ON THE WELL-POSEDNESS OF THE RATIONAL COVARIANCE EXTENSION PROBLEM*

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Abstract. In this paper, we give a new proof of the solution of the rational covariance extension problem, an interpolation problem with historical roots in potential theory, and with recent application in speech synthesis, spectral estimation, stochastic systems theory, and systems identification. The heart of this problem is to parameterize, in useful systems theoretical terms, all rational, (strictly) positive real functions having a specified window of Laurent coefficients and a bounded degree. In the early 1980's, Georgiou used degree theory to show, for any fixed "Laurent window", that to each Schur polynomial there exists, in an intuitive systems-theoretic manner, a solution of the rational covariance extension problem. He also conjectured that this solution would be unique, so that the space of Schur polynomials would parameterize the solution set in a very useful form. In a recent paper, this problem was solved as a corollary to a theorem concerning the global geometry of rational, positive real functions. This corollary also asserts that the solutions are analytic functions of the Schur polynomials.

After giving an historical motivation and a survey of the rational covariance extension problem, we give a proof that the rational covariance extension problem is well-posed in the sense of Hadamard, i.e a proof of existence, uniqueness and continuity of solutions with respect to the problem data. While analytic dependence on the problem data is stronger than continuity , this proof is much more streamlined and also applies to a broader class of nonlinear problems.

The paper concludes with a discussion of open problems.

1. Introduction

This paper is motivated by the study of the rational covariance extension problem, a problem with historical roots going back to work by Carathéodory and Schur in potential theory [11, 12, 34]. In a recent paper [7] this problem was solved as a corollary to a theorem concerning the global geometry of positive real, or rational Carathéodory functions. These complex-valued functions are analytic and bounded in either the interior or the exterior of the unit disc, and therefore have real parts which are bounded harmonic functions in this region. Carathéodory's interest was in classifying all bounded, positive harmonic functions with prescribed first n Fourier coefficients on the unit circle. This problem was also studied by Toeplitz [35] and

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Schur [34], who was able to develop a complete parameterization of the class of such interpolants defining meromorphic functions $v(z)$, which are positive real. We defer to Section 2 for further discussion of Schur parameters and the associated classical orthogonal polynomials related to this problem.

Our interest in this problem was motivated by its connection to speech synthesis [13], spectral estimation [21, 29], stochastic systems theory [22], and systems identification [27]. Since these application areas focus principally on mathematical models for devices, such as circuits, which can be physically realized with a finite number of active elements, the covariance extension problem in these contexts insists that the solution to the Carathéodory extension problem be rational, as well as being positive real. Indeed, rational, positive real functions also arise in circuit theory as the mathematical models for the impedance, or transfer function, of an RLC network, where the degree of the rational function is precisely the sum of the number of capacitors and inductors and where the positivity reflects the fact that the network resistors are positive.

For these reasons, systems-theoretic formulations of the Carathéodory extension problem insist on rationality as well, hence the emphasis on the *rational* covariance extension problem. For historical reasons, it is interesting to contrast this problem with another rational interpolation problem arising in linear systems theory, the deterministic partial realization problem [23, 24, 33, 19]. In this problem, one insists on rational interpolants which are not necessarily positive real. As it turns out, if one suppresses positivity it is possible to give explicit parameterizations of all rational interpolants having a bounded degree; see Section 3. On the other hand, the Schur parameterization gives a solution to the problem if one suppresses rationality. The combination of these two design requirements has made this problem more elusive for several decades, despite its importance in stochastic system theory, spectral analysis and speech synthesis, see Section 3.

In general, from any rational, positive real function interpolating given Laurent coefficients, one may form its real part on the unit circle, which will define a rational spectral density interpolating the given correlation (or Laurent) coefficients. It is for this reason that the rational covariance extension problem has applications in spectral analysis and speech synthesis, since the stable, minimum phase spectral factor of this density will shape white noise into a process with the given (observed) correlation coefficients. In this connection, there is one well-known solution to the rational covariance extension problem, which also has a pleasant interpretation in terms of the Schur parameterization, the maximum entropy filter introduced by Burg [4] in 1967. This gives rise to a rational spectral density, and hence a shaping filter, with no finite zeros. In many applications, it turns out to be important to be able to design filters with prescribed zeros and which shape processes with observed correlation coefficients. Indeed, the open question as to which zeros can be prescribed, and in which manner, has been a limiting factor in filter design. In practice, however, we would require more, e.g., that the parameters of the shaping filter should be uniquely determined by, and should depend continuously on, the problem data, so that small variations in problem data would give rise to small variations in the solution.

In this paper, we give a new proof of the following consequence of recent work on the geometry of positive real functions, a proof which bypasses the more detailed geometric analysis of [7].

Theorem 1.1. *Suppose one is given a finite sequence of real numbers*

$$c_0, c_1, c_2, \dots, c_n \tag{1.1}$$

which is positive in the sense that the Toeplitz matrix

$$T_n = \begin{bmatrix} c_0 & c_1 & c_2 & \dots & c_n \\ c_1 & c_0 & c_1 & \dots & c_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & c_{n-2} & \dots & c_n \end{bmatrix} \tag{1.2}$$

is positive definite. Then, to each Schur polynomial $\sigma(z)$, i.e., a monic polynomial with all roots having modulus less than one, there corresponds a unique Schur polynomial $a(z)$ such that, for some suitable uniquely defined positive number ρ,

$$w(z) = \rho \frac{\sigma(z)}{a(z)} \tag{1.3}$$

satisfies the interpolation condition

$$w(z)w(1/z) = 1 + \sum_{i=1}^{\infty} \hat{c}_i (z^i + z^{-i})\,; \qquad \hat{c}_i = c_i \quad \textit{for} \quad i = 1, 2, \dots, n \tag{1.4}$$

Moreover, this one-one correspondence is a homeomorphism.

As we shall describe in more detail, this theorem can be expressed in terms of a mapping f between two Euclidean n-spaces which gives a global framework for studying the problem $f(a) = \sigma$. In general, the problem of finding solutions to $f(x) = y$ for a continuous function f has been formalized by Hadamard in the concept of well-posedness. More explicitly, such a problem is said to be well-posed provided:

(i) f is surjective
(ii) f is injective
(iii) f has a continuous inverse.

Concerning (i) in the context of the rational covariance extension problem, in a very innovative paper [17], Georgiou applied degree theoretic methods to show that any Schur polynomial was possible as the numerator of a spectral factor which interpolated the given covariance data. Moreover, he conjectured that (ii) would also hold for this problem. In [7], an answer, in the affirmative, to Georgiou's conjecture was derived as a consequence of a deeper result about the geometry of the space of rational, positive real functions of degree at most n. Intuitively, the geometric result asserts that interpolation and filtering define two complementary foliations, or partitions, of this space. From the complementarity one can deduce that a certain Jacobian matrix is always nonsingular, which together with methods from degree theory (see Section 3) implies (i),(ii), and (iii).

The theorem as stated in [7] actually asserts that f is surjective, injective and that the data $\rho, a_1, a_2, \dots, a_n$ even depend analytically on $\sigma_1, \sigma_2, \dots, \sigma_n$ but, as we shall sketch in Section 3, the proof of analyticity requires a good deal more involved proof than is needed for continuity. Indeed, the principal contribution of the present paper is to give a vastly streamlined proof of well-posedness, which bypasses both analyticity and a study of the geometry of the space of positive real functions. In contrast, our proof begins by demonstrating uniqueness using the residue theorem and then applies a result which asserts that uniqueness implies existence for a general class of nonlinear problems. The fact that uniqueness of solutions to the corresponding systems of equations would imply existence is known for polynomial maps, and is indeed familiar for linear transformations. We prove that proper, continuously differentiable, injective maps are homeomorphisms, a result which may be known but which may also prove to be of independent interest.

The paper itself is organized as follows. In Section 2, we begin with a review of the Carathéodory extension problem for rational functions, rather than for meromorphic functions as was treated in the classical literature. Not surprisingly, the rational covariance extension problem can also be stated in the language of classical analysis and in Section 3, we recast this basic problem in terms of the analysis of a function between two Euclidean spaces of the same dimension and give an abbreviated survey of the use of degree theory to solve the rational covariance extension problem. In Section 4, we provide a more direct proof that the rational covariance extension problem is well-posed in the sense of Hadamard. We conclude our paper in Section 5 with a discussion of some open problems in this area.

2. Preliminaries on the rational covariance extension problem

Given a finite sequence of real numbers

$$c_0, c_1, c_2, \dots, c_n \tag{2.1}$$

which is positive in the sense that the Toeplitz matrix

$$T_n = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_0 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & c_{n-2} & \cdots & c_n \end{bmatrix} \tag{2.2}$$

is positive definite, consider the class of infinite extensions

$$c_{n+1}, c_{n+2}, c_{n+3}, \dots \tag{2.3}$$

of (2.1) with the properties that the function $v(z)$ defined by

$$v(z) = \frac{1}{2}c_0 + c_1 z^{-1} + c_2 z^{-2} + \dots \tag{2.4}$$

in the neighborhood of infinity is

(i) *rational* of at most degree n
(ii) *strictly positive real*, i.e., it is analytic for $|z| \geq 1$ and satisfies

$$v(z) + v(z^{-1}) > 0 \tag{2.5}$$

at each point of the unit circle.

Since, by assumption, we must have $c_0 > 0$, it is no restriction to normalize the problem by setting $c_0 := 1$. This will be done in the rest of the paper. Removing the rationality condition (i) required in systems theory, this becomes a classical interpolation problem studied by Carathéodory [11, 12], Toeplitz [35] and Schur [34]. Indeed, using what are now known as *Schur parameters*, Schur introduced a complete parameterization of the class of extensions defining meromorphic functions $v(z)$, analytic for $z \geq 1$ and satisfying $\Re v(z) \geq 0$ there. Such functions are called *Carathéodory functions*. Clearly all $v(z)$ satisfying (i) and (ii) are Carathéodory functions.

More precisely, recall that the Szegö polynomials

$$\varphi_t(z) \;=\; z^t + \varphi_{t1}z^{t-1} + \cdots + \varphi_{tt} \tag{2.6}$$

are monic polynomials orthogonal on the unit circle [1, 20], which can be determined recursively [1] via the *Szegö-Levinson equations*

$$\varphi_{t+1}(z) \;\; = z\varphi_t(z) - \gamma_t\varphi_t^*(z) \quad \varphi_0(z) = 1 \tag{2.7a}$$

$$\varphi_{t+1}^*(z) \;\; = \varphi_t^*(z) - \gamma_t z\varphi_t(z) \quad \varphi_0^*(z) = 1, \tag{2.7b}$$

where $\gamma_0, \gamma_1, \gamma_2, \dots$ are the Schur parameters

$$\gamma_t \;=\; \frac{1}{r_t}\sum_{k=0}^{t} \varphi_{t,t-k}c_{k+1}, \tag{2.8}$$

and where $(r_0, r_1, r_2, \dots)$ are generated by

$$r_{t+1} = (1-\gamma_t^2)r_t \quad r_0 = 1. \tag{2.9}$$

Similarly, the Szegö polynomials

$$\psi_t(z) \;=\; z^t + \psi_{t1}z^{t-1} + \cdots + \psi_{tt} \tag{2.10}$$

of the second kind are obtained from (2.7) by merely exchanging γ_t for $-\gamma_t$ everywhere.

For each t, the Schur parameters $\gamma_0, \gamma_1, \dots, \gamma_{t-1}$ are uniquely determined by the covariance parameters $c_1, c_2, \dots, c_t$ via (2.7), (2.8) and (2.9). Conversely, it can be shown that $c_1, c_2, \dots, c_t$ are uniquely determined by $\gamma_0, \gamma_1, \dots, \gamma_{t-1}$ so that there is a bijective correspondence between partial covariance and Schur sequences of the same length [34]. Moreover the function $v(z)$ having the Laurent expansion

$$v(z) = \frac{1}{2} + c_1z^{-1} + c_2z^{-2} + c_3z^{-3} + \dots \tag{2.11}$$

for $|z| > 1$ is a Carathéodory function if and only if

$$|\gamma_t| < 1 \quad \text{for } t = 0, 1, 2, \dots, \tag{2.12}$$

and, as was shown by Schur [34], (2.11) and (2.12) provide us with complete parameterization of all meromorphic Carathéodory functions. As for the covariance extension problem, $c_1, c_2, \dots, c_n$ are fixed, and hence $\gamma_0, \gamma_1, \dots, \gamma_{n-1}$ are fixed too. The assumption that the Toeplitz matrix T_n is positive definite is equivalent to the condition that $|\gamma_t| < 1$ for $t = 0, 1, \dots, n-1$. Covariance extension is then equivalent to selecting the remaining Schur parameters

$$\gamma_n, \gamma_{n+1}, \gamma_{n+2}, \cdots \tag{2.13}$$

arbitrarily subject to the positivity constraint (2.12). An important special case, the *maximum entropy solution*, is obtained by setting all Schur parameters (2.13) equal to zero, a choice that certainly satisfies (2.12). This yields the rational Carathéodory function

$$v(z)=\frac{1}{2}\frac{\psi_n(z)}{\varphi_n(z)}, \tag{2.14}$$

where $\varphi_n(z)$ and $\psi_n(z)$ are the degree n Szegö polynomials of first and second kind respectively.

In general, however, the extension (2.13) will yield a Carathéodory function v which can only be guaranteed to be meromorphic, not rational of degree at most n as required in our case, and, as pointed out in [7], there is no way to characterize the rational solutions by a finite number of inequalities. Indeed, adding rationality changes the character of the problem considerably.

One of the important earlier approaches to providing a parameterization of rational solutions to the interpolation problem was discovered independently by Georgiou [17] and Kimura [26]. They introduced the parameterization

$$v(z)=\frac{1}{2}\frac{\psi_n(z)+\alpha_1\psi_{n-1}(z)+\cdots+\alpha_n\psi_0(z)}{\varphi_n(z)+\alpha_1\varphi_{n-1}(z)+\cdots+\alpha_n\varphi_0(z)}, \tag{2.15}$$

where $\alpha_1,\alpha_2,\cdots,\alpha_n$ are real numbers. In fact, for each choice of $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$, the rational function (2.15), which has degree at most n, interpolates the given partial covariance sequence $1,c_1,c_2,\dots,c_n$, although $v(z)$ need not be strictly positive real. As seen from (2.14), the choice $\alpha=0$ yields to the maximum entropy solution. The $2n$ variables (α,γ), with $\gamma:=(\gamma_0,\gamma_1,\dots,\gamma_{n-1})$ being the fixed Schur parameters, are merely a birational change of coordinates in the $\mathbb{R}^{2n}$ space defined by the coefficients $(a_1,\dots,a_n,b_1,\dots,b_n)$ of

$$\begin{aligned} a(z) &:= z^n+a_1z+\cdots+a_nz^n=\varphi_n(z)+\alpha_1\varphi_{n-1}(z)+\cdots+\alpha_n && \text{(2.16a)}\\ b(z) &:= z^n+b_1z+\cdots+b_nz^n=\psi_n(z)+\alpha_1\psi_{n-1}(z)+\cdots+\alpha_n. && \text{(2.16b)}\end{aligned}$$

(See [9].) Given the partial covariance sequence $1,c_1,c_2,\dots,c_n$, and, hence, equivalently, $\gamma_0,\gamma_1,\dots,\gamma_{n-1}$, let $\mathcal{P}_n(\gamma)$ denote the subset of all $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)\in\mathbb{R}^n$ which renders the rational function (2.15) strictly positive real. The set $\mathcal{P}_n(\gamma)$ contains $\alpha=0$, the maximum entropy solution, but it otherwise cannot be described in simple terms [9].

The Schur and the Georgiou-Kimura parameterizations reflect the dichotomy between rationality and positivity: While the Schur parameterization gives an elegant parameterization of all positive real meromorphic functions which interpolate the covariance data, the Georgiou-Kimura parameterization gives an elegant parameterization of all rational functions which interpolate the covariance data. Imposing the second constraint on either parameterization apparently leads to a very hard problem.

Although an explicit description, say by inequalities in α, of $\mathcal{P}_n(\gamma)$ is elusive, it was shown in [5] that $\mathcal{P}_n(\gamma)$ is diffeomorphic to Euclidean space of dimension n, i.e. to $\mathbb{R}^n$. As an example, taking $\gamma=0$, the space $\mathcal{P}_n(0)$ can be identified with the space $\mathcal{S}_n$ of all Schur polynomials of degree n and thus with $\mathbb{R}^n$, as can also be seen from identifying the space of (unordered) roots in the disc with the space of (unordered) roots in the complex plane.

We now turn to the consequences of $v(z)$ being both rational and positive real. First note that, in view of condition (i), if $v(z)$ is rational of at most degree n, $v(z)$ may be represented as

$$v(z) = \frac{1}{2}\frac{b(z)}{a(z)}, \tag{2.17}$$

where $a(z)$ and $b(z)$ are monic polynomials of degree n, which may of course have common factors. Furthermore, $v(z)$ is strictly positive real if and only if the pseudo-polynomial

$$a(z)b(z^{-1}) + a(z^{-1})b(z) > 0 \tag{2.18}$$

on the unit circle and

$$a(z) = 0 \quad \Rightarrow \quad |z| < 1. \tag{2.19}$$

Since the function $1/v(z)$ is strictly positive real if and only if $v(z)$ is, we also have

$$b(z) = 0 \quad \Rightarrow \quad |z| < 1, \tag{2.20}$$

and (2.18) and (2.20) are sufficient for $v(z)$ to be positive real.

In view of (2.5),

$$\Phi(z) := v(z) + v(z^{-1}) \tag{2.21}$$

is a rational spectral density which is positive on the unit circle. It is well-known that it has a unique *minimum phase spectral factor*, i.e., a rational function $w(z)$ analytic for $|z| \geq 1$ and finite and nonzero at infinity such that

$$w(z)w(z^{-1}) = v(z) + v(z^{-1}). \tag{2.22}$$

It is immediately seen that $w(z)$ has the form

$$w(z) = \rho\frac{\sigma(z)}{a(z)} \tag{2.23}$$

for some strictly positive real number and monic polynomial $\sigma(z)$ of degree n. In other words,

$$a(z)b(z^{-1}) + a(z^{-1})b(z) = \rho^2\sigma(z)\sigma(z^{-1}), \tag{2.24}$$

ρ being the normalizing factor allowing all polynomials to be monic. The rational function $w(z)$ is called a *modeling filter* of the partial sequence (2.1).

For example, the maximum entropy solution, obtained by setting $a = \varphi_n$ and $b = \psi_n$ in (2.17), has the modeling filter

$$w(z) = \rho\frac{z^n}{\varphi_n(z)},$$

and thus it has the property that the corresponding spectral density (2.21) has no finite zeros.

In general, it is reasonable to expect that different rational extensions will correspond to different modeling filters. The main issue is to find a useful, systems-theoretic parameterization of all possible modeling filters, a problem to which we shall return in the next section.

3. A survey of degree theoretic methods for the rational covariance extension problem

In order to address the rational covariance extension problem, Georgiou [17] launched an investigation of which zeros could be prescribed using degree theory as a tool for studying the existence of solutions to nonlinear equations. In 1983, he proved that any Schur polynomial was possible as the numerator of a spectral factor which interpolated the given covariance data. More precisely, Georgiou proved the following result.

Theorem 3.1. *Suppose one is given a finite sequence of real numbers*

$$c_0, c_1, c_2, \dots, c_n \tag{3.1}$$

which is positive in the sense defined in Section 2. Then, to each Schur polynomial

$$\sigma(z) = z^n + \sigma_1 z^{n-1} + \cdots + \sigma_n, \tag{3.2}$$

there corresponds a Schur polynomial

$$a(z) = z^n + a_1 z^{n-1} + \cdots + a_n \tag{3.3}$$

such that, for some suitable uniquely defined positive number ρ,

$$w(z) = \rho \frac{\sigma(z)}{a(z)} \tag{3.4}$$

satisfies the interpolation condition

$$w(z)w(1/z) = 1 + \sum_{i=1}^{\infty} \hat{c}_i (z^i + z^{-i})\,; \qquad \hat{c}_i = c_i \quad \text{for} \quad i = 1, 2, \dots, n. \tag{3.5}$$

Georgiou also conjectured that there is a unique such $a(z)$ so that there is a complete parameterization for the solutions of the rational covariance extension problem posed above in terms the zeros of the modeling filters, a conjecture which is answered in the affirmative in [7]. In this section, we shall provide a sketch first of Georgiou's approach, in a slightly different geometric context, and then of the solution given in [7] to the rational covariance extension problem.

We begin with a brief review of some basic facts from degree theory. Suppose more generally that $U, V \subset \mathbb{R}^{n+1}$ are open connected subsets and that

$$F : U \to V$$

is a (C^1) function on U. Recall that a function $F : U \to V$ is said to be *proper* if, and only if, $F^{-1}(K)$ is compact for every compact K subset of V. We are interested in solutions to the equation

$$y = F(x). \tag{3.6}$$

For $x \in U$, we denote the Jacobian matrix of F at x by $\mathrm{Jac}_x(F)$. A point $y \in V$ is called a *regular value* for F if either

(a) $F^{-1}(y)$ is empty; or
(b) for each $x \in F^{-1}(y)$, $\mathrm{Jac}_x(F)$ is nonsingular.

Regular values not only exist but, according to Sard's Theorem [31], are dense. Since for a regular value y of type (b), $F^{-1}(y)$ is finite, we may then compute the finite sum

$$\deg_y(F) = \sum_{F(x)=y} \text{sign} \det \text{Jac}_x(F). \tag{3.7}$$

If y is a regular value of type (a), we set $\deg_y(F) = 0$.

The main conclusions of degree theory [31] relate to the solvability of equations and may be summarized as follows:

(i) The degree, $\deg_y(F)$, of F with respect to y is independent of the choice of regular value y.

(ii) Therefore, we may define the degree of F as

$$\deg(F) = \deg_y(F)$$

for any regular y.

(iii) If H is a jointly continuous map from $U \times [0,1] \to V$ such that $H(x,0) = F(x)$ and $H(x,1) = G(x)$, then

$$\deg(F) = \deg(G).$$

Remark 3.2. From the definition of degree, it is clear that in general degree theory cannot be used to enumerate solutions to the equation (3.6) since $\det \text{Jac}_x(F)$ can assume either positive or negative values. One well-known exception, when the degree actually corresponds to the number of solutions arises in the computation of the degree of complex polynomials, for which the degree equals the algebraic degree of the polynomial. Indeed, the Cauchy-Riemann equations imply that the Jacobian determinant

$$\det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = u_x^2 + v_y^2$$

of a complex analytic function can only assume non-negative values. In particular, for a regular value y one has

$$\deg_y(F) = \sum_{F(x)=y} \text{sign} \det \text{Jac}_x(F) = \#\{x | F(x) = y\}.$$

As we shall see, a similar situation prevails here: Positivity of the covariance sequence in fact implies a similar positivity condition on the Jacobian determinant. This non-trivial fact, which perhaps reflects the interrelating complex analysis and probability theory, underlies our proof of uniqueness, which itself is shown using complex analytic methods.

As a prelude to an application of degree theory, we shall first need to set up the domain, the range and the mapping associated to the rational covariance extension problem. For any $a \in \mathcal{S}_n$, define the operator $S(a) : \mathcal{V}_n \to \mathcal{W}_n$ from the $n+1$-dimensional vector space $\mathcal{V}_n$ of polynomials of degree at most n into the $n+1$-dimensional vector space $\mathcal{W}_n$ of symmetric pseudo-polynomials of degree at most n via

$$S(a)b = \frac{1}{2}[a(z)b(z^{-1}) + a(z^{-1})b(z)]. \tag{3.8}$$

In view of the unit circle version of Orlando's formula [15], $S(a)$ is a nonsingular linear transformation for all $a \in \mathcal{S}_n$. (Also see, e.g., [14] where a determinental expression is given in terms of a.) Let $\mathcal{D}_n \subset \mathcal{W}_n$ be the space of pseudo-polynomials

$$d(z) = d_0 + d_1(z + z^{-1}) + \cdots + d_n(z^n + z^{-n}) \tag{3.9}$$

of degree at most n which are positive on the unit circle. Then, for any $d \in \mathcal{D}_n$, $\lambda S(a)b = d$ uniquely defines a strictly positive real function $v(z) = \frac{1}{2}\frac{b(z)}{a(z)}$, where $\lambda \in \mathbb{R}_+$ is a normalizing factor chosen so that $b(z)$ is monic and hence $c_0 = 1$. If $\gamma = 0$, this problem reduces to spectral factorization. In fact, in this case

$$\varphi_t(z) = \psi_t(z) = z^t$$

so that $b(z) = a(z)$, and consequently $\lambda S(a)b = d$ reduces to

$$\lambda a(z)a(z^{-1}) = d(z). \tag{3.10}$$

With this in mind, let us consider the continuous map $F_\gamma : \mathbb{R}^{n+1} \to \mathcal{W}_n$ defined by

$$F_\gamma(\lambda, \alpha) = \frac{1}{2}\lambda[a(z)b(z^{-1}) + a(z^{-1})b(z)], \tag{3.11}$$

with $a(z)$ and $b(z)$ depending on $\alpha \in \mathbb{R}^n$ via (2.16), which maps the $n+1$-dimensional Euclidean space $\mathbb{R}^{n+1}$ into the $n+1$-dimensional Euclidean space $\mathcal{W}_n$ defined above. Now restricting F_γ to $\mathbb{R}_+ \times \mathcal{P}_n(\gamma)$, where $\mathbb{R}_+ = \{\lambda \in \mathbb{R} \mid \lambda > 0\}$, we obtain the function

$$f_\gamma : \mathbb{R}_+ \times \mathcal{P}_n(\gamma) \to \mathcal{D}_n. \tag{3.12}$$

In fact, $(\lambda, \alpha) \in \mathbb{R}_+ \times \mathcal{P}_n(\gamma)$ implies that $v(z) = \frac{1}{2}\frac{b(z)}{a(z)}$ is strictly positive real, and hence $f_\gamma(\lambda, \alpha)$ is positive on the unit circle. In the same way, restricting F_γ to to the closure $\overline{\mathbb{R}_+ \times \mathcal{P}_n(\gamma)}$, we obtain an extension of f_γ to the boundary, namely

$$\bar{f}_\gamma : \overline{\mathbb{R}_+ \times \mathcal{P}_n(\gamma)} \to \overline{\mathcal{D}_n}. \tag{3.13}$$

Clearly the functions f_γ and $\bar{f}_\gamma$ are continuous.

Georgiou's application of degree theory to the rational covariance extension problem entailed an implicit verification that the mapping in question is proper. The proof we present below is similar, but is posed instead in the slightly different geometric framework used in [7].

Lemma 3.3. *The map f_γ is a proper map.*

Proof. We note that the boundary $\partial\mathcal{P}_n(\gamma)$ of $\mathcal{P}_n(\gamma)$ consists of those α for which (2.15) is positive real but not strictly positive real, and the boundary $\partial\mathcal{D}_n$ of $\mathcal{D}_n$ consists of those pseudo-polynomials which are nonnegative on the unit circle and have at least one zero there (including the zero pseudo-polynomial). Since

$$\partial(\mathbb{R}_+ \times \mathcal{P}_n(\gamma)) = \left(\{0\} \times \overline{\mathcal{P}_n(\gamma)}\right) \cup (\mathbb{R}_+ \times \partial\mathcal{P}_n(\gamma)),$$

we therefore have

$$\bar{f}_\gamma(\partial(\mathbb{R}_+ \times \mathcal{P}_n(\gamma))) \subset \partial\mathcal{D}_n. \tag{3.14}$$

Consequently, $\bar{f}_\gamma$ maps the interior into the interior and the boundary into the boundary. Suppose $K \subset \mathcal{D}_n$ is compact. Since $\bar{f}_\gamma$ is continuous, $\bar{f}_\gamma^{-1}(K)$ is closed. But, in view of (3.14), $\bar{f}_\gamma^{-1}(K) = f_\gamma^{-1}(K)$, so it only remains to show that $f_\gamma^{-1}(K)$ is bounded. Clearly α must be bounded for (2.16a) and (2.16b) to be Schur polynomials. Moreover, if d_0 is the constant term of $d \in \mathcal{D}_n$,

$$d_0 = \lambda^2(\alpha_n^2 + r_1\alpha_{n-1}^2 + \cdots + r_n). \tag{3.15}$$

Therefore, since $r_n > 0$ by (2.9) and (2.12) and d_0 attains a maximum on the compact set $f_\gamma^{-1}(K)$, λ is bounded also. $\square$

Using property (iii), or, more precisely, evaluating the degree of f_γ via a homotopy deformation from the case of general γ to that of $\gamma = 0$, Georgiou was able to show:

Theorem 3.4. *For all $d \in \mathcal{D}_n$,*

$$\deg_d(f_\gamma) = 1.$$

In particular, f_γ is surjective.

An alternative computation of the degree of f_γ using property (ii) is given in [7]. This proof consisted in showing that $\deg_d(f_\gamma) = 1$ where d is the maximum entropy filter.

We now show how to refine these degree theoretic calculations in order to prove a strong form of well-posedness, as in [7]:

Theorem 3.5. *Suppose one is given a finite sequence of real numbers*

$$c_0, c_1, c_2, \dots, c_n \tag{3.16}$$

which is positive in the sense defined above. Then, to each Schur polynomial $\sigma(z)$ there corresponds a unique Schur polynomial $a(z)$ such that, for some suitable uniquely defined positive number ρ,

$$w(z) = \rho\frac{\sigma(z)}{a(z)} \tag{3.17}$$

satisfies the interpolation condition

$$w(z)w(1/z) = 1 + \sum_{i=1}^{\infty} \hat{c}_i(z^i + z^{-i})\,; \qquad \hat{c}_i = c_i \quad \textit{for} \quad i = 1, 2, \dots, n \tag{3.18}$$

Moreover, this one-one correspondence is an analytic diffeomorphism.

As in the case of complex polynomials, it turns out that the degree can also be used to count the number of solutions as well.

Theorem 3.6. *For each $(\lambda, \alpha) \in \mathbb{R}_+ \times \mathcal{P}_n(\gamma)$, $Jac_{(\lambda,\alpha)}(f_\gamma)$ is nonsingular.*

This fundamental fact has several corollaries. First, since the Jacobian of f_γ is always nonsingular, every value of f_γ is a regular value. Moreover, since the open manifold $\mathbb{R}_+ \times \mathcal{P}_n(\gamma)$ is connected, the sign of the determinant cannot change and therefore there cannot be cancellations among the summands in the calculation of the degree at a regular value. This shows that $\deg_d(f_\gamma) \neq 0$, giving an independent proof of Georgiou's Theorem.

Corollary 3.7 (Georgiou). *The map f_γ is surjective.*

Furthermore, since it has been shown that the degree is 1, we obtain

$$1 = \deg_d(f) = \#\{(\lambda,\alpha) | f_\gamma(\lambda,\alpha) = d\}$$

for all $d \in \mathcal{D}_n$. Therefore, we obtain the following result.

Corollary 3.8 ([7]). *The map f_γ is injective. Moreover, by the Implicit Function Theorem, f_γ is an analytic diffeomorphism.*

Because the everywhere nonsingularity of a Jacobian is not universal, we conclude this section with a sketch of the proof of nonsingularity in our particular case, referring to [7] for further details. In order to compute the Jacobian effectively, we need to obtain an intrinsic description of the tangent vectors to $\mathbb{R}_+ \times \mathcal{P}_n(\gamma)$ at a point (λ, α). For this reason, it is more convenient to first consider $\mathbb{R}_+ \times \mathcal{P}_n(\gamma)$ as a submanifold of $\mathbb{R}_+ \times \mathcal{P}_n$, where $\mathcal{P}_n$ denotes the open subset of $\mathbb{R}^{2n}$ consisting of those pairs (α, γ) for which (2.15) is positive real. It is then most convenient to express the tangent vectors to $\mathcal{P}_n$ as pairs of polynomials and, therefore, we shall express the point (α, γ) in terms of (a, b), the monic denominator, numerator pair of polynomials for the rational function $v(z)$ defined by (2.15).

Our strategy will be then to first compute the Jacobian in directions tangent to $\mathbb{R}_+ \times \mathcal{P}_n$ at a point (λ, a, b) and then to determine which such tangent vectors are in fact tangent to the submanifold $\mathbb{R}_+ \times \mathcal{P}_n(\gamma)$ at a point (λ, a, b) in $\mathbb{R}_+ \times \mathcal{P}_n(\gamma)$. We shall then determine what it means for tangent vectors to $\mathbb{R}_+ \times \mathcal{P}_n(\gamma)$ to be annihilated by the Jacobian of f_γ.

Denoting the tangent space to $\mathbb{R}_+ \times \mathcal{P}_n(\gamma)$ at (λ, a, b) by $T_{(\lambda,a,b)}\, \mathbb{R}_+ \times \mathcal{P}_n(\gamma)$ and the tangent space to $\mathcal{P}_n(\gamma)$ at (a, b) by $T_{(a,b)}\, \mathcal{P}_n(\gamma)$, there is a natural direct sum decomposition

$$T_{(\lambda,a,b)}\, \mathbb{R}_+ \times \mathcal{P}_n(\gamma) \simeq T_\lambda\, \mathbb{R}_+ \oplus T_{(a,b)}\, \mathcal{P}_n(\gamma)$$

Hence, for a tangent vector $(\mu, u, v) \in T_{(\lambda,a,b)}\, \mathbb{R}_+ \times \mathcal{P}_n(\gamma)$, the Jacobian of f at (λ, a, b) becomes

$$\mathrm{Jac}_{(\lambda,a,b)}(f)(\mu, u, v) = S(a)(\lambda^2 v + \lambda\mu b) + S(b)(\lambda^2 u + \lambda\mu a). \tag{3.19}$$

For simplicity of notation, we define polynomials, having degree less or equal to n, via

$$p = \lambda^2 u + \lambda\mu a \tag{3.20a}$$

$$q = \lambda^2 v + \lambda\mu b \tag{3.20b}$$

In this notation, the tangent vector (μ, u, v) is annihilated by the Jacobian of f_γ if and only if

$$S(a)q + S(b)p = 0.$$

Note also that to say $p = q = 0$ is to say that $\mu = 0$ and that $u = v = 0$. We next need to characterize those tangent vectors (μ, u, v) which are tangent to $\mathbb{R}_+ \times \mathcal{P}_n(\gamma)$.

Lemma 3.9 ([7]). *For any $(a, b) \in \mathcal{P}_n(\gamma)$,*

$$T_{(a,b)}\, \mathcal{P}_n(\gamma) = \{(u, v) \mid av - bu = r,\ \deg r \le n - 1\} \tag{3.21}$$

We next observe that if (u, v) is tangent to $\mathcal{P}_n(\gamma)$, p and q also satisfy

$$aq - bp = r, \quad \deg r \leq n - 1.$$

In this language, Theorem 3.6 is a direct consequences of the following observation, which is referred to in [7] as the Transversality Lemma.

Lemma 3.10 ([7])**.** *There are no nonzero polynomials p and q of degree at most n such that*

$$S(a)q + S(b)p = 0 \tag{3.22}$$

and

$$aq - bp = r, \tag{3.23}$$

where r is a polynomial of degree less than n.

Proof. Suppose that p and q are polynomials of at most degree n satisfying (3.22) and (3.23). We want to prove that $p = q = 0$. To this end, first note that, in view of (3.23), the function

$$g(z) := \frac{q(z)}{b(z)} - \frac{p(z)}{a(z)} = \frac{r(z)}{a(z)b(z)}$$

has relative degree at least $n + 1$ and is analytic outside a disc contained in the open unit disc so that it has the Laurent expansion

$$g(z) = g_0 z^{-n-1} + g_1 z^{-n-2} + \cdots \tag{3.24}$$

there. Likewise $g(z^{-1})$ is analytic in an open disc containing the closed unit disc, and in this region it has the Taylor expansion

$$g(z^{-1}) = g_0 z^{n+1} + g_1 z^{n+2} + \cdots$$

Now a simple calculation shows that

$$g(z) - g(z^{-1}) = \frac{h(z)}{b(z)a(z^{-1})} - \frac{h(z^{-1})}{a(z)b(z^{-1})}$$

where

$$h(z) := a(z^{-1})q(z) + b(z)p(z^{-1})$$

so that

$$g(z) - g(z^{-1}) = -h(z^{-1})\frac{d(z, z^{-1})}{a(z)a(z^{-1})b(z)b(z^{-1})}$$

and therefore

$$\int_{|z|=1} |h(z)|^2 \frac{d(z, z^{-1})}{|a(z)|^2|b(z)|^2}\frac{dz}{z} = \int_{|z|=1} h(z)[g(z^{-1}) - g(z)]\frac{dz}{z} \tag{3.25}$$

However, h is a pseudo-polynomial of degree less than or equal to n, i.e.

$$h(z) = h_0 + h_1(z + z^{-1}) + \cdots + h_n(z^n + z^{-n})$$

and therefore $h(z)g(z^{-1})z^{-1}$ is holomorphic, having no poles in the an open disc containing the closed unit disc. Similarly, the Laurent expansion of $h(z)g(z^{-1})z^{-1}$ in the region where (3.24) holds has only negative powers of z of order larger than one. Consequently (3.25) is zero, which implies that $h(e^{i\theta}) \equiv 0$, because $d(z, z^{-1})$, $|a(z)|^2$

and $|b(z)|^2$ are all positive on the unit circle. Therefore, by the identity theorem, $h \equiv 0$ in the whole complex plane so that

$$g(z) = g(z^{-1}).$$

But $g(z)$ has only negative powers of z and $g(z^{-1})$ only nonnegative powers of z in an annulus containing the unit circle and hence $g \equiv 0$. Since, therefore, $r \equiv 0$, we have

$$q(z) = \frac{b(z)}{a(z)}p(z)$$

which substituted into (3.22) yields

$$\left[\frac{b(z)}{a(z)} + \frac{b(z^{-1})}{a(z^{-1})}\right]\left[a(z)p(z^{-1}) + a(z^{-1})p(z)\right] = 0.$$

Since $(a, b) \in \mathcal{P}_n(\gamma)$, the first factor is positive on the unit circle and so

$$a(e^{i\theta})p(e^{-i\theta}) + a(e^{-i\theta})p(e^{i\theta}) = 0$$

for all θ, and therefore, by the identity theorem,

$$S(a)p = 0.$$

Since a is a Schur polynomial, and hence has no reciprocal roots, the unit circle version of Orlando's formula [15] (also see [14] and [10, Lemma 5.5]) implies that p, and hence q, is identically zero. $\square$

4. Well-posedness of the the rational covariance extension problem

In this section, we give a streamlined proof of the fact that the rational covariance extension problem is well-posed. We first remark that Theorem 1.1 is actually a consequence of the fact that f_γ is a homeomorphism. To see this note that if f_γ is a homeomorphism then it is in particular a homeomorphism for $\gamma = 0$ so that the map $f_0 : \mathbb{R}_+ \times \mathcal{P}_n(\gamma) \to \mathcal{D}_n$ defined via

$$f_0(\mu, \sigma) = \mu\sigma(z)\sigma(1/z)$$

is a homeomorphic bijection. Then the commutative diagram

$$\begin{array}{ccc} \mathbb{R}_+ \times \mathcal{P}_n(0) & \xrightarrow{g} & \mathbb{R}_+ \times \mathcal{P}_n(\gamma) \\ f_0 \searrow & & \nearrow f_\gamma^{-1} \\ & \mathcal{D}_n & \end{array}$$

defines a homeomorphic bijection g under which

$$\frac{1}{2}\lambda[a(z)b(1/z) + a(1/z)b(z)] = \mu\sigma(z)\sigma(1/z).$$

Setting $\rho^2 := \mu/\lambda$, this is equivalent to

$$\frac{1}{2}\frac{b(z)}{a(z)} + \frac{1}{2}\frac{b(1/z)}{a(1/z)} = \rho^2\frac{\sigma(z)}{a(z)}\frac{\sigma(1/z)}{a(1/z)},$$

where

$$\frac{1}{2}\frac{b(z)}{a(z)} = \frac{1}{2} + c_1 z + \cdots + c_n z^{-n} + \ldots$$

interpolates the given partial covariance sequence so that

$$w(z) = \rho \frac{\sigma(z)}{a(z)}$$

is a modeling filter. Therefore, proving Theorem 1.1 is equivalent to proving that f_γ is a homeomorphism.

We next give a direct proof of injectivity.

Theorem 4.1. *The map f_γ is injective.*

Proof. Suppose that there are two points in $\mathbb{R}_+ \times \mathcal{P}_n(\gamma)$, namely $(\lambda_1, \alpha^{(1)})$ and $(\lambda_2, \alpha^{(2)})$, which f_γ sends to the same $d \in \mathcal{D}_n$, and let $(a_1(z), b_1(z))$ and $(a_2(z), b_2(z))$ be the corresponding polynomials (2.16). Then, for $j = 1, 2$,

$$\frac{d(z)}{\lambda_j a_j(z) a_j(1/z)} = \frac{1}{2}\frac{b_j(z)}{a_j(z)} + \frac{1}{2}\frac{b_j(1/z)}{a_j(1/z)}. \tag{4.1}$$

Since $\alpha^{(j)} \in \mathcal{P}_n(\gamma)$ for $j = 1, 2$,

$$\frac{1}{2}\frac{b_j(z)}{a_j(z)} = \frac{1}{2} + c_1 z^{-1} + c_2 z^{-2} + \cdots + c_n z^{-n} + O(z^{-n-1}) \tag{4.2}$$

outside some circle $|z| = r$ of radius $r < 1$. Similarly,

$$\frac{1}{2}\frac{b_j(1/z)}{a_j(1/z)} = \frac{1}{2} + c_1 z + c_2 z^2 + \cdots + c_n z^n + O(z^{n+1}) \tag{4.3}$$

inside a circle $|z| = r^{-1}$ of radius $r^{-1} > 1$, and hence (4.1) equals the sum of the two power series (4.2) and (4.3) in an open annulus containing the unit circle. Consequently,

$$h(\theta) := \frac{d(e^{i\theta})}{\lambda_1 |a_1(e^{i\theta})|^2} - \frac{d(e^{i\theta})}{\lambda_2 |a_2(e^{i\theta})|^2} = \sum_{-\infty}^{\infty} h_k e^{ik\theta}, \tag{4.4}$$

where $h_k = 0$ for $k = 0, \pm 1, \pm 2, \ldots, \pm n$. On the other hand,

$$g(\theta) := \lambda_1 |a_1(e^{i\theta})|^2 - \lambda_2 |a_2(e^{i\theta})|^2 = \sum_{-n}^{n} g_k e^{ik\theta}, \tag{4.5}$$

and therefore

$$\int_{-\pi}^{\pi} h(\theta) h(\theta) d\theta = 0,$$

or, in other words,

$$\int_{-\pi}^{\pi} \frac{d(e^{i\theta})}{\lambda_1 \lambda_2 |a_1(e^{i\theta})|^2 |a_2(e^{i\theta})|^2} |g(\theta)|^2 d\theta = 0. \tag{4.6}$$

Since $a_1(z)$ and $a_2(z)$ have no zeros on the unit circle and $d(\theta) > 0$ for all $\theta \in [-\pi, \pi]$, (4.6) implies that $g(\theta) \equiv 0$, i.e.,

$$\lambda_1 |a_1(e^{i\theta})|^2 = \lambda_2 |a_2(e^{i\theta})|^2 \quad \text{for all } \theta \in [-\pi, \pi]. \tag{4.7}$$

Therefore, by the identity theorem,

$$\lambda_1 a_1(z) a_1(1/z) = \lambda_2 a_2(z) a_2(1/z) \tag{4.8}$$

in the whole complex plane. But it is well-known that a polynomial spectral factorization problem can only have one Schur solution, and hence we must have $(\lambda_1, \alpha^{(1)}) = (\lambda_2, \alpha^{(2)})$, as claimed. $\square$

Recall that the original degree theoretic proof of injectivity [7], sketched in the previous section, also gave an independent proof of surjectivity as well. There are of course precedents for the equivalence of, or the interrelationship between, these two fundamental properties:

- The first arises in linear algebra, where it is known for a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ that uniqueness of solutions implies existence of solutions, and the existence of an inverse.
- In 1960, D. J. Newman [32] discovered the theorem that every injective polynomial map $p : \mathbb{R}^2 \to \mathbb{R}^2$ is surjective, and hence is a homeomorphism.
- Bialynicki-Birula and Rosenlicht [3] proved that every injective polynomial map $p : \mathbb{R}^n \to \mathbb{R}^n$ is surjective. The inverse, although continuous, is not necessarily a polynomial map.

In our context, we would ask whether such theorems might hold for differentiable maps.

Theorem 4.2. *Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1, proper map. If f is injective, then f is surjective. Moreover, f is a homeomorphism onto $\mathbb{R}^n$ if, and only if, f is proper and injective.*

Proof. We shall prove that for an injective C^1, proper map, we must have $|\deg(f)| = 1$. Since f is injective, several simplifications occur. First, a value of f, $y = f(x)$, is regular if and only if $\det \mathrm{Jac} f(x)$ is nonzero. Moreover, if such an x exists, then

$$\deg(f) = \operatorname{sign} \det \mathrm{Jac} f(x),$$

from which it would follow that $|\deg(f)| = 1$. We conclude the proof by showing that such an x must exist.

Lemma 4.3. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 map which is injective, then* $\det$ *$Jacf(x) \neq 0$ for some $x \in \mathbb{R}^n$.*

Proof. To see this, we suppose that the maximum rank of $\mathrm{Jac} f(x)$ over $x \in \mathbb{R}^n$ is $n-r$ and that this maximum rank is achieved a the point $x_0 \in \mathbb{R}^n$. Since rank $\mathrm{Jac} f(x)$ is lower semicontinuous in x and since rank $\mathrm{Jac} f(x)$ achieves its maximum at x_0, we must have

$$\text{rank } \mathrm{Jac} f(x) = n - r$$

for all x in a neighborhood of x_0. Therefore, by the Implicit Parameterization Theorem [2, p.32], in a neighborhood of x_0 there exists an r-dimensional submanifold passing through x_0, on which f is constant. Since f is injective, we must have that $r = 0$, so that $\det \mathrm{Jac} f(x) \neq 0$ in some neighborhood of x_0 as was to be shown. $\square$

Therefore, f is a continuous bijection from $\mathbb{R}^n$ to $\mathbb{R}^n$, Moreover, since f is proper, f maps closed sets to closed sets, so that f^{-1} is also continuous. This concludes the proof of the theorem. $\square$

Remark 4.4. Alternatively, one might expect to prove the theorem using Sard's Theorem, from which Lemma 4.3 would itself follow if one knew that for some ball B, the measure of $f(B)$ is positive. On the other hand, applying the change of variables formula

$$\int_{f(B)} h(y)dy = \int_B h(f(x))|\det \mathrm{Jac} f(x)|dx$$

to the case where h is the characteristic function of $f(B)$, one sees that

$$\mu(f(B)) = \int_B |\det \mathrm{Jac} f(x)|dx,$$

so that to say that the measure of $f(B)$ is positive for some ball B is to say that $\det \mathrm{Jac} f(x) \neq 0$ for some $x \in B$. As a matter of fact, by applying Lemma 4.3 to an arbitrary ball B, one can derive the conclusion of Sard's Theorem for injective C^1 maps, viz., that the sets of regular points and regular values are dense.

Remark 4.5. We note that this theorem also follows from Brouwer's Theorem on Invariance of Domain.

Remark 4.6. We shall next complete the proof of well-posedness. We remark that the use of degree theory in this proof is in sharp contrast with the methods of [17] and of [7]. Indeed, in [17] the degree is shown to be equal to one by a homotopy deformation to the spectral factorization problem, for which the degree is evaluated at a particular polynomial. Similarly, in [7], the degree is evaluated at the maximum entropy filter. In the proof we give here, we show that for any injective, proper C^1 map there is a dense set of points which correspond to regular values, and we can evaluate the absolute value of the degree without choosing any particular point. A final point of difference is that, in this proof, we are only showing that the set of regular points is dense, rather than showing everywhere nonsingularity of the Jacobian, an assertion which requires much more effort but which, of course, would also yield analytic dependence of the solution on the problem data.

Corollary 4.7. *The map f_γ is a homeomorphic bijection.*

Proof. In view of Theorem 4.1, in order to apply Theorem 4.2, it only remains to show that the map is proper and that the domain and range are Euclidean spaces. We have shown that f_γ is proper in Lemma 3.3. The fact that the open manifold $\mathbb{R}_+ \times \mathcal{P}_n(\gamma)$ is diffeomorphic to Euclidean space follows of course from the same assertion about $\mathcal{P}_n(\gamma)$. As remarked in Section 2 , that $\mathcal{P}_n(\gamma)$ is diffeomorphic to Euclidean space was shown in [5]. The proof in [5] uses the Brown-Stallings criterion [30] which asserts that an n-manifold is diffeomorphic to Euclidean n-space if and only if every compact subset has a Euclidean neighborhood. Finally, by spectral factorization, the open manifold $\mathcal{D}_n$ is diffeomorphic to $\mathbb{R}_+ \times \mathcal{P}_n(0)$ which is a product of Euclidean spaces as above; see also Section 2 for an alternative proof. □

5. Some open problems

In this section, we discuss several open problems related to the rational covariance extension problem.

The minimal stochastic partial realization problem. The minimal stochastic partial realization problem consists in determining, from the data $c_0, c_1, c_2, \dots, c_n$, a positive real, rational interpolant $v(z)$ having minimal degree. This minimal *positive* degree can be different from the minimal *algebraic* degree of a rational function which interpolates the data, but which is not positive real. This latter problem is known as the deterministic partial realization problem, since realizing a minimal degree rational function in state-space form would give a minimal realization interpolating the data $c_0, c_1, c_2, \dots, c_n$ when viewed as Hankel parameters.

One would expect the solution to the rational covariance extension problem to shed some light on the longstanding, open problem of determining the minimal positive degree of a partial sequence $c_0, c_1, c_2, \dots, c_n$. One such recent discovery has occurred through the development of an associated Riccati-type equation, the Covariance Extension Equation, whose unique positive semi-definite solution has as its rank the minimum dimension of a stochastic linear realization of the given rational covariance extension $v(z)$ [6]. This gives an elegant characterization of the positive degree of the interpolant. However, in order to compute the positive degree of the sequence one, would still need to minimize this degree over all choices of the zero, or the σ, polynomial. Nonetheless, the parameterization of all partial stochastic realizations by the zeroes of the shaping filter w should give some new insights into this problem.

Computational methods for solving indefinite Riccati-type equations. In both its form as a complete parameterization of rational extensions to a given covariance sequence and as an indefinite Riccati-type equation, one of problems which remains open is that of developing effective computational methods for the approximate solution of the Riccati-type equation arising as the Covariance Extension Equation.

The rational covariance extension problem for non-strictly positive real data. We have solved the rational covariance extension problem in terms of rational functions v which interpolate the data, which have all poles in the interior of the unit disc, and which have a strictly positive real-part on the unit circle. In certain applications of signal processing, it is desirable to allow zeroes of the shaping filter which are either very close to, or lie on, the unit circle. Thus, it would be very desirable to allow for rational functions v which interpolate the data, which are positive real and which have their zeroes or poles inside the closed unit disc. We should note that such functions may lead to spectral densities which are not H^2 on the the unit circle, and which do not have an annulus of convergence in the complex plane. Not unrelated is the fact that although the problem is probably well-posed for v, it does not seem to be well-posed for w.

The multivariable case. The rational covariance extension problem appears to be wide-open in the case of sequences of matrix data $C_0, C_1, C_2, \dots, C_n$ and their realization, or interpolation, by multivariable stochastic systems.

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