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A HAMILTONIAN APPROACH TO THE FACTORIZATION OF THE MATRIX RICCATI EQUATION*

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To L.E. Zachrisson, in memoriam

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In this note the theory of Hamiltonian systems and an idea due to L.E. Zachrisson is used to obtain the factorizations of the matrix Riccati difference equation on which the derivations [8, 9, 11, 12, 13] of fast (non-Riccati) algorithms are based. Although we are primarily interested in discrete-time Riccati equations, for comparison, the corresponding continuous-time result is briefly discussed.

Key words: Matrix Riccati Equation, Fast Algorithms, Hamiltoniam Systems, Canonical Equations, Linear-Quadratic Regulator Problem, Kalman Filtering.

1. Introduction

Consider the $n \times n$ matrix difference equation

$$P(t+1) - P(t) = \Lambda(P(t)); \quad P(0) = P_0 \tag{1.1}$$

$t = 0, 1, 2, \ldots$, where

$$\Lambda(P) = FPF' - P - (FPH' + G)(HPH' + S)^{-1}(FPH' + G)' + Q. \tag{1.2}$$

Here F, G, H, S and Q are constant matrices of dimensions $n \times n$, $n \times m$, $m \times n$, $m \times m$ and $n \times n$, respectively; S, Q and P_0 are symmetric. (Prime denotes transpose.) Althouth it is actually sufficient to assume that S is invertible, for simplicity we shall here take S to be either positive or negative definite. Moreover we assume that, for some integer $t_1 > 0$,

$$R(t) = HP(t)H' + S \tag{1.3}$$

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is nonsingular for all $t = 0, 1, 2, \dots, t_1$. We impose no positivity assumptions on Q and P_0.

Such matrix Riccati difference equations play an important role in systems theory, and they arise in a certain class of variational problems. During the last decade there has been a considerable interest in so-called fast algorithms to replace the Riccati equation when n is large and when only a few linear combinations of the columns of the solution matrices $\{P(t); t = 0, 1, 2, \dots\}$ are required [8, 9, 11, 12, 13, 15]. It is not hard to see that, if $\Lambda(P_0)$ has rank r, there is a factorization

$$P(t+1) - P(t) = V(t)Z(t)V(t)', \tag{1.4}$$

where V is an $n \times r$ and Z an $r \times r$ matrix sequence. It can be shown that these sequences satisfy simple difference equations, and when r is small there are potentially some computational advantages with such a procedure. This algorithm was first presented in [9] for the special type of Riccati equation occurring in Kalman filtering of stationary stochastic processes. The generalized version considered here appeared in [8, 13].

The primary purpose of this note is to attempt to place this factorization result in a natural theoretical framework. The approaches of [9, 11, 12] proceed from basic principles in stochastic processes, but, by their very nature, these methods are limited to the situation which corresponds to stationary processes. The derivation of [8, 13] does cover the general situation described above, but it is based on *ad hoc* matrix manipulations which give very little insight into what is going on.

In this note we derive these results in terms of the Hamiltonian equations of the variational problem corresponding to the Riccati equation (1.1). The basic idea of this approach was suggested to us by L.E. Zachrisson. We begin by deriving the continuous-time version of the factorization result (this is the framework in which Zachrisson's idea [16] was formulated), since this problem is considerably simpler. We hope that this detour will help the reader to understand the basic idea of the approach and to see what to look for in the discrete-time setting.

A first version of this result appeared in [1].

2. A preliminary study: The continuous-time case

For the sake of comparison we shall first consider the matrix Riccati differential equation

$$\dot{P}(t) = \Lambda(P(t)); \quad P(0) = P_0 \tag{2.1}$$

where

$$\Lambda(P) = FP + PF' - (PH' + G)S^{-1}(PH' + G) + Q \tag{2.2}$$

and F, G, H, S, Q and P_0 are constant matrices defined as in Section 1. This is the continuous-time counterpart of (1.1). It can be shown [2] that there is a $t_1 > 0$ such that (2.1) has a unique solution P on $[0, t_1]$.

To obtain the required factorization we shall proceed as in [16]. Consider the control problem to find a square-integrable m-dimensional vector function u so as to minimize (if S is positive definite)

$$\tfrac{1}{2}x(0)'P_0x(0) + \tfrac{1}{2}\int_0^{t_1} [x(t)'Qx(t) + 2x(t)'Gu(t) + u(t)'Su(t)]\,dt, \tag{2.3}$$

where the n-dimensional vector function x is the solution of

$$-\dot{x} = F'x + H'u; \quad x(t_1) = a. \tag{2.4}$$

By a standard completion-of-squares argument it is easy to see that there really exists a (unique) optimal control; we shall call it u^*. Let the corresponding solution of (2.4) be denoted x^*.

Next define the Hamiltonian function corresponding to this control problem:

$$\mathcal{H}(t, x, u, y) = y'(F'x + H'u) + \tfrac{1}{2}(x'Qx + 2x'Gu + u'Su).$$

(See e.g. [6].) Then the corresponding canonical equations are

$$\begin{cases} \dot{x}^*(t) = -\dfrac{\partial\mathcal{H}}{\partial y}(t, x^*(t), u^*(t), y(t)); \quad x^*(t_1) = a, & (2.5a) \\[2ex] \dot{y}(t) = \dfrac{\partial\mathcal{H}}{\partial x}(t, x^*(t), u^*(t), y(t)); \quad y(0) = P_0x^*(0). & (2.5b) \end{cases}$$

Here (2.5a) is merely (2.4) rewritten, while (2.5b) defines the adjoint function y; (2.5b) can be written

$$\dot{y} = Fy + Qx^* + Gu^*; \quad y(0) = P_0x^*(0). \tag{2.6}$$

Now, the Pontryagin Maximum Principle [6] states that

$$\frac{\partial\mathcal{H}}{\partial u}(t, x^*(t), u^*(t), y(t)) = 0,$$

i.e.,

$$u^*(t) = -S^{-1}[Hy(t) + G'x^*(t)], \tag{2.7}$$

which inserted into (2.4) and (2.6) yields the following form of the canonical equations

$$\begin{cases} \dot{x}^* = -A'x^* + H'S^{-1}Hy; \quad x^*(t_1) = a, & (2.8a) \\ \dot{y} = [Q - GS^{-1}G']x^* + Ay; \quad y(0) = P_0x^*(0) & (2.8b) \end{cases}$$

where $A := F - GS^{-1}H$. Therefore

$$\begin{cases} x^*(t) = X(t)x_0 & (2.9a) \\ y(t) = Y(t)x_0 & (2.9a) \end{cases}$$

where $x_0 := x^*(0)$ and X and Y are $n \times n$-matrix functions satisfying

$$\begin{cases} \dot{X} = -A'X + H'S^{-1}HY; \quad X(0) = I, & (2.10a) \\ \dot{Y} = [Q - GS^{-1}G']X + AY; \; Y(0) = P_0. & (2.10b) \end{cases}$$

It is well known [2, 6] and easy to check that

$$P(t) = Y(t)X(t)^{-1} \tag{2.11}$$

is the unique solution of the Riccati equation (2.1) on $[0, t_1]$. Consequently

$$y(t) = P(t)x^*(t) \tag{2.12}$$

for all $t \in [0, t_1]$.

Using (2.7) and (2.12) it can now be seen that

$$\mathcal{H}(t, x^*(t), u^*(t), y(t)) =$$

$$= \tfrac{1}{2}x^*(t)'[FP(t) + P(t)F' - (P(t)H' + G)S^{-1}(P(t)H' + G)' + Q]x^*(t)$$

and consequently, in view of (2.1) and (2.9),

$$\mathcal{H}(t, x^*(t), u^*(t), y(t)) = \tfrac{1}{2}x_0'M(t)x_0, \tag{2.13}$$

where the $n \times n$-matrix function M is defined by

$$M(t) = X(t)'\dot{P}(t)X(t) \tag{2.14}$$

But, for each $a \in \mathbf{R}^n$, (2.13) is constant [6]; this is the energy conservation condition. Hence, since $x_0 = X^{-1}(t_1)a$, (2.13) is constant for all $x_0 \in \mathbf{R}^n$, i.e., $M(t) \equiv \Lambda(P_0)$ is constant. Therefore,

$$\dot{P}(t) = \bar{X}(t)\Lambda(P_0)\bar{X}(t)' \tag{2.15}$$

where $\bar{X}(t) := [X(t)']^{-1}$. Now, if rank $\Lambda(P_0) = r$, there exist two constant matrices N and Σ of dimensions $n \times r$ and $r \times r$, respectively, such that $\Lambda(P_0) = N\Sigma N'$; for example, Σ may be chosen as the signature matrix. Then (2.15) is the required factorization, In fact,

$$\dot{\bar{X}}(t) = [F - (P(t)H' + G)S^{-1}H]\bar{X}; \quad \bar{X}(0) = I. \tag{2.16}$$

To see this, use (2.11) and the fact that $(d/dt)[X^{-1}] = -X^{-1}\dot{X}X^{-1}$. Hence, defining the $n \times r$-matrix function $Q := \bar{X}N$, we obtain

$$\begin{cases} \dot{P} = Q\Sigma Q'; \quad P(0) = P_0, & (2.17a) \\ \dot{Q} = [F - (PH' + G)S^{-1}H]Q; \quad Q(0) = N. & (2.17b) \end{cases}$$

This result was first presented in this form in [7], where (2.17) was derived by differentiating the matrix Riccati equation. For Riccati equations corresponding to Kalman filtering of stationary processes, an independent proof based on backward innovations was given in [10] (also see [11]). It should however be noted that the factorization (2.15) appeared already in [3], even though its possible use for deriving fast algorithms was not observed; another procedure based on the Hamiltonian formulations was used there.

3. The discrete-time control problem

We now return to the problem stated in the introduction. In analogy with Section 2 we consider the following control problem. Find a sequence $\{u(1), u(2), \dots, u(t_1)\}$ of m-dimensional vectors minimizing (if S is positive definite) or maximizing (if S is negative definite) the functional

$$\tfrac{1}{2}x(0)'P_0x(0) + \tfrac{1}{2}\sum_{t=1}^{t_1}[x(t)'Qx(t) + 2x(t)'Gu(t) + u(t)'Su(t)] \tag{3.1}$$

subject to

$$x(t) = F'x(t+1) + H'u(t+1); \quad x(t_1) = a \tag{3.2}$$

for $t = 0, 1, 2, \dots, t_1 - 1$. As described in Section 2 we can show that there indeed exists a unique optimal control sequence $\{u^*(1), u^*(2), \dots, u^*(t_1)\}$; let $\{x^*(0), x^*(1), \dots, x^*(t_1)\}$ be the corresponding solution of (3.2).

The Hamiltonian function is defined as

$$\mathcal{H}(t, x, u, y) = y'(F'x + H'u - x) + \tfrac{1}{2}(x'Qx + 2x'Gu + u'Su)$$

and the canonical equations read

$$\begin{cases} x^*(t) - x^*(t+1) = \dfrac{\partial\mathcal{H}}{\partial y}(t, x^*(t+1), u^*(t+1), y(t)); \quad x^*(t_1) = a, & (3.3a) \\ y(t+1) - y(t) = \dfrac{\partial\mathcal{H}}{\partial x}(t, x^*(t+1), u^*(t+1), y(t)); \quad y(0) = P_0x^*(0). & (3.3b) \end{cases}$$

Here (3.3a) is the optimal version of (3.2); and (3.3b), which can be written

$$y(t+1) = Fy(t) + Qx^*(t+1) + Gu^*(t+1); \quad y(0) = P_0x^*(0) \tag{3.4}$$

defines the adjoint sequence $\{y(0), y(1), \dots, y(t_1)\}$. Then the Maximum Principle [4] requires that

$$\frac{\partial\mathcal{H}}{\partial u}(t, x^*(t+1), u^*(t+1), y(t)) = 0,$$

i.e.,

$$u^*(t+1) = -S^{-1}[Hy(t) + G'x^*(t+1)]. \tag{3.5}$$

Inserting this into (3.3) yields

$$\begin{cases} x^*(t) = A'x^*(t+1) - H'S^{-1}Hy(t); & x^*(t_1) = a, \quad (3.6a) \\ y(t+1) = Ay(t) + [Q - GS^{-1}G']x^*(t+1); & y(0) = P_0x^*(0). \quad (3.6b) \end{cases}$$

where
$$A = F - GS^{-1}H. \tag{3.7}$$

Now, following the procedure of Section 2, we would like to consider the system of $n \times n$-matrix difference equations

$$A'X(t+1) = X(t) + H'S^{-1}HY(t); \qquad X(0) = I, \tag{3.8a}$$

$$Y(t+1) = [Q - GS^{-1}G']X(t+1) + AY(t); \quad Y(0) = P_0, \tag{3.8b}$$

$(t = 0, 1, 2, \dots, t_1)$. However, as seen from the following proposition, in general (3.8) will have no solution.

Proposition 3.1. *The system* (3.8) *has a solution* $\{(X(t), Y(t));\ t = 0, 1, \dots, t_1\}$ *if and only if the matrix A is nonsingular.*

The proof is based on the following lemma.

Lemma 3.1. *Let P be any symmetric* $n \times n$ *matrix and let* $R := HPH' + S$ *be nonsingular. Then the matrix* $[1 + H'S^{-1}HP]$ *is full rank, and its inverse is* $[I - H'R^{-1}HP]$.

Proof. It is easy to check that

$$[I + H'S^{-1}H'P][I - H'R^{-1}HP] = I.$$

Hence the two matrices on the left side are full rank.

Proof of Proposition 3.1. Assume that (3.8) has a solution. Then, in view of condition (1.3) and Lemma 3.1,

$$A'X(1) = I + H'S^{-1}HP_0, \tag{3.9}$$

is full rank. But this can happen only if A is nonsingular. This takes care of the 'only if' part. The 'if' part is trivial.

The following lemma, which will also be needed in Section 4, will help us understand the significance of the condition that A be nonsingular.

Lemma 3.2. *The matrix Riccati equation* (1.1) *can be written*

$$P(t+1) - A[P(t) - P(t)H'R(t)^{-1}HP(t)]A' = Q - GS^{-1}G' \tag{3.10}$$

where $A := F - GS^{-1}H$ *and R is defined by* (1.3).

Proof. Since $R = HPH' + S$, it is not hard to see that

$$R^{-1} = S^{-1} - R^{-1}HPH'S^{-1}. \tag{3.11}$$

Now expand (1.1) and use (3.11) in the terms $FPH'R^{-1}G'$ and $GR^{-1}G'$ (twice in the latter) to obtain (3.10).

From (3.10) we can see that, if $a \in \ker A'$ (the null space of A'), $P(t)a$ is constant for $t = 1, 2, \ldots, t_1$, i.e., a is an *invariant direction* of the Riccati equation. Consequently, if A is singular, (1.1) has nontrivial invariant directions, and hence the dimension n of the Riccati equation can be reduced to eliminate these [5, 14]. It is no major restriction in generality to assume that this reduction has already been performed and that therefore A is nonsingular. This will be done in the rest of this paper. However, it should be pointed out that by adjusting the initial conditions of (3.8) so that they have the same rank as A and using pseudo-inverses in the sequel, we can dispense with this assumption, but such a strategy would only obscure the basic ideas of this paper.

Subject to the assumption that A is nonsingular, the system (3.6) has the solution

$$\begin{cases} x^*(t) = X(t)x_0, & (3.12a) \\ y(t) = Y(t)x_0, & (3.12b) \end{cases}$$

where $x_0 := x^*(0)$. The following proposition describes the connection between the matrix Riccati equation and the canonical equations.

Proposition 3.2. *Let A be nonsingular and let $\{(X(t), Y(t));\ t = 0, 1, \ldots, t_1\}$ be the unique solution of* (3.8). *Then $X(t)$ is nonsingular for all $t = 0, 1, \ldots, t_1$, and the solution of the matrix Riccati equation* (1.1) *is given by $P(t) = Y(t)X(t)^{-1}$.*

Proof. We first show that (3.8) is still satisfied with Y exchanged for PX, where P is the solution of the Riccati equation. Inserting $Y = PX$ into (3.8a) yields

$$A'X(t+1) = [I + H'S^{-1}HP(t)]X(t), \tag{3.13}$$

which, in view of Lemma 3.1 and condition (1.3), may be written

$$X(t) = [I - H'R(t)^{-1}HP(t)]A'X(t+1). \tag{3.14}$$

Next insert $Y = PX$ and (3.14) into (3.8b) to obtain

$$\{P(t+1) - A[P(t) - P(t)H'R(t)^{-1}HP(t)]A' - Q + GS^{-1}G'\}X(t+1) = 0,$$

which is the identity (Lemma 3.2). Hence, by uniqueness, $Y = PX$. Next, note that, if $X(t)$ is nonsingular, by (3.13) and Lemma 3.1, so is $X(t+1)$. Hence, since $X(0) = I$, $X(t)$ is nonsingular for $t = 0, 1, \ldots, t_1$.

Corollary 3.1. *Let A be nonsingular. For $t = 0, 1, \ldots, t_1$, let $\bar{X}(t) := [X(t)']^{-1}$.*

Then $\bar{X}$ satisfies the recursion

$$\bar{X}(t+1) = [F - (FP(t)H' + G)R(t)^{-1}H]\bar{X}(t); \quad \bar{X}(0) = I. \tag{3.15}$$

Proof. This follows immediately from (3.14) and (3.11).

4. Factorization of the matrix Riccati difference equation

In order to obtain a factorization of type (1.4), we shall take a closer look at the Hamiltonian function $\mathcal{H}$. A discrete-time counterpart of (2.13) reads

$$\begin{aligned}&\mathcal{H}(t, x^*(t+1), u^*(t+1), y(t))\\ &\quad = \tfrac{1}{2}x_0'M(t)x_0 + \tfrac{1}{2}[x^*(t+1) - x^*(t)]'P(t)[x^*(t+1) - x^*(t)].\end{aligned} \tag{4.1}$$

where the $n \times n$ matrix sequence M is defined by

$$M(t) = X(t+1)'[P(t+1) - P(t)]X(t+1). \tag{4.2}$$

To see this we need the following lemma.

Lemma 4.1. *Let $z \in \mathbf{R}^n$ be arbitrary. Then*

$$\begin{aligned}&\mathcal{H}(t, x^*(t+1), u^*(t+1), z)\\ &\quad = z'[x^*(t) - x^*(t+1)] + \tfrac{1}{2}y(t+1)'x^*(t+1) - \tfrac{1}{2}y(t)'x^*(t).\end{aligned} \tag{4.3}$$

Proof. Using the expression (3.5) for $u(t+1)$ we obtain

$$\begin{aligned}&\mathcal{H}(t, x^*(t+1), u^*(t+1), z)\\ &\quad = z'[x^*(t) - x^*(t+1)] + \tfrac{1}{2}y(t)'H'S^{-1}Hy(t)\\ &\qquad + \tfrac{1}{2}x^*(t+1)'(Q - GS^{-1}G')x^*(t+1)\end{aligned} \tag{4.4}$$

which together with the canonical equations (3.6) yields (4.3).

Remembering that

$$y(t) = P(t)x^*(t), \tag{4.5}$$

(4.1) is an immediate consequence of this lemma. However, unlike its continuous-time counterpart, (4.1) is not constant, nor is M. On the other hand

$$\mathcal{H}(t, x^*(t+1), u^*(t+1), \tfrac{1}{2}[y(t+1) + y(t)]) = \text{constant}. \tag{4.6}$$

In fact, from Lemma 4.1 it follows that

$$\mathscr{H}(t, x^*(t+1), u^*(t+1), \tfrac{1}{2}[y(t+1)+y(t)])$$

$$= \tfrac{1}{2}y(t+1)'x^*(t) - \tfrac{1}{2}y(t)'x^*(t+1). \qquad (4.7)$$

Then, using the canonical equations (3.6), it is not hard to check that the constancy holds for each $x_0 \in \mathbf{R}$. Therefore, again taking (4.5) into account,

$$X(t+1)'[P(t+1) - P(t)]X(t) = \text{constant},$$

i.e.,

$$P(t+1) - P(t) = \bar{X}(t+1)X(1)'\Lambda(P_0)\bar{X}(t)', \qquad (4.8)$$

where $\bar{X}$ is defined as in Corollary 3.1. If rank $\Lambda(P_0)$ is small, this does lead to a fast procedure, but, as we shall see in the end of this section, the corresponding algorithm will be more complicated. This is the algorithm obtained from the nonsymmetric factorization mentioned in [13, p. 320]; also cf. [8, 13].

To retain symmetry we shall discard the energy constancy and instead derive a recursion for the M-sequence (4.2). As in [8, 9, 11, 12, 13] the basic idea here is the shift-invariance caused by the constant coefficients of (1.1). If we use the reformulated Riccati equation (3.10) to eliminate $[Q - GS^{-1}G']$ in (4.4), we merely obtain (4.3) again. However, if we first shift (3.10) one step backward in time we get the following recursion for M.

Proposition 4.1. *Let A be nonsingular. Then the solution of the matrix Riccati equation* (1.1) *is given by*

$$P(t+1) - P(t) = \bar{X}(t+1)M(t+1)M(t)\bar{X}(t+1)'; \quad P(0) = P_0, \qquad (4.9)$$

where $\bar{X}$ is given by (3.15) *and M satisfies the recursion*

$$\begin{cases} M(t+1) = M(t) + M(t)\bar{X}(t+1)'H'R(t)^{-1}H\bar{X}(t+1)M(t), \\ M(0) = X(1)'\Lambda(P_0)X(1). \end{cases} \qquad (4.10)$$

Proof. Exchange $[Q - GS^{-1}G']$ in (4.4) by the left member of (3.10) *with t exchanged for $t-1$*. In the term corresponding to the second term in (3.10) express $x^*(t+1)$ in terms of $x^*(t)$ by using (3.6a) and (4.5). Then, by replacing $H'P(t)H$ by $R(t) - S$ in the resulting expression and again using (4.5), we obtain

$$\mathscr{H}(t, x^*(t+1), u^*(t+1), 0) =$$

$$= \tfrac{1}{2}x^*(t+1)'P(t)x^*(t+1) - \tfrac{1}{2}x^*(t)'P(t-1)x^*(t)$$

$$+ \tfrac{1}{2}x^*(t)'[P(t) - P(t-1)]H'R(t-1)^{-1}H[P(t) - P(t-1)]x^*(t),$$

which compared with (4.3) yields (4.10), since these relations hold for all $x_o \in \mathbf{R}^n$. To see this use (3.12), (4.2) and (4.5). Relation (4.9) follows directly from (4.2).

Another symmetric factorization is obtained by considering

$$\hat{M}(t) = X(t)'[P(t+1) - P(t)]X(t) \tag{4.11}$$

instead of M. By instead shifting (3.10) one step forward in time we obtain

Proposition 4.2. *Let A be nonsingular. Then the solution of the matrix Riccati equation* (1.1) *is given by*

$$P(t+1) - P(t) = \bar{X}(t)\hat{M}(t)\bar{X}(t)'; \quad P(0) = P_0$$

where $\bar{X}$ is given by (3.15) *and $\hat{M}$ satisfies the recursion*

$$\begin{aligned} &\hat{M}(t+1) = \hat{M}(t) - \hat{M}(t)\bar{X}(t)H'R(t)^{-1}H\bar{X}(t)'\hat{M}(t), \\ &\hat{M}(0) = \Lambda(P_0). \end{aligned} \tag{4.13}$$

Proof. Proceed precisely as in the proof of Proposition 4.1 except that t should be exchanged for $t+1$ in applying (3.10) to (4.4).

These factorizations can now be used to derive the non-Riccati algorithms of [8, 9, 11, 12, 13]. Analogously to the continuous-time case, let $r := \operatorname{rank} \Lambda(P_0)$; hence $\operatorname{rank} M(0) = r$, and there is a (nonunique) factorization $M(0) = N\Sigma N'$, where N is $n \times r$ and Σ is $r \times r$. Then Proposition 4.1 yields the algorithm

$$P(t+1) = P(t) + V(t)Z(t)V(t)'; \quad P(0) = P_0 \tag{4.14}$$

where V is determined by

$$\begin{cases} U(t+1) = U(t) + FV(t)Z(t)V(t)'H'; \quad U(0) = FP_0H' + G, & (4.15a) \\ V(t+1) = [F - U(t+1)R(t+1)^{-1}H]V(t); \quad V(0) = \bar{X}(1)N, & (4.15b) \\ Z(t+1) = Z(t) + Z(t)V(t)'H'R(t)^{-1}HV(t)Z(t); \quad Z(0) = \Sigma, & (4.15c) \\ R(t+1) = R(t) + HV(t)Z(t)V(t)'H'; \quad R(0) = HP_0H' + S. & (4.15d) \end{cases}$$

To see this define $U(t) := FP(t)H' + G$ and $V(t) := \bar{X}(t+1)N$, and note that $M(t) = NX(t)N'$. Then (4.15) follows from (4.9), (4.10), (3.15) and (1.3).

Likewise, setting $\Lambda(P_0) = N\Sigma N'$, Proposition 4.2 yields the algorithm

$$P(t+1) = P(t) + V(t)Z(t)Z(t)V(t)'; \quad P(0) = P_0 \tag{4.16}$$

where V, now defined by $V(t) := \bar{X}(t)N$, is determined via

$$\begin{cases} U(t+1) = U(t) + FV(t)Z(t)V(t)'H'; \quad U(0) = FP_0H' + G, & (4.17a) \\ V(t+1) = [F - U(t)R(t)^{-1}H]V(t); \quad V(0) = N, & (4.17b) \\ Z(t+1) = Z(t) - Z(t)V(t)'H'R(t)^{-1}HV(t)Z(t); \quad Z(0) = \Sigma, & (7.17c) \\ R(t+1) = R(t) + HV(t)Z(t)V(t)'H'; \quad R(0) = HP_0H' + S. & (4.17d) \end{cases}$$

Here the definitions of V and Z have changed $[M(t) = NZ(t)N']$ while U and R remain the same.

Here the algorithms have been derived under the assumption that A, as defined by (3.7), is nonsingular, a condition which is satisfied if the Riccati equation (1.1) has no invariant directions. This assumption is for convenience only, and the algorithms (4.14)–(4.17) hold without this condition. The derivation above can be modified so that $\bar{X}$ is exchanged for the Moore–Penrose pseudo-inverse $X^{\#}$.

It can now be seen that the unsymmetric factorization (4.8) leads to an algorithm with two V-recursions and no M-recursion. For the interesting case where r is small such an algorithm will have more equations than (4.15) and (4.17).

In the present form the algorithms (4.15) and (4.17) first appeared in [8], although the basic structure of the algorithms had already been presented in [9], where the Riccati equation corresponding to Kalman-filtering of stationary processes was considered. With respect to (4.15) this situation corresponds to $P_0 = Q = 0$ and with respect to (4.17) to P_0 satisfying a Liapunov equation. It could be argued that these are the only *natural* cases for which the algorithms may be fast. Finally, Propositions 4.1 and 4.2 should be compared with the two factorizations in the main lemma of [13].

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