13}$$

respectively, where $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \ldots\}$ are the Schur parameters defined by

$$\gamma_t = \frac{1}{r_t}\left[\sum_{i=0}^{t-1} \phi_{t,t-i} c_{i+1} + c_{t+1}\right] \tag{14}$$

$$r_{t+1} = (1 - \gamma_t^2) r_t; \quad r_0 = 1$$

Here $\{\phi_{tk}\}$ are the coefficients of

$$\phi_t(z) = z^t + \phi_{t1} z^{t-1} + \ldots + \phi_{tt} \tag{15}$$

and $\phi_t^*(z) := z^t \phi_t(1/z)$ is the reversed polynomial

$$\phi_t^*(z) = \phi_{tt} z^t + \ldots + \phi_{t1} z + 1 \tag{16}$$

Therefore, in each of (12) (13), the second equation is equivalent to the first.

Now let $\mathscr{A}_+(n)$ be the set of all $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ such that (11) is positive real. Kimura has raised the question of whether $\mathscr{A}_+(n)$ is star-shaped, i.e. whether

$$\alpha \in \mathscr{A}_+(n) \Rightarrow \lambda\alpha \in \mathscr{A}_+(n) \quad \text{for all } \lambda \in [0, 1]$$

Simulations in lower dimensions seem to support such an assertion. In the case $n = 1$, we have

$$a(z) = z - \gamma_0 + \alpha$$

which is stable if and only if

$$|\alpha - \gamma_0| < 1 \tag{17}$$

Moreover

$$b(z) = z + \gamma_0 + \alpha$$

and hence condition (10 b) can be written

$$(\alpha^2 + 1 - \gamma_0^2) + 2\alpha \cos\theta > 0 \quad \text{for } \theta \in [0, 2\pi]$$

i.e.

$$-\alpha^2 - 1 + \gamma_0^2 < 2\alpha < \alpha^2 + 1 - \gamma_0^2$$

which holds if and only if

$$-(1 - |\gamma_0|) < \alpha < (1 - |\gamma_0|) \tag{18}$$

Any α satisfying (18) also satisfies (17) and hence $\mathscr{A}_+(1)$ is the interval defined by (18).

More generally, there is a strong relationship between the geometry of $\mathscr{A}_+(n)$ and the geometry of the following set of Schur polynomials

$$\mathscr{S}_{\mathbb{R}}(n) = \{p(z) = z^n + p_1 z^{n-1} + \dots + p_n : p_i \in \mathbb{R}, p(z) = 0 \Rightarrow |z| < 1\}$$

$$\mathscr{S}_{\mathbb{C}}(n) = \{q(z) = z^n + q_1 z^{n-1} + \dots + q_n : q_i \in \mathbb{C}, q(z) = 0 \Rightarrow | < 1\}$$

Of course, if the rational function (8 *a*) is positive real, then $b(z), a(z) \in \mathscr{S}_{\mathbb{R}}(n)$. In particular, it is plausible that by allowing the Schur parameters to tend to zero and allowing only stable cancellations, we would have

$$\forall\, |\gamma_i| < 1, \quad \mathscr{A}_+(n) \text{ is convex} \Rightarrow \mathscr{S}_{\mathbb{R}}(n) \text{ is convex} \tag{19 a}$$

The relationship between $\mathscr{A}_+(n)$ and $\mathscr{S}_{\mathbb{C}}(n)$ is known in the literature in various guises. Given that $a(z) \in \mathscr{S}_{\mathbb{R}}(n)$, to say that the rational function (8 *a*) is positive real is to say, by the 'maximum modulus principle', that the harmonic function

$$\operatorname{Re}\{b(z)/2a(z)\}, \quad |z| \geqslant 1$$

which achieves its minimum value on $|z| = 1$, takes only positive values. That is, for any $\mu \in \mathbb{C}^+$

$$\frac{1}{1+\mu}(b(z) + \mu(z)) \in \mathscr{S}_{\mathbb{C}}(n)$$

From this observation, we can easily see that

$$\mathscr{S}_{\mathbb{C}}(n) \text{ is convex} \Rightarrow \mathscr{A}_+(n) \text{ is convex, for all } |\gamma_i| < 1 \tag{10 b}$$

Thus, convexity of $\mathscr{A}_+(n)$ is 'sandwiched' between convexity of the real and of the complex Schur polynomials. From the principle (19) we can easily see that

(i) for $n \geqslant 3$, $\mathscr{A}_+(n)$ is not convex for all $|\gamma_i| < 1$ since $\mathscr{S}_{\mathbb{R}}(n)$ is not convex;
(ii) $\mathscr{A}_+(1)$ is convex for all $|\gamma_0| < 1$, since $\mathscr{S}_{\mathbb{C}}(1)$ is convex.

The remaining case, $n = 2$, has a special and quite interesting feature. Although $\mathscr{S}_{\mathbb{R}}(2)$ is convex, as it turns out, $\mathscr{S}_{\mathbb{C}}(2)$ is not. If $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are sufficiently close to 1 for the polynomials

$$p_1(z) = z^2 - \varepsilon_1 \exp(-i\varepsilon_2), \quad p_2(z) = z^2 - 2\varepsilon_3 z + \varepsilon_3^2$$

we have $p_1, p_2 \in \mathscr{S}_{\mathbb{C}}(2)$ but

$$\lambda p_1 + (1 - \lambda) p_2 \notin \mathscr{S}_{\mathbb{C}}(2)$$

for a λ in a subinterval of (0, 1). Indeed, the analysis in this case promises to be both tedious and decidedly non-trivial. While there is evidence for convexity on the basis of a large number of simulations, our preliminary analysis raises the question as to whether $\mathscr{A}_+(2)$ may be star-shaped yet not convex.

In § 4, we shall exploit in more rigorous detail a principle similar to (19) which relates the star-shapedness of $\mathscr{A}_+(n)$ about the maximum entropy solution to the star-shapedness of $\mathscr{S}_{\mathbb{R}}(n)$ about the monomial z^n.

3. $\mathscr{A}_+(n)$ is euclidean

In this section, we shall prove a positive result concerning the geometry of $\mathscr{A}_+(n)$, asserting that there exists a smooth, one-to-one mapping

$$F: \mathscr{A}_+(n) \to \mathbb{R}^n \tag{20 a}$$

onto euclidean space with a smooth inverse

$$F^{-1}: \mathbb{R}^n \to \mathscr{A}_+(n) \tag{20 b}$$

More precisely, for fixed $\gamma_0, \ldots, \gamma_{n-1}$ the set of α such that $b(z)$, $a(z)$ in the rational function (8 a) has only stable common factors in an open subset of $\mathbb{R}^n$ (see Dayawansa and Ghosh 1988). Furthermore, the set of all such α values for which (8 a) has strictly positive real part on $|z| \leqslant 1$ is again open. Therefore $\mathscr{A}_+(n)$ is an open subset of $\mathbb{R}^n$. So it makes sense to say that the function F in (20) is differentiable. We note that the existence of such an F would, of course, be immediate if $\mathscr{A}_+(n)$ were convex and would imply that $\mathscr{A}_+(n)$ is convex, and hence star-shaped, in many co-ordinate systems. Being an open subset of $\mathbb{R}^n$, $\mathscr{A}_+(n)$ is, in the language of differential geometry, a smooth n-manifold and the existence of a smooth F in (20) is just the assertion that $\mathscr{A}_+(n)$ is diffeomorphic to $\mathbb{R}^n$.

Quite analogous to determining convexity, it is a difficult task to decide whether two smooth n-manifolds are diffeomorphic; however, Milnor (1964) gives an exposition of several tools which are useful in understanding whether a smooth n-manifold M is diffeomorphic to $\mathbb{R}^n$. Among these criteria are those given in the following lemmas.

Lemma 1 (Brown–Stallings)

A smooth n-manifold M is diffeomorphic to $\mathbb{R}^n$ if and only if for any compact subset $K \subset M$, there is an open subset, $U \subset M$, diffeomorphic to $\mathbb{R}^n$ such that $K \subset U$.

In our proof we shall also need to recognize spaces of polynomials which are diffeomorphic to $\mathbb{R}^n$. For $\Omega \subset \mathbb{C}$ any subset we denote by $\mathscr{P}_\Omega(n)$ is the space of real monic polynomials $p(z)$, of degree n, with all roots lying in Ω.

Lemma 2

Suppose $\Omega \subset \mathbb{C}$ is a self-conjugate open subset with a simple, closed, rectifiable, orientable curve as boundary. Then $\mathscr{P}_\Omega(n)$ is diffeomorphic to $\mathbb{R}^n$.

Proof

By the Riemann mapping theorem, Ω is complex-analytically diffeomorphic to the open unit disc $\mathbb{D}$. Moreover, since Ω is self-conjugate, we can choose the diffeomorphic T_1 to be real; i.e. for $z_i \in \Omega$

$$\bar{z}_1 = z_2 \quad \text{implies} \quad \overline{T_1(z_1)} = T_1(z_2) \tag{21}$$

Next, choose any real diffeomorphism T_2 of $\mathbb{D}$ to $\mathbb{C}$. Composing, we obtain a real diffeomorphism

$$T = T_2 \circ T_1 : \Omega \to \mathbb{C}$$

Now the roots of any $p \in \mathscr{P}_\Omega$ determines an unordered n-triple $(\lambda_1, \ldots, \lambda_n)$, i.e. a *divisor* of degree n, of points $\lambda_i \in \mathbb{C}$, not necessarily distinct. We denote this divisor by $\mathscr{D}_p$ and the space of such divisors is the so-called *symmetric product* $\Omega^{(n)}$ of Ω.

For example, using the elementary theory of symmetric functions, we see that the symmetric products $\Omega^{(n)}$ for $\Omega = \mathbb{R}$ and $\mathbb{C}$, are diffeomorphic, respectively, to $\mathbb{R}^n$ and $\mathbb{C}^n$. In general, the symmetric product $\Omega^{(n)}$ is always a smooth n-manifold; in fact, $\Omega^{(n)}$ is diffeomorphic to $\mathscr{P}_\Omega(n)$. Moreover, the projection

$$\Pi_n : \Omega^n \to \Omega^{(n)}$$

is smooth, and any diffeomorphism

$$T : \Omega_1^{(n)} \to \Omega_2^{(n)}$$

is induced by a unique permutation-invariant diffeomorphism

$$\tilde{T} : \Omega^n \to \Omega^n$$

In particular, if $T : \Omega_1 \to \Omega_2$ is a diffeomorphism, then the induced map

$$\bar{T} : \Omega_1^{(n)} \to \Omega_2^{(n)}$$

defined on divisors of degree n via

$$\bar{T}(\lambda_1, \ldots, \lambda_n) = T(\lambda_1), \ldots, T(\lambda_n))$$

is a diffeomorphism.

Suppose, then, that Ω is as in Lemma 2 and T is a real diffeomorphism of Ω with $\mathbb{C}$. T induces, as above, a real diffeomorphism

$$\bar{T} : \Omega^{(n)} \to \mathbb{C}^{(n)} \simeq \mathbb{C}^n \tag{22}$$

and since T is real, $\bar{T}$ maps self-conjugate divisors to self-conjugate divisors. That is,

$$\bar{T} : \mathscr{P}_\Omega^{(n)} \to \mathscr{P}_\mathbb{C}^{(n)}$$

is a diffeomorphism. However, by the 'fundamental theorem of algebra', $\mathscr{P}_\mathbb{C}(n)$ is the n-dimensional euclidean space of real monic polynomials of degree n. □

Theorem 1

$\mathscr{A}_+(n)$ is diffeomorphic to $\mathbb{R}^n$.

Proof

First note that $\alpha \in \mathscr{A}_+(n)$ if and only if $b + \mu a \in \mathscr{S}_\mathbb{C}(n)$ for all $\mu \in \mathbb{C}^+$. Let $\gamma_0, \ldots, \gamma_{n-1}$ be fixed. Then given α, the polynomial $\dfrac{1}{1+\mu}(b + \mu a)$ and its divisor of roots $D_\mu(\mathscr{D}_b)$ are determined uniquely. Thus, $\alpha \in \mathscr{A}_+(n)$ if and only if two conditions are satisfied:

(i) $\mathscr{D}_b \subset \mathbb{D}$

(ii) $D_\mu(\mathscr{D}_b) \subset \mathbb{D}$ for all $\mu \in \overline{\mathbb{C}^+}$

Of course, $D_0(\mathscr{D}_b) = \mathscr{D}_b$. We note that since $b/2a$ is strictly positive real, the roots of $b + \mu a$ are bounded away from the unit circle; i.e.

$$D_\mu(\mathscr{D}_b) \subset \mathbb{D}(0; \rho) \quad \rho < 1 \tag{23}$$

Moreover, by a standard compactness argument, if $K \subset \mathscr{A}_+(n)$ is a compact subset, then for all $\alpha \in K$, (23) holds where ρ may be chosen (uniformly) less than one. Therefore, any compact $K \subset \mathscr{A}_+(n)$ determines, and is determined by, a compact set of divisors $\mathscr{D}_b$ satisfying (i) and (ii) and hence (23). Now let K be an arbitrary but fixed compact subset of $\mathscr{A}_+(n)$. Of course, $\mathscr{A}_+(n) \neq K$ and therefore there exists a relatively compact subset $V \subset \mathscr{A}_+(n)$ such that

$$K \subset V \subset \bar{V} \subset \mathscr{A}_+(n), \quad \text{dist}\,(K, \bar{V}) > 0$$

hold. In particular, (i), (ii) and (23), for some $\rho < 1$, hold for all divisors $\mathscr{D}_b$ corresponding to $\alpha \in \bar{V}$. Noting that the sets

$$D_K = \{z \in \mathbb{D}: z \in D_\mu(\mathscr{D}_b), b \in K, \mu \in \mathbb{C}^+\}$$

$$D_{\bar{V}} = \{z \in \mathbb{D}: z \in D_\mu(\mathscr{D}_b), b \in \bar{V}, \mu \in \overline{\mathbb{C}^+}\}$$

are each self-conjugate subsets of $\mathbb{D}(0; \rho)$, it is possible to choose a simple, closed, rectifiable, orientable curve γ (also self-conjugate) such that $\gamma \subset \mathscr{D}_{\bar{V}} - \mathscr{D}_K$. Choosing Ω to be the interior of γ, i.e. the bounded component of $\mathbb{C} - \gamma$ (as we may by the 'Jordan separation theorem') we set $U = \mathscr{P}_\Omega(n)$. Since $\mathscr{D}_K \subset \Omega \subset \mathscr{D}_{\bar{V}}$ and $\bar{V} \subset \mathscr{A}_+(n)$, we see that

$$K \subset U \subset \mathscr{A}_+(n)$$

and, by Lemma 2, U is diffeomorphic with $\mathbb{R}^n$. By the Brown–Stallings criterion, $\mathscr{A}_+(n)$ is therefore diffeomorphic with $\mathbb{R}^n$. □

4. Higher dimensional case

We begin with an elementary but useful observation. Consider the pencil of polynomials

$$p_0(z) + \lambda p_1(z), \quad 0 \leqslant \lambda \leqslant 1 \tag{24}$$

Such an interval of polynomials arises in studying the star-shapedness of the Kimura–Georgiou parametrization and the Hurwitz or Schur properties of polytopes of polynomials. Indeed it is known (Bartlett *et al.* 1988) that in order to test the stability of a polytope of polynomials, it is sufficient to test the stability of each edge. Of course, testing the stability properties of (24) can be done using Nyquist criteria or root–locus methods by restricting the gain values to lie in the interval [0, 1]. However, it is more convenient to note that the root loci of (24) can, in fact, be interpreted as an 'unrestricted' root–locus plot.

Lemma 3

The root loci, $z_i(\lambda)$, of (24) are the root loci $z_i(k)$ of

$$p_0(z) + k(p_0(z) + p_1(z)) \quad 0 \leqslant k \leqslant \infty \tag{25}$$

Proof

Since

$$p_0(z) + \lambda p_1(z) = (1 - \lambda)p_0(z) + \lambda(p_0(z) + p_1(z))$$

to say (24) holds is to say that (25) holds when

$$k = \lambda/(1-\lambda)$$

so that for $0 \leqslant \lambda < 1$, we have $0 \leqslant k < \infty$.

Example 1

Consider, for $0 \leqslant \varepsilon \leqslant 1$, the interval of polynomials

$$p_\varepsilon(\lambda, z) = z^3 + \lambda(3\varepsilon z^2 + 3\varepsilon^2 z + \varepsilon^3) \quad 0 \leqslant \lambda \leqslant 1 \tag{26}$$

Using Lemma 1 and standard generalized root-locus methods involving Newton diagrams, we see that for λ near zero the roots $z_i(\lambda)$, $i = 1, 2, 3$, depart from zero in a Butterworth pattern of order three; viz., their tangent directions have the constant phases $-\pi$, $\pm\pi/3$. Also, for λ near one the roots $z_i(\lambda)$, $i = 1, 2, 3$, approach the point $-\varepsilon$ in a Butterworth pattern of order three with asymptotic phases of 0, $2\pi/3$, $4\pi/3$. In particular, while $p_1(1, z)$ is marginally Schur and $p_1(0, z)$ is a Schur polynomial, two of the roots of $p_1(\lambda, z)$ lie outside the unit disc if λ is sufficiently close to one.

For Example 1, we may conclude the following lemma.

Lemma 4

There is a non-empty open interval $(\varepsilon_0, 1)$ such that for ε contained in $(\varepsilon_0, 1)$ the following assertions hold:

(i) $p_\varepsilon(0, z)$ and $p_\varepsilon(1, z)$ are Schur polynomials;

(ii) for a non-empty open interval $I_\varepsilon = (\lambda_-(\varepsilon), \lambda_+(\varepsilon)) \subset [0, 1]$, the polynomial

$$p_\varepsilon(\lambda, z) = (1-\lambda)p_\varepsilon(0, z) + \lambda p_\varepsilon(1, z)$$

is not Schur for $\lambda \in I_\varepsilon$.

We now state the main result of this section, which proves that the Kimura–Georgiou parametrization is not star-shaped about the maximum entropy solution.

Theorem 2

There is a non-empty open subset

$$U = \{(\gamma_1, \gamma_2, \gamma_3, \alpha_1, \alpha_2, \alpha_3)\} \subset \mathbb{R}^6$$

such that

(i) the Schur parameters γ_i satisfy $|\gamma_i| < 1$;

(ii) each rational function

$$g_\alpha(z) = \frac{\psi_3(z) + \alpha_1\psi_2(z) + \alpha_2\psi_1(z) + \alpha_3}{\phi_3(z) + \alpha_1\phi_2(z) + \alpha_2\phi_1(z) + \alpha_3}$$

is positive real; and

(iii) for each g_α there exists a non-empty open subinterval $I_\alpha \subset [0, 1]$ such that for $\lambda \in I_\alpha$ the rational function $g_{\lambda\alpha}(z)$ is not positive real.

Remark 1

In particular, there is an 'open set' of counter-examples to a Kharitonov-like property for positive reality in the Kimura–Georgiou parametrization. Moreover, this gives another proof that the positive real functions do not form a convex set in the Kimura–Georgiou parametrization. However, it is very tempting to speculate that such a Kharitonov property might hold if the Schur parameters are sufficiently close in modulus to unity.

Proof of Theorem 2

We note first that as the γ_i tend towards zero the Szegö polynomials of the first and second kind ϕ_m and ψ_m tend to the monomial z^m. In particular, for γ_i sufficiently small

$$\phi_3(z) + \alpha_1\phi_2(z) + \alpha_2\phi_1(z) + \alpha_3 \tag{27}$$

and

$$\psi_3(z) + \alpha_1\psi_2(z) + \alpha_2\psi_1(z) + \alpha_3 \tag{28}$$

will be Schur polynomials if

$$z^3 + \alpha_1 z^2 + \alpha_2 z + \alpha_3 \tag{29}$$

is a Schur polynomial. Similarly, if (29) has two roots lying outside the closed unit disc, for γ_i sufficiently small, both (27) and (28) will have two roots lying outside the closed unit disc. Therefore, choosing

$$\alpha_1 = 3\varepsilon, \quad \alpha_2 = 3\varepsilon^2, \quad \alpha_3 = \varepsilon^3 \tag{30}$$

from Lemma 4(i) there is an open set of γ_i, sufficiently small, and an open interval $(\varepsilon_0, 1)$ of ε values such that

(i) both (27) and (28) are Schur polynomials; and

(ii) the ratio of (27) and (28) has real part close to unity on the unit circle and hence is positive real.

Moreover, by Lemma 4(ii), for each choice of α corresponding to an ε contained in $(\varepsilon_0, 1)$, there is a non-empty open interval $I_\alpha \subset [0, 1]$ such that

(iii) the rational function

$$g_\lambda(z) = \frac{\psi_3(z) + \lambda\alpha_1\psi_2(z) + \lambda\alpha_2\psi_1(z) + \lambda\alpha_3}{\phi_3(z) + \lambda\alpha_1\phi_2(z) + \lambda\alpha_2\phi_1(z) + \lambda\alpha_3}$$

is not positive real for any $\lambda \in I_\alpha$. Indeed both the numerator and denominator will have two roots outside the closed unit disc.

Since these conclusions are valid for arbitrary small deviations in $\alpha_1, \alpha_2, \alpha_3$, we conclude that there exists an open subset

$$U = \{(\gamma_1, \gamma_2, \gamma_3, \alpha_1, \alpha_2, \alpha_3)\} \subset \mathbb{R}^6$$

such that the conclusions of the theorem hold. □

We now give a numerical example, illustrating Theorem 2. First, we note that for the polynomials

$$p_0(z) = z^3 \quad p_1(z) = 3\varepsilon z^2 + 3\varepsilon^2 z + \varepsilon^3$$

the pencil

$$p_0(z) + \lambda p_1(z) \quad 0 \leqslant \lambda \leqslant 1$$

consists entirely of Schur polynomials only for a proper subinterval $[0, \varepsilon_0]$ contained in $[0, 1]$. Rough calculations put ε_0 between 0·8 and 0·9. In particular, it is easily checked that choosing $\varepsilon = 0{\cdot}95$ gives an interval of polynomials with end points being Schur polynomials but for which an interval of λ yields '*non-Schur*' polynomials defined α_i via (30).

Furthermore, choosing the Schur parameters

$$\gamma_1 = 0{\cdot}0001, \quad \gamma_2 = 0{\cdot}0002, \quad \gamma_3 = 0{\cdot}0003$$

and constructing the corresponding Szegö polynomials $\phi_i, \psi_i, i = 1, \ldots, 3$, we obtain the positive real rational functions

$$g_1(z) = \frac{\psi_3(z) + 2{\cdot}7\psi_2(z) + 2{\cdot}43\psi_1(z) + 0{\cdot}729}{\phi_3(z) + 2{\cdot}7\phi_2(z) + 2{\cdot}43\phi_1(z) + 0{\cdot}729} \tag{31}$$

for which

$$g_\lambda(z) = \frac{\psi_3(z) + 2{\cdot}7\lambda\psi_2(z) + 2{\cdot}43\lambda\psi_1(z) + 0{\cdot}729\lambda}{\phi_3(z) + 2{\cdot}7\lambda\phi_2(z) + 2{\cdot}43\lambda\phi_1(z) + 0{\cdot}729\lambda} \tag{32}$$

is not positive real for λ in the interval (0·708, 0·98). For example, it is easily seen from (32) that, for $\lambda = 0{\cdot}84$, it is not positive real, while for $\lambda = 0$ and for $\lambda = 1$ it is positive real.

While convexity of $\mathscr{S}_{\mathrm{R}}(n)$ for $n \leqslant 2$ rules out the existence of such counter-examples, for $n \geqslant 3$ a similar construction to Example 1, *mutatis mutandis*, gives a proof of the following extension of Theorem 2.

Theorem 3

For $n \geqslant 3$, there is an open subset $U \subset \mathbb{R}^{2n}$ of parameters $(\alpha_1, \ldots, \alpha_n, \quad \gamma_0, \ldots, \gamma_{n-1})$ such that

(i) $|\gamma_j| < 1 \quad j = 1, \ldots, n-1$

(ii) $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathscr{A}_+(n)$

Moreover, for each $(\alpha, \gamma) \in U$ there exists an open subinterval $I_{\alpha,\gamma} \subset (0, 1)$ such that

(iii) $\lambda\alpha \notin \mathscr{A}_+(n)$ for $\lambda \in I_{\alpha,\gamma}$.

In particular, for $n \geqslant 3$, $\mathscr{A}_+(n)$ is in general not star-shaped about the maximum entropy filter.

Acknowledgment

This work was partially supported by the Air Force Office of Scientific Research, the National Science Foundation, and the Swedish Board of Technical Development.

We thank Hidenori Kimura, for bringing the problem described in § 1 to our attention, and Martin Hagström and Anders Rantzer for technical assistance.

References

Bartlett, A. C., Hollot, C. V., and Lin, H., 1988, *Mathematics of Control, Signals and Systems*, **1**, 61–71.

Dayawansa, W. P., and Ghosh, B. K., 1988, *SIAM Journal of Control and Optimisation*, **26**, 1149–1174.

GEORGIOU, T. T., 1983, Partial realization of covariance sequences. Ph.D. dissertation, Department of Electrical Engineering, University of Florida; 1987, *I.E.E.E. Transactions on Acoustics, Speech, and Signal Processing*, **35**, 438–449.

GRENANDER, U., and SZEGÖ, G., 1958, *Toeplitz Forms and their Applications* (Berkeley: University of California Press).

KIMURA, H., 1983, A canonical form for partial realization of covariance sequences. Technical Report 83-01, Department of Control Engineering, Osaka University, Japan; 1986, *Modelling, Identification and Robust Control*, edited by C. I. Byrnes and A. Lindquist (North-Holland), pp. 499–513.

MILNOR, J., 1964, *Lectures in Modern Mathematics*, Vol. 1, edited by T. L. Saaty (J. Wiley and Sons), pp. 165–183.