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ON THE DUALITY BETWEEN FILTERING AND NEVANLINNA–PICK INTERPOLATION*

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Abstract. Positive real rational functions play a central role in both deterministic and stochastic linear systems theory, as well as in circuit synthesis, spectral analysis, and speech processing. For this reason, results about positive real transfer functions and their realizations typically have many applications and manifestations.

In this paper, we study certain manifolds and submanifolds of positive real transfer functions, describing a fundamental geometric duality between filtering and Nevanlinna–Pick interpolation. Not surprisingly, then, this duality, while interesting in its own right, has several corollaries which provide solutions and insight into some very interesting and intensely researched problems. One of these is the problem of parameterizing all rational solutions *of bounded degree* of the Nevanlinna–Pick interpolation problem, which plays a central role in robust control, and for which the duality theorem yields a complete solution. In this paper, we shall describe the duality theorem, which we motivate in terms of both the interpolation problem and a fast algorithm for Kalman filtering, viewed as a nonlinear dynamical system on the space of positive real transfer functions.

We also outline a new proof of the recent solution to the rational Nevanlinna–Pick interpolation problem, using an algebraic topological generalization of Hadamard's global inverse function theorem.

Key words. Nevanlinna–Pick interpolation, filtering, positive real functions, foliations, degree constraint

AMS subject classifications. 30E05, 47N70, 58C99, 93B29, 93E11

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1. Introduction.

Modulo a conformal equivalence, the classical Nevanlinna–Pick problem amounts to determining a function which is *positive real*, i.e., is analytic and has nonnegative real part in $\mathbb{D}^c := \{z \in \mathbb{C} \mid |z| > 1\}$, and which satisfies the interpolation condition

$$f(z_k) = w_k \quad \text{for } k = 0, 1, \dots, n, \tag{1.1}$$

where $z_0, z_1, \dots, z_n \in \mathbb{D}^c$ and $w_0, w_1, \dots, w_n \in \mathbb{C}$. This problem has a solution if and only if the associated Pick matrix P is positive semidefinite. It is unique if P is singular, and there are infinitely many solutions if $P > 0$ (see [35, 33]). We are interested in a particular subset of these solutions, namely those which are rational of degree at most n, and we shall refer to the problem of determining these as the *Nevanlinna–Pick problem with degree constraints* [12].

For simplicity, in this paper we shall consider the special case that the interpolation points are all distinct and fixed and with $z_0 = \infty$. Then the Pick matrix becomes

$$P = \left[\frac{w_k + \bar{w}_\ell}{1 - z_k^{-1} \bar{z}_\ell^{-1}} \right]_{k,\ell=0}^{n}.$$

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Moreover, we assume that the sets $z_0, z_1, \dots, z_n$ and $w_0, w_1, \dots, w_n$ are self-conjugate so that only real interpolants f need to be considered. We also normalize the problem by setting

$$w_0 = 1$$

so that $f(\infty) = 1$. Finally, we assume that the interpolant is *strictly positive real* in the sense that

$$f(e^{i\theta}) + f(e^{-i\theta}) > 0 \quad \text{for all } \theta \in [-\pi, \pi].$$

Any such function can, in a unique fashion, be written as

$$f(z) + f(z^{-1}) = v(z)v(z^{-1}), \tag{1.2}$$

where v is a *minimum-phase spectral factor* having all zeros in the open unit disc. These zeros will be called *the spectral zeros* of f. As we have remarked above, there are several conformal equivalents of this problem, including Nevanlinna–Pick interpolation for bounded-real, or Schur, functions. Indeed, even for positive real functions there are two conventions, one dealing with interpolation problems inside the unit disc and one outside the disc, as considered here. Our convention is motivated by the desire to have spectral factors which are stable and minimum-phase and therefore may be realized, in control engineering terms, by a stable discrete-time linear system.

We shall show that the space of all strictly positive real, rational functions of at most degree n, $\mathcal{P}_n$, admits two foliations: an *interpolation foliation* with one leaf for each choice of interpolation values $w_1, w_2, \dots, w_n$ satisfying the Pick condition, and a *filtering foliation* with one leaf for each choice of spectral zeros. These foliations are complementary, each pair of leaves with one from each foliation intersecting in one point under nonzero angle. This result is analogous to that obtained in [6] for the case that $z_0 = z_1 = \cdots = z_n = \infty$, the rational covariance extension problem. We note that the corresponding decompositions for the space of functions which are positive real, rather than strictly positive real, are not necessarily disjoint, nor are the equivalence classes necessarily smooth manifolds. For these reasons, we shall work with strictly positive real functions.

More generally, in section 6 we also prove that $\mathcal{P}_n$ is diffeomorphic to $\mathcal{W}_n^+ \times \mathcal{S}_n$, where $\mathcal{W}_n^+$ is the space of all $w_1, w_2, \dots, w_n$ satisfying the Pick condition, and $\mathcal{S}_n$ is the space of (real) *Schur polynomials* of degree n, i.e., real monic polynomials of degree n with all zeros in the open unit disc. Since, in addition, it can be shown that both $\mathcal{W}_n^+$ and $\mathcal{S}_n$ are diffeomorphic to $\mathbb{R}^n$, this implies that $\mathcal{P}_n$ is Euclidean of dimension $2n$.

2. Preliminaries. Let H_2 be the Hardy space of all real functions which are analytic in the exterior of the unit disc, $\mathbb{D}^c := \{z \in \mathbb{C} \mid |z| < 1\}$, and have square-integrable radial limits

$$\lim_{r \to +1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta < \infty$$

on the boundary. Denoting by L_2 the space of all real functions which are square-integrable on the unit circle, we may identify H_2 with the subspace of L_2 consisting of those functions with vanishing positively indexed Fourier coefficients. More precisely, for $f \in H_2$,

$$f(z) = f_0 + f_1 z^{-1} + f_2 z^{-2} + \cdots .$$

Similarly, let $\bar{H}_2$ be the conjugate Hardy space of L_2-functions which are analytic in the open unit disc and thus have vanishing negatively indexed Fourier coefficients so that

$$f(z) = f_0 + f_1 z + f_2 z^2 + \cdots$$

for $f \in \bar{H}_2$. Hence, if $f^*(z) := f(z^{-1})$, $f \in H_2$ if and only if $f^* \in \bar{H}_2$.

The space L_2 is a Hilbert space with inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})g^*(e^{i\theta})d\theta.$$

Next, given the interpolation points $z_1, z_2, \dots, z_n$, define the Blaschke product

$$B(z) := \prod_{k=1}^{n} \frac{1 - z_k^{-1} z}{z - \bar{z}_k^{-1}}.$$

As is well known, the subspace BH_2 is invariant under the shift z^{-1}. In order to set notation, we remark that BH_2 is the kernel of the evaluation operator $E : H_2 \to \mathbb{C}^n$ defined

$$E(f) = \begin{bmatrix} f(z_1) \\ \vdots \\ f(z_n) \end{bmatrix},$$

and, if $z_0 = \infty$, $z^{-1}BH_2$ is the kernel of

$$\hat{E}(f) = \begin{bmatrix} f(z_0) \\ f(z_1) \\ \vdots \\ f(z_n) \end{bmatrix}.$$

In this paper, the coinvariant subspaces $H(B) := H_2 \ominus BH_2$,

$$\mathcal{K} := H(z^{-1}B) = H_2 \ominus z^{-1}BH_2, \quad \text{and} \quad \mathcal{L} := z^{-1}H(B) \tag{2.1}$$

will play an important part. They are all finite-dimensional. In fact, given the polynomial

$$\tau(z) = \prod_{k=1}^{n} (z - \bar{z}_k^{-1}), \tag{2.2}$$

$\mathcal{K}$ consists of all rational functions

$$r(z) = \frac{\pi(z)}{\tau(z)}$$

for which the polynomial π is of degree at most n, and hence $\mathcal{K}$ is $(n+1)$-dimensional. The spaces $H(B)$ and $\mathcal{L}$ are n-dimensional subspaces of $\mathcal{K}$. In particular, $\mathcal{L}$ consists of those rational functions $r \in \mathcal{K}$ for which $r(\infty) = 0$. We shall also need the subset $\mathcal{R}$ of functions in $r \in \mathcal{K}$ with the property that $r - 1 \in \mathcal{L}$ and r is *minimum-phase* in the sense that the numerator polynomial π has all its zeros in the open unit disc.

In fact, to say that $r \in \mathcal{R}$ is to say that $\pi \in \mathcal{S}_n$, the n-dimensional space of (monic) Schur polynomials defined in section 1.

Finally, we shall need the subspace

$$\mathcal{Q} := \mathcal{K} + \mathcal{K}^*, \tag{2.3}$$

in terms of which we have the orthogonal decomposition

$$L_2 = zB^*\bar{H}^2 \oplus \mathcal{Q} \oplus z^{-1}BH^2 \tag{2.4}$$

and the subspace $\mathcal{D} \subset \mathcal{Q}$ defined as

$$\mathcal{D} := \{Q = q + q^* \mid q \in \mathcal{K}\}. \tag{2.5}$$

An important convex $(n+1)$-dimensional subset $\mathcal{D}_n^+$ of $\mathcal{D}$ consists of those $D \in \mathcal{D}$ which are positive real, i.e., satisfy the condition that $D(e^{i\theta}) > 0$ for all $\theta \in [-\pi, \pi]$. Also define the n-dimensional subset $\mathcal{Z}_n^+$ of $\mathcal{D}_n^+$ of all $D \in \mathcal{D}_n^+$ which are normalized so that $D(1) = 1$. It is immediately seen that $\mathcal{Z}_n^+$ is also convex.

The following lemma is a trivial modification of the unit circle version of Orlando's formula [15] (also see [5, Lemma 5.5]).

LEMMA 2.1. *Let $a \in \mathcal{R}$, and define $S(a) : \mathcal{K} \to \mathcal{D}$ to be the linear mapping defined by*

$$S(a)v = av^* + a^*v.$$

Then $\ker S(a) = 0$.

3. The interpolation foliation. Any rational function f of degree at most n has a representation

$$f(z) = \frac{b(z)}{a(z)}, \quad a, b \in \mathcal{K}. \tag{3.1}$$

If, in addition, f is strictly positive real, the zeros of the rational functions a and b in (3.1) must be located in the open unit disc. Therefore, if we also assume that $f(\infty) = 1$, it is no restriction to choose $a, b \in \mathcal{R}$. Consequently, we define $\mathcal{P}_n$ to be the space of all pairs (a, b) with $a, b \in \mathcal{R}$ such that f is strictly positive real. The following result was established in [6]. We note that $\mathcal{R}$ is diffeomorphic to Euclidean space $\mathbb{R}^n$ because $\mathcal{S}_n \simeq \mathbb{R}^n$ [4].

PROPOSITION 3.1. *The space $\mathcal{P}_n$ is a smooth, connected, real manifold of dimension $2n$.*

Next, denote by $\mathcal{W}_n^+$ the space of all $w \in \mathbb{C}^n$ with components $w_1, w_2, \dots, w_n \in \mathbb{C}$ satisfying the Pick condition $P > 0$ and forming a self-conjugate set.

PROPOSITION 3.2. *$\mathcal{W}_n^+$ is a smooth, connected, real manifold of dimension n.*

Proof. It is clear that $\mathcal{W}_n^+$ is a smooth manifold having real dimension n. From the form of the Pick matrix, one can also see that $\mathcal{W}_n^+$ is convex and hence connected. ∎

Let $\eta : \mathcal{P}_n \to \mathcal{W}_n^+$ be the restriction of the evaluation operator E to $\mathcal{P}_n$. Then, for each $w \in \mathcal{W}_n^+$,

$$\mathcal{P}_n(w) = \eta^{-1}(w) \tag{3.2}$$

is the space of all $f \in \mathcal{P}_n$ satisfying the interpolation condition (1.1) corresponding to w.

THEOREM 3.3. *The connected components of the sets* $\{\mathcal{P}_n(w) \mid w \in \mathcal{W}_+\}$ *form the leaves of an* n*-dimensional foliation of* $\mathcal{P}_n$.

Remark 3.4. Below, we shall prove that the submanifold $\mathcal{P}_n(w)$ is actually connected. This fact is a nontrivial consequence of the transversality lemma we shall prove in section 5.

To prove this theorem, we need to show that η is a submersion [26], i.e., that the Jacobian $\mathrm{Jac}(\eta)|_{(a,b)}$ is everywhere surjective. To this end, for any $u, v \in \mathcal{L}$, first form the directional derivative of f in the direction (u, v), i.e.,

$$D_{(u,v)}f = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[\frac{b + \epsilon v}{a + \epsilon u} - \frac{b}{a}\right] = \frac{av - bu}{a^2}.$$

Then, the directional derivative of η in the direction (u, v) is

$$D_{(u,v)}\eta = \begin{bmatrix} D_{(u,v)}f(z_1) \\ D_{(u,v)}f(z_2) \\ \vdots \\ D_{(u,v)}f(z_n) \end{bmatrix},$$

which is zero if and only if

$$av - bu = rB, \quad \text{where } r \in \mathcal{L}.$$

Consequently,

$$\ker \mathrm{Jac}(\eta)|_{(a,b)} = \{(u, v) \in \mathcal{L} \times \mathcal{L} \mid av - bu \in B\mathcal{L}\}.$$

LEMMA 3.5. *The tangent space of* $\mathcal{P}_n(w)$ *at* (a, b) *has dimension* n *and is given by*

$$T_{(a,b)}\mathcal{P}_n(w) = \{(u, v) \in \mathcal{L} \times \mathcal{L} \mid av - bu \in B\mathcal{L}\}.$$

Proof. The tangent vectors of $\mathcal{P}_n(w)$, as defined by (3.2), are precisely the vectors in the nullspace of the Jacobian of η at (a, b). For simplicity of notation, denote this space by V. To prove that $\dim V = n$, let $M_{(a,b)} : V \to B\mathcal{L}$ be the mapping $M_{(a,b)}(u, v) = av - bu$. Let n_0 be the number of common zeros of a and b. Then there are three proper rational functions, each taking the value 1 at infinity, namely θ of degree n_0 and $\tilde{a}$ and $\tilde{b}$ of degree $n - n_0$, such that $a = \theta\tilde{a}$ and $b = \theta\tilde{b}$ and $\tilde{a}$ and $\tilde{b}$ have no nontrivial common factors. Now, if $(u, v) \in \ker M_{(a,b)}$, we have $av - bu = 0$, and hence

$$\frac{v}{u} = \frac{b}{a} = \frac{\tilde{b}}{\tilde{a}},$$

so there must be a rational function ϑ of degree n_0 vanishing at infinity such that $u = \vartheta\tilde{a}$ and $v = \vartheta\tilde{b}$. Consequently, since ϑ is completely arbitrary,

$$\dim \ker M_{(a,b)} = n_0.$$

Moreover, for $(u, v) \in V$,

$$av - bu = \theta(\tilde{a}v - \tilde{b}u) = Br \quad \text{for some } r \in \mathcal{L}.$$

Therefore, since $\dim \mathcal{L} = n$ and θ is fixed of degree n_0,

$$\dim M_{(a,b)}(V) = n - n_0.$$

Therefore, by complementarity between rank and nullity,

$$\dim V = \dim M_{(a,b)}(V) + \dim \ker M_{(a,b)} = n,$$

as claimed. □

Proof of Theorem 3.3. Since the Jacobian $\mathrm{Jac}(\eta)$ is a linear map from the $2n$-dimensional tangent space of $\mathcal{P}_n$ to the n-dimensional tangent space of $\mathcal{W}_+$, complementarity of rank and nullity for $\ker \mathrm{Jac}(\eta)$ and the fact that $\dim \ker \mathrm{Jac}(\eta) = n$ (Lemma 3.5) imply that the range of $\mathrm{Jac}(\eta)$ has dimension n. Hence η is a submersion, proving the statement of the theorem [26, p. 2]. □

4. The filtering foliation. The following lemma is a trivial reformulation of results presented in [28, 29] concerning a fast filtering algorithm for Kalman filtering [27] (see also [5]).

LEMMA 4.1. *Given any $(a, b) \in \mathcal{P}_n$, consider the dynamical system*

$$a_{t+1}(z) = \frac{1}{2(1+\gamma_t)}[(1+z)a_t(z) + (1-z)b_t(z)], \quad a_0(z) = a(z),$$

$$b_{t+1}(z) = \frac{1}{2(1-\gamma_t)}[(1-z)a_t(z) + (1+z)b_t(z)], \quad b_0(z) = b(z), \tag{4.1}$$

where

$$\gamma_t = \left(z\frac{b_t(z) - a_t(z)}{2} \right)\Big|_{z=\infty}. \tag{4.2}$$

Then, for $t = 0, 1, 2, \ldots,$

$$(a_t, b_t) \in \mathcal{P}_n \tag{4.3}$$

and

$$r_t S(a_t) b_t = S(a)b, \quad \textit{where} \quad r_t = \prod_{k=0}^{t-1}(1-\gamma_k^2). \tag{4.4}$$

Moreover, as $t \to \infty$, $\gamma_t \to 0$, $r_t \to r_\infty$, and

$$(a_t, b_t) \to (\sigma, \sigma), \quad \textit{where } \sigma \in \mathcal{R}. \tag{4.5}$$

The parameters (4.2) are the Schur parameters (reflection coefficients) corresponding to the function f, and, consequently, $|\gamma_t| < 1, t = 0, 1, 2, \ldots,$ whenever f is strictly positive real. The connection to the Schur algorithm and Kalman filtering is explained in the appendix, where, for convenience, an independent proof of Lemma 4.1 is given. For initial conditions $(a, b) \notin \mathcal{P}_n$, the fast filtering algorithm exhibits much more complicated (and interesting) dynamical behavior, which is investigated in detail in [5]. Here, however, we are only interested in its behavior on the set $\mathcal{P}_n$.

In view of (3.1), we have

$$f(z) + f(z^{-1}) = \frac{S(a)b}{aa^*}, \tag{4.6}$$

and hence Lemma 4.1 implies that

$$f(z) + f(z^{-1}) = r_\infty \frac{\sigma(z)\sigma(z^{-1})}{a(z)a(z^{-1})}, \tag{4.7}$$

showing that the spectral factor in (1.2) is

$$v(z) = \sqrt{r_\infty}\frac{\sigma(z)}{a(z)}.$$

We note that $\mathcal{P}_n$ is invariant under the dynamical system (4.1); i.e., whenever the initial condition $(a, b) \in \mathcal{P}_n$, the iterates $(a_t, b_t) \in \mathcal{P}_n$. Moreover, the dynamical system (4.1) converges to the limit point (σ, σ) along the invariant manifold (4.4) [5]. Hence, the equilibrium set is

$$\mathcal{P}_n(\hat{w}), \quad \text{where } \hat{w} := \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{4.8}$$

Furthermore, (4.8) defines the center manifold for the dynamical system (4.1) evolving on $\mathcal{P}_n$, and no equilibrium in the center fold has a nontrivial unstable manifold. The invariant set (4.4) may also be written as

$$\rho_t S(a_t)b_t = S(\sigma)\sigma, \quad \text{where } \rho_t = \Pi_{k=t}^{\infty}(1 - \gamma_k^2)^{-1}.$$

Then, for each $\sigma \in \mathcal{R}$,

$$\mathcal{W}^s(\sigma) = \{(a, b) \in \mathcal{P}_n \mid \rho S(a)b = S(\sigma)\sigma \quad \text{for some } \rho \in \mathbb{R}_+\} \tag{4.9}$$

is the stable manifold in $\mathcal{P}_n$ through (σ, σ). In view of (4.6), $S(a)b$ is positive on the unit circle for all $(a, b) \in \mathcal{P}_n$, and hence we can eliminate the variable ρ in $\rho S(a)b = S(\sigma)\sigma$ by dividing by $[S(a)b](1)$. Therefore, we define the mapping $h : \mathcal{P}_n \to \mathcal{Z}_n^+$ as

$$h(a, b) = \frac{S(a)b}{[S(a)b](1)}, \tag{4.10}$$

where $\mathcal{Z}_n^+$ is the n-dimensional convex space defined in section 2. Then, the manifold (4.9) may also be written as

$$\mathcal{W}^s(\sigma) = h^{-1}(\kappa(\sigma)), \tag{4.11}$$

where

$$\kappa(\sigma) := \frac{S(\sigma)\sigma}{[S(\sigma)\sigma](1)} \in \mathcal{Z}_n^+.$$

Theorem 4.2. *The connected components of the sets* $\{\mathcal{W}^s(\sigma) \mid \sigma \in \mathcal{R}\}$ *form the leaves of an* n*-dimensional foliation of* $\mathcal{P}_n$.

Remark 4.3. The stable manifolds $\mathcal{W}^s(\sigma)$ are in fact connected. In this paper we shall sketch a proof of this fact based on the transversality lemma.

For the proof we need to show that the Jacobian $\text{Jac}(h)|_{(a,b)}$ has full rank. To this end, we compute the directional derivative of h in the direction (u, v) for arbitrary $u, v \in \mathcal{L}$ as

$$D_{(u,v)}h = \lim_{\epsilon \to 0} \frac{h(a + \epsilon u, b + \epsilon v) - h(a, b)}{\epsilon} = \frac{S(a)q + S(b)p}{[S(a)b](1)}, \tag{4.12}$$

where

$$p = u - \mu b, \quad q = v - \mu a, \quad \mu = \frac{1}{2}\left[\frac{S(a)v + S(b)u}{S(a)b}\right](1). \tag{4.13}$$

In this computation, we have also used the fact that $S(a)b = S(b)a$.

LEMMA 4.4. *The tangent space of* $\mathcal{W}^s(\sigma)$ *at* (a, b) *has dimension* n *and is given by*

$$T_{(a,b)}\mathcal{W}^s(\sigma) = \{(u, v) \in \mathcal{L} \times \mathcal{L} \mid S(a)q + S(b)p = 0\},$$

where $p, q \in \mathcal{K}$ *depend on* (u, v) *as in* (4.13).

Proof. The tangent space $T_{(a,b)}\mathcal{W}(\sigma)$ is precisely the kernel of the Jacobian $\text{Jac}(h)|_{(a,b)}$ of $h^{-1}(\kappa(\sigma))$, i.e., the space of (u, v) for which the directional derivative (4.12) is zero. This yields the expression of the lemma. Since the n algebraic equations contained in

$$h(a, b) = \kappa(\sigma)$$

are obtained by eliminating the variable ρ from the $n+1$ algebraic equations contained in

$$\rho S(a)b = S(\sigma)\sigma,$$

$T_{(a,b)}\mathcal{W}(\sigma)$ has the same dimension as $\ker \text{Jac}(F)|_{(\rho,a,b)}$, where $F : \mathbb{R}_+ \times \mathcal{P}_n \to \mathcal{D}_+$ is defined as

$$F(\rho, a, b) = \rho S(a)b.$$

Now, the directional derivative of F in the direction $(\lambda, u, v) \in \mathbb{R} \times \mathcal{L} \times \mathcal{L}$ is given by

$$D_{(\lambda,u,v)}F(\rho, a, b) = S(a)[\rho v + \lambda b] + S(b)u,$$

so $T_{(a,b)}\mathcal{W}(\sigma)$ has the same dimension as

$$W := \{(r, u) \in \mathcal{K} \times \mathcal{L} \mid S(a)r + S(b)u = 0\}.$$

Then, exactly the same proof as in [5, Lemma 5.11] shows that $\dim W = n$. ◻

We note in passing that Lemma 5.11 in [5] also shows that if we extend $\mathcal{W}(\sigma)$ outside the positive region $\mathcal{P}_n$ we encounter singularities where the rank of the Jacobian is deficient precisely at the points where a and b have common reciprocal zeros.

Proof of Theorem 4.2. Given Lemma 4.4 and hence that $\ker \text{Jac}(h) = n$, the rest of the proof is completely analogous to that of Theorem 3.3. ◻

5. The transversality lemma and the geometry of positive real functions. The following result is modeled after the corresponding result in [6, Lemma 4.5].

THEOREM 5.1 (transversality lemma). *Let $\mathcal{K}$ and $\mathcal{L}$ be the spaces defined in* (2.1)*. Then there are no nonzero p and q in $\mathcal{K}$ such that*

$$aq - bp \in B\mathcal{L} \tag{5.1}$$

and

$$S(a)q + S(b)p = 0. \tag{5.2}$$

Proof. We want to prove that if $p \in \mathcal{K}$ and $q \in \mathcal{K}$ satisfy (5.1) and (5.2), then $p = q = 0$. To this end, first note that (5.2) may be written as

$$h(z) + h(z^{-1}) = 0, \tag{5.3}$$

where

$$h(z) := a(z^{-1})q(z) + b(z)p(z^{-1}),$$

and that $h \in \mathcal{Q}$, where $\mathcal{Q}$ is defined by (2.3). Moreover, in view of (5.1),

$$g(z) := \frac{q(z)}{b(z)} - \frac{p(z)}{a(z)} = B(z)\frac{r(z)}{a(z)b(z)}, \quad \text{where } r \in \mathcal{L}.$$

Since $a(\infty) = b(\infty) = 1$ and $r(\infty) = 0$, the rational function $\frac{r}{ab}$ has a Laurent expansion

$$\frac{r(z)}{a(z)b(z)} = c_1 z^{-1} + c_2 z^{-2} + c_3 z^{-3} + \cdots$$

about infinity which holds on and outside the unit circle, and hence $g \in z^{-1}BH^2$. Therefore, $g^* \in zB^*\bar{H}^2$, and, consequently, by (2.4), both g and g^* are orthogonal to $\mathcal{Q}$ and hence to h. In particular,

$$\langle h, g - g^* \rangle = 0. \tag{5.4}$$

Now, a simple calculation shows that

$$g - g^* = \frac{h}{ba^*} - \frac{h^*}{ab^*} = \frac{S(a)b}{aa^*bb^*}h,$$

where (5.3) has been used to obtain the second equality, and therefore (5.4) yields

$$\left\langle h, \frac{S(a)b}{aa^*bb^*}h \right\rangle = 0.$$

However, since $(a, b) \in \mathcal{P}_n$ is positive real, $S(a)b$ is positive on the unit circle, and so is aa^*bb^*. Hence h must be zero, implying that $g = g^*$, i.e., g is constant and thus contained in $\mathcal{Q}$. But g is orthogonal to $\mathcal{Q}$, so g must be zero also. Then

$$q(z) = \frac{b(z)}{a(z)}p(z),$$

which, substituted into (5.2), yields

$$\left[\frac{b}{a}+\frac{b^*}{a^*}\right][ap^*+a^*p]=0.$$

Since $(a,b)\in\mathcal{P}_n$, the first factor is positive on the unit circle, and so

$$a(e^{i\theta})p(e^{-i\theta})+a(e^{-i\theta})p(e^{i\theta})=0$$

for all θ, and therefore, by the identity theorem,

$$S(a)p=0.$$

However, by Lemma 2.1, $S(a)$ has full rank, so p, and hence q, are zero. □

The transversality lemma has the following important consequence.

LEMMA 5.2. *Suppose that the point* $(a,b)\in\mathcal{P}_n$ *lies on the submanifolds* $\mathcal{P}_n(w)$ *and* $\mathcal{W}^s(\sigma)$*. Then*

$$T_{(a,b)}\mathcal{P}_n(w)\cap T_{(a,b)}\mathcal{W}^s(\sigma)=0.$$

Proof. Taking $(u,v)\in T_{(a,b)}\mathcal{P}_n(w)\cap T_{(a,b)}\mathcal{W}(\sigma)$, we see from Lemma 4.4 that (5.2) holds with p and q defined by (4.13). Moreover, since

$$aq-bp=av-\mu ab-bu+\mu ab=av-bu$$

for this choice of p and q, (5.1) also holds by Lemma 3.5. Hence, by Theorem 5.1, we must have $p=q=0$. But then evaluating at ∞, we obtain from (4.13) that $\mu=p(\infty)=q(\infty)$, which a fortiori must be zero, hence implying that $(u,v)=0$. □

It remains to show that the submanifolds $\mathcal{W}^s(\sigma)$ and $\mathcal{P}_n(w)$ are connected and thus constitute the leaves of the filtering foliation and the interpolation foliation, respectively.

COROLLARY 5.3. *The stable manifolds* $\{\mathcal{W}^s(\sigma)\mid\sigma\in\mathcal{R}\}$ *are diffeomorphic to* $\mathcal{W}_n^+$ *and thus connected. In particular, the stable manifolds of the fast filtering algorithm* (4.1) *decompose the space* $\mathcal{P}_n$ *into the leaves of a foliation.*

Proof. Consider again the mapping

$$\eta:\ \mathcal{P}_n\to\mathcal{W}_n^+$$

with $\eta^{-1}(w)=\mathcal{P}_n$. The restriction η_σ of η to $\mathcal{W}^s(\sigma)$ is a map of n-manifolds

$$\eta_\sigma:\ \mathcal{W}^s(\sigma)\to\mathcal{W}_n^+.$$

We claim that

$$\det\operatorname{Jac}(\eta_\sigma)|_{(a,b)}\neq 0$$

for all $(a,b)\in\mathcal{W}^s(\sigma)$. To prove this, we need to show that the directional derivative

$$D_{(u,v)}\eta_\sigma=\operatorname{Jac}(\eta_\sigma)\begin{bmatrix}u\\v\end{bmatrix}$$

is zero for any $(u,v)\in T_{(a,b)}\mathcal{W}^s(\sigma)$ only if $(u,v)=0$. However,

$$\ker\operatorname{Jac}(\eta_\sigma)\subset\ker\operatorname{Jac}(\eta)=T_{(a,b)}\mathcal{P}_n(w)$$

(Lemma 3.5), and hence this follows from Lemma 5.2. To proceed, we also need to show that η_σ is *proper*, i.e., that the inverse image $\eta_\sigma^{-1}(K)$ is compact for each compact set in the range space.

LEMMA 5.4. *The mapping η_σ is proper.*

Proof. Suppose $w_k \to w$ in $\mathcal{W}_n^+$ with $w_k = \eta_\sigma(a_k, b_k)$. Since $\mathcal{P}_n$ and hence $\mathcal{W}^s(\sigma)$ are relatively compact, the sequence (a_k, b_k) has a cluster point (a, b) in $\overline{\mathcal{W}^s(\sigma)} \subset \overline{\mathcal{P}}_n$, where a and b have all their zeros in the closed unit disc. We need to show that $(a, b) \in \mathcal{W}^s(\sigma)$. Now, suppose instead that $(a, b) \in \partial\mathcal{W}^s(\sigma)$, the boundary of $\mathcal{W}^s(\sigma)$. Then $(a, b) \in \partial\mathcal{P}_n$. In fact, if $(a, b) \in \mathcal{P}_n$, then, by Theorem 4.2, $(a, b) \in \mathcal{W}^s(\hat{\sigma})$ for some $\hat{\sigma} \in \mathcal{R}$ such that $\hat{\sigma} \neq \sigma$. But then

$$\frac{S(a)b}{[S(a)b](1)} = \kappa(\hat{\sigma}) \neq \kappa(\sigma),$$

which is impossible by continuity. Now, the boundary $\partial\mathcal{P}_n$ consists of those (a, b) for which either $S(a)b$ has a zero on the unit circle or $S(a)b$ is identically zero. Since the zeros of $S(a_k)b_k$ are fixed and therefore independent of k and lie inside the unit disc, $S(a)b$ cannot have zeros on the unit circle without being identically zero. Therefore, the function $f = b/a$ has the property $f + f^* = 0$, and hence f must have all poles and zeros on the unit circle. Then, it is well known [23] and easy to check that f takes the form

$$f(z) = \prod_{k=1}^{m} \frac{z - \mu_k}{z + \mu_k}, \quad |\mu_k| = 1, \quad m \leq n,$$

and, consequently,

$$F(z) = \frac{f(z) - 1}{f(z) + 1}$$

is a Blaschke product of degree m. Thus, modulo a trivial conformal equivalence, Corollary 2.3 in [16, p. 9] states that the rank of the corresponding Pick matrix equals m. Therefore, since $m < n + 1$, the Pick matrix is singular, and the corresponding value vector w must lie in the boundary of $\mathcal{W}_n^+$, contrary to assumption. Consequently, $(a, b) \notin \partial\mathcal{W}^s(\sigma)$, and thus $(a, b) \in \mathcal{W}^s(\sigma)$ as claimed. □

Since η_σ is proper and has a nowhere vanishing Jacobian, $\eta_\sigma^{-1}(w)$ is a finite set with cardinality δ, which is independent of w [30]. Therefore, $\eta_\sigma : \mathcal{W}^s(\sigma) \to \mathcal{W}_n^+$ is a δ-fold covering $\mathcal{W}_n^+$ [30]. Consider the point $\hat{w} \in \mathcal{W}_n^+$ defined by (4.8). For $(a, b) \in \mathcal{W}^s(\sigma)$, to say that $\eta_\sigma(a, b) = \hat{w}$ is to say that $a = b$. Since (a, a) is an equilibrium for the fast filtering algorithm of Lemma 4.1 and lies on the stable manifold of the equilibrium (σ, σ), we must have $(a, a) = (\sigma, \sigma)$, or $(a, b) = (\sigma, \sigma)$. Therefore, $\delta = 1$ and the map $\eta_\sigma : \mathcal{W}^s(\sigma) \to \mathcal{W}_n^+$ is a diffeomorphism. □

COROLLARY 5.5. *The submanifolds $\{\mathcal{P}_n(w) \mid w \in \mathcal{W}_+\}$ are connected. In particular, Nevanlinna–Pick interpolation defines a foliation of the space $\mathcal{P}_n$.*

Proof. Suppose $(a^{(1)}, b^{(1)})$ and $(a^{(2)}, b^{(2)})$ lie in $\mathcal{P}_n(w)$. Since $\mathcal{P}_n$ is connected, there is a continuous path $\gamma : [0, 1] \to \mathcal{P}_n$ with $\gamma(0) = (a^{(1)}, b^{(1)})$ and $\gamma(1) = (a^{(2)}, b^{(2)})$. Composing γ with η, we obtain a closed curve

$$\tilde{\gamma} = \eta \circ \gamma : \quad [0, 1] \to \mathcal{W}_+$$

with initial (and final) point w, i.e., $w = \eta(a^{(i)}, b^{(i)}),\ i = 1, 2$. Since $\mathcal{W}_n^+$ is convex, it is simply connected and therefore $\tilde{\gamma}$ can be contracted to the "constant curve" w

through a homotopy $\tilde{H}$ [22]; i.e.,

$$\tilde{H} : [0,1] \times [0,1] \to \mathcal{W}_+$$

jointly continuous and satisfying

$$\begin{aligned}\tilde{H}(r,0) &= \tilde{\gamma}(r),\\ \tilde{H}(r,1) &= w,\\ \tilde{H}(0,t) &= w,\\ \tilde{H}(1,t) &= w.\end{aligned}$$

We now construct a lifting of the homotopy $\tilde{H}$ to a homotopy H, with values in $\mathcal{P}_n$, covering $\tilde{H}$; i.e., $\eta \circ H = \tilde{H}$. Returning first to the curve γ, each point $\gamma(r)$ lies in a unique stable manifold, which we denote by $\mathcal{W}^s(\sigma(r))$. Since η_σ is a diffeomorphism for each σ, for each r fixed we can lift the curve $\tilde{H}_r$, defined as $\tilde{H}_r(t) = \tilde{H}(r,t)$ for $t \in [0,1]$, to a curve in $\mathcal{W}^s(\sigma(r))$ covering $\tilde{H}_r$ by defining $H_r(t) = \eta^{-1}_{\sigma(r)}(\tilde{H}_r(t))$. Note that H_r is a curve lying in $\mathcal{P}_n$ with initial point $\gamma(r)$. Now define $H : [0,1]\times[0,1] \to \mathcal{P}_n$ via

$$H(r,t) = H_r(t) = \eta^{-1}_{\sigma(r)}(\tilde{H}(r,t)).$$

We claim that H is jointly continuous. To see this, suppose $(r_k, t_k) \to (r,t)$ and set

$$(a_k, b_k) = H(r_k, t_k), \quad (a,b) = H(r,t).$$

We next note that $w_k := \tilde{H}(r_k, t_k) \to \tilde{H}(r,t) =: \tilde{w}$, $\gamma(r_k) \to \sigma(r)$, and consequently that $\sigma(r_k) \to \sigma(r)$, as $k \to \infty$. To prove that H is jointly continuous, it suffices to prove that every neighborhood of (a,b) contains the points (a_k, b_k) for all k sufficiently large. Now, $(a,b) \in \mathcal{P}_n(\tilde{w})$, and, using the implicit function theorem, we can choose neighborhoods $N(a,b)$ which are rectangular in the sense that a neighborhood of (a,b) in $\mathcal{P}_n(\tilde{w})$ serves as the vertical axis, while the horizontal axes consist of unique "slices" consisting of n-manifolds to which the restriction of η will be a diffeomorphism.

That is, the horizontal slices will be open subsets of $\mathcal{W}^s(\sigma)$. Since $\sigma(r_k) \to \sigma(r)$, and since the foliation defined by the stable manifolds of the fast filtering algorithm is itself defined by a submersion, such a neighborhood $N(a,b)$ will intersect $\mathcal{W}^s(\sigma(r_k))$ for all k sufficiently large. Now, (a_k, b_k) is the endpoint of the unique curve $H_{r_k}(t)$ for $t \in [0, t_k]$ in $\mathcal{W}^s(\sigma(r_k))$ covering $\tilde{H}_{r_k}$. Similarly, for any $\bar{t}$ satisfying $0 \le \bar{t} < t_k$, (a_k, b_k) is the endpoint of the unique curve in $\mathcal{W}^s(\sigma(r_k))$ covering $\tilde{H}_{r_k}$ on $[\bar{t}, t_k]$. Since $\eta(N(a,b))$ is open, there exists a $\bar{t}$, $0 \le \bar{t} \le t_k$, for all k sufficiently large so that

$$\tilde{H}_{r_k}[\bar{t}, t_k] \subset \eta(N(a,b)).$$

In particular, since $\eta_{\sigma(r_k)}$ is a (global) diffeomorphism, there exist unique lifts γ_k of these curves in $\mathcal{P}_n$ which lie in $\mathcal{W}^s(\sigma(r_k)) \cap N(a,b)$ and cover $\tilde{H}_{r_k}$ on $[\bar{t}, t_k]$ and have initial points $\eta^{-1}_{\sigma(r_k)}(\tilde{H}(r_k, \bar{t}))$. Since such liftings are unique, it follows that γ_k and H_{r_k} coincide on the subinterval $[\bar{t}, t_k]$, and therefore

$$H_{r_k}(t) \subset N(a,b) \quad \text{for } t \in [\bar{t}, t_k].$$

Consequently,

$$(a_k, b_k) = H_{r_k}(t_k) \in N(a,b)$$

for all k sufficiently large.

We have established that H is jointly continuous. The mapping H also satisfies

$$\begin{aligned} &H(r,0) = \gamma(r), \\ &H(r,1) \subset \mathcal{P}_n(w) \quad \text{for } 0 \le r \le 1, \\ &H(0,t) = \gamma(0) = (a^{(1)}, b^{(1)}), \\ &H(1,t) = \gamma(1) = (a^{(2)}, b^{(2)}). \end{aligned}$$

In particular, $H(\cdot,1)$ is a continuous path in $\mathcal{P}_n(w)$ joining $(a^{(1)}, b^{(1)})$ and $(a^{(2)}, b^{(2)})$. Since these points are arbitrary in $\mathcal{P}_n(w)$, this manifold is path connected and hence connected. □

Remark 5.6. The foliation by stable manifolds does, of course, define an integrable connection on the distribution tangent to the interpolation foliation, and it is tempting to believe that we can deduce a path-lifting result from the existence of this connection. At this point in the proofs we do not, however, know whether $\eta : \mathcal{P}_n \to \mathcal{W}_n^+$ is a fiber bundle or even a fibration. Moreover, η is definitely not proper, so one could at best expect a path lifting on a sufficiently small subinterval. For this reason, we directly established the homotopy lifting property for curves. We remark that it is possible to go further, showing that $\eta : \mathcal{P}_n \to \mathcal{W}_n^+$ is a fibration. In this case, one could then deduce path connectedness of the fiber from the fact that $\mathcal{W}_n^+$ is simply connected, using the long exact homotopy sequence of the fibration. Since we only needed the sequence for curves and connected components, we instead used a constructive approach to defining the boundary operator in the sequence.

6. Main results. Another consequence of the transversality lemma is that the leaves of the interpolation foliation intersect the leaves of the filtering foliation transversely; i.e., the two foliations are complementary. Actually, a much deeper relationship exists between these foliations, having several interesting corollaries.

THEOREM 6.1. *The filtering foliation and the interpolation foliation are complementary. Moreover, each leaf $\mathcal{P}_n(w)$ intersects each leaf $\mathcal{W}^s(\sigma)$ of the filtering foliation in one, and only one, point in $\mathcal{P}_n$.*

The first assertion follows immediately from Lemma 5.2 after it has been established that $\mathcal{P}_n(w)$ and $\mathcal{W}^s(\sigma)$ are connected and so are the leaves of respective foliation (Corollary 5.5 and Corollary 5.3). Consequently, there are two complementary foliations of $\mathcal{P}_n$, namely,

$$\mathcal{F}_1 : \quad \mathcal{P}_n = \bigcup_{w \in \mathcal{W}_+} \mathcal{P}_n(w), \tag{6.1}$$

indexed by the interpolation values $w \in \mathcal{W}_+$, and

$$\mathcal{F}_2 : \quad \mathcal{P}_n = \bigcup_{\sigma \in \mathcal{R}} \mathcal{W}(\sigma), \tag{6.2}$$

indexed by the equilibrium points (4.8) of the dynamical system (4.1), or, equivalently, by the spectral zeros in the form of a point in $\mathcal{S}_n$. This suggests that, given a set of admissible interpolation values and a set of stable spectral zeros, there is a unique solution of the Nevanlinna–Pick problem represented by the intersection between the corresponding leaves of the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$. This is precisely the second assertion of the theorem and is a consequence of Proposition 6.3 to be proven below.

To this end, first note that the fact that the filtering foliation and the interpolation foliation are complementary says that this uniqueness does occur to first order, in the following sense.

LEMMA 6.2. *Let* $h_w : \mathcal{P}_n(w) \to \mathcal{Z}_n^+$ *be the restriction of* h, *defined by* (4.10), *to* $\mathcal{P}_n(w)$. *Then, for each* $(a,b) \in \mathcal{P}_n(w)$, *the Jacobian matrix Jac*(h_w) *of* h_w *is nonsingular.*

Proof. To prove this, we need to show that the directional derivative

$$D_{(u,v)}h = \text{Jac}(h_w)\begin{bmatrix} u \\ v \end{bmatrix}$$

is zero for any $(u,v) \in T_{(a,b)}\mathcal{P}_n(w)$ only if $(u,v) = 0$. But this follows from Lemma 5.2 precisely as in the proof of Corollary 5.3. ◻

It is interesting to note that the duality between interpolation and filtering is reflected in a symmetry between the restricted mappings

$$\eta_\sigma : \mathcal{W}^s(\sigma) \to \mathcal{W}_n^+$$

and

$$h_w : \mathcal{P}_n(w) \to \mathcal{Z}_n^+.$$

Recall that η_σ is the restriction of $\eta : \mathcal{P}_n \to \mathcal{W}_n^+$ to $\mathcal{W}^s(\sigma) = h^{-1}(\kappa(\sigma))$, and h_w is the restriction of $h : \mathcal{P}_n \to \mathcal{Z}_n^+$ to $\mathcal{P}_n(w) = \eta^{-1}(w)$. Moreover, we have the following result.

PROPOSITION 6.3. *The mappings* η_σ *and* h_w *are diffeomorphisms. In particular, each choice of* σ *and* w *determines and is determined by precisely one element* $(a,b) \in \mathcal{P}_n$.

Proof. We have already shown in the proof of Corollary 5.3 that η_σ is a diffeomorphism. Concerning h_w, we first establish properness.

LEMMA 6.4. *The mapping* h_w *is proper.*

Proof. To show this, consider a sequence (κ_k) in $\mathcal{Z}_n^+$ with $\kappa_k = h_w(a_k, b_k)$ which converges to $\kappa \in \mathcal{Z}_n^+$ as $k \to \infty$, and prove that any cluster point (a,b) of (a_k, b_k) lies in $\mathcal{P}_n(w)$. Since $\mathcal{P}_n$ is relatively compact, $(a,b) \in \overline{\mathcal{P}_n(w)} \in \overline{\mathcal{P}_n}$. Now, suppose (a,b) is not in $\mathcal{P}_n(w)$ but in the boundary $\partial\mathcal{P}_n(w)$. Then $(a,b) \in \partial\mathcal{P}_n$ because if $(a,b) \in \mathcal{P}_n$, then, by Theorem 3.3, $(a,b) \in \mathcal{P}_n(\hat{w})$ for some $\hat{w} \neq w$, which contradicts continuity of $\eta(a,b)$. But if $(a,b) \in \partial\mathcal{P}_n$, then $S(a)b$ either has a zero on the unit circle or is identically zero, while $a, b \in \mathcal{R}$ of course remain nonzero. Therefore, if there is no zero at $z = 1$, $\kappa_k \to \partial\mathcal{Z}_n^+$ and if $[S(a)b](1) = 0$, then $\kappa_k \to \infty$, contradicting the assumption that $\kappa \in \mathcal{Z}_n^+$ in both cases. ◻

Since h_w is a proper map with nonvanishing Jacobian (Lemma 6.2), $h_w : \mathcal{P}_n(w) \to \mathcal{Z}_n^+$ is a δ-fold covering. Since $\mathcal{P}_n(w)$ is connected and $\mathcal{Z}_n^+$ is convex, and hence simply connected, the number, δ, of sheets must be one [30]. Therefore, h_w is a diffeomorphism. ◻

This concludes the proof of Theorem 6.1.

These geometric implications of the transversality lemma allow us to give an alternative geometric proof and amplification of the following result in [20], where, however, spectral zeros on the unit circle are also allowed, and in [12], using convex analysis to the minima of a functional defined using generalized entropy gains.

COROLLARY 6.5 (spectral zero assignability theorem for Nevanlinna–Pick interpolation). *Suppose* w *determines a positive definite Pick matrix. The positive real* *interpolants* (a, b) *in* $\mathcal{P}_n(w)$ *can be uniquely determined by a choice of stable spectral zeros.*

This corollary shows that the spectral zeros are design parameters which can be used, for example, in designing robust bounded real closed loop systems. This result also holds in the case that all the interpolation points $z_0 = z_1 = \cdots = z_n = \infty$, a situation of great interest in signal processing, spectral analysis, and stochastic systems [6, 7, 8, 9, 10, 11] (see also [17, 18], where the first proofs of existence were presented). In this case, the design parameter is intuitively very appealing, since it represents a choice of zeros for shaping filters which can shape white noise into a process matching a finite window of covariance data.

The following theorem, finally, is also a consequence of the transversality lemma. Here $\simeq$ denotes "diffeomorphic," and $\mathcal{S}_n$ is the space of Schur polynomials of degree n introduced in section 1.

THEOREM 6.6. *The space* $\mathcal{P}_n$ *is Euclidean of dimension* $2n$. *More specifically,*

$$\mathcal{P}_n \simeq \mathcal{W}_n^+ \times \mathcal{Z}_n^+ \simeq \mathcal{W}_n^+ \times \mathcal{S}_n,$$

where $\mathcal{W}_n^+$, $\mathcal{Z}_n^+$, *and* $\mathcal{S}_n$ *are all diffeomorphic to* $\mathbb{R}^n$.

For the proof we need the following "folk theorem," for which we have been unable to find a direct reference.

LEMMA 6.7. *An open, convex set* $D \in \mathbb{R}^n$ *is diffeomorphic to* $\mathbb{R}^n$.

It is well known and easy to see that an open convex set $D \in \mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$ [3, p. 2]. Except for $n = 4$, this implies that D is also diffeomorphic to $\mathbb{R}^n$, so the problem is only for $n = 4$ [31, p. 5]. However, convexity gives us much more, and it is simpler to give a direct proof. The following is an outline of a proof provided by O. Viro.[1]

Convexity allows us to construct a C^∞-function $\varphi : \mathbb{R}^n \to [0, 1]$, such that $\varphi(0) = 1$, $\varphi(x) > 0$ in D, and $\varphi(x) = 0$ outside D, which is monotonely nondecreasing along any ray $\{\lambda y \mid \|y\| = 1, \lambda \geq 0\}$. (We place the origin inside D.) In fact, for each supporting hyperplane H_k to D, one can construct a function φ_k which is zero in the half-space not containing D and which is monotonely nonincreasing along the normal direction from the origin, with the value one on a parallel hyperplane $\hat{H}_k$ and in the whole half-space beyond it. If D is a polytope, there are finitely many supporting hyperplanes H_k, and we may take $\varphi(x) = \prod_k \varphi_k(x)$. In general, we choose the hyperplanes H_k on a dense set of the boundary and let the distances d_k between each pair $\hat{H}_k$ and H_k be a sequence which tends to zero. Then, only a finite number of φ_k are different from one at any point in D, and hence the construction still works. The function $\psi : D \to \mathbb{R}^n$, with $\psi(0) = 1$ and $\psi(x) = x/\varphi(x)$ otherwise, is then a diffeomorphism. In fact, the monotonicity implies that the Jacobian does not vanish in D.

Proof of Theorem 6.6. Since $\mathcal{W}_n^+$ and $\mathcal{Z}_n^+$ are open and convex sets, they are diffeomorphic to $\mathbb{R}^n$ by Lemma 6.7. In the case of $\mathcal{Z}_n^+$, this can also be seen from the facts that $\mathcal{Z}_n^+ \simeq \mathcal{S}_n$ [6, p. 1849] and $\mathcal{S}_n \simeq \mathbb{R}^n$ [4]. Then, the rest follows from Proposition 6.3. □

Appendix. Fast Kalman filtering and the Schur algorithm.

Modulo a trivial reformulation, Lemma 4.1 is proven in [27, 28, 29] in the context of Kalman filtering, using the Szegö polynomials orthogonal on the unit circle and the Levinson recursion. Obviously, the recursion (4.1) is related to the Schur algorithm

[1] Similar ideas of a proof have also been suggested to us by H. Shapiro and M. Benedicks.

[1], as was established in, for example, [13]. In this appendix, we give a simple proof in this context.

Given any $(a,b) \in \mathcal{P}_n$ and the corresponding strictly positive real function $f = b/a$, define

$$\varphi(z) := \frac{f(z)-1}{f(z)+1} = \frac{b(z)-a(z)}{b(z)+a(z)} =: \frac{P(z)}{Q(z)},$$

which is a *Schur function* in the sense that it maps the exterior of the unit disc, $\mathbb{D}^c$, into the open unit disc $\mathbb{D}$. The Schur algorithm

$$\varphi_{t+1}(z) = z\frac{\varphi_t(z)-\varphi_t(\infty)}{1-\varphi_t(\infty)\varphi_t(z)}, \quad \varphi_0(z) = \varphi(z), \tag{A.1}$$

defines a sequence $\varphi_t(z)$, $t = 0, 1, 2, \dots$, of Schur functions, and the Schur parameters

$$\gamma_t = \varphi_{t+1}(\infty), \quad t = 0, 1, 2, \dots, \tag{A.2}$$

are less than one in modulus [1].

Proposition A.1. *For* $t = 0, 1, 2, \dots,$

$$\varphi_{t+1}(z) = \frac{zP_t(z)}{Q_t(z)}, \tag{A.3}$$

where P_t *and* Q_t *are polynomials satisfying the recursions*

$$\begin{cases} Q_{t+1}(z) = Q_t(z) - \gamma_t z P_t(z), & Q_0(z) = Q(z), \\ P_{t+1}(z) = zP_t(z) - \gamma_t Q_t(z), & P_0(z) = P(z). \end{cases} \tag{A.4}$$

Here Q_t *is of degree* n *having leading coefficient*

$$r_t = \prod_{k=0}^{t-1}(1-\gamma_k^2). \tag{A.5}$$

Proof. Clearly,

$$\varphi_1(z) = z\varphi_0(z) = \frac{zP(z)}{Q(z)},$$

so (A.3) holds for $t = 0$. Now let $t \geq 1$, and suppose that

$$\varphi_t(z) = \frac{zP_{t-1}(z)}{Q_{t-1}(z)}.$$

Then, the Schur algorithm (A.1) together with (A.2) yields

$$\varphi_{t+1}(z) = z\frac{zP_{t-1}(z) - \gamma_{t-1}Q_{t-1}(z)}{Q_{t-1}(z) - \gamma_{t-1}zP_{t-1}(z)} = \frac{zP_t(z)}{Q_t(z)},$$

and hence (A.3) holds for $t = 1, 2, \dots$ by induction. Moreover, (A.3) and (A.4) yield

$$\frac{Q_{t+1}(z)}{Q_t(z)} = 1 - \gamma_t\varphi_{t+1}(z),$$

which, evaluated at $z = \infty$, becomes $1 - \gamma_t^2$ by (A.2). But $|\gamma_t| < 1$, and hence $\deg Q_{t+1} = n$ whenever $\deg Q_t = n$. Since $\deg Q_0 = n$, it thus follows by induction that $\deg Q_t = n$ for $t = 0, 1, 2, \ldots$. More precisely, r_t, given by (A.5), is the leading coefficient of $Q_t(z)$. $\square$

The recursion (A.4) is precisely the fast algorithm for Kalman filtering [27] in the formulation of [28]. In fact, suppose $\{y_0, y_1, y_2, \ldots\}$ is a stationary stochastic process with spectral density

$$\Phi(z) = \frac{1}{2}[f(z) + f(z^{-1})],$$

where $f = b/a$ has a minimal realization

$$f(z) = 1 + 2h(zI - F)^{-1}g.$$

Then the linear least squares estimate $\hat{y}_t$ of y_t given $y_0, y_1, \ldots, y_{t-1}$ is generated by the Kalman filter

$$\begin{cases} \hat{x}_{t+1} = F\hat{x}_t + k_t(y_t - \hat{y}_t), \\ \hat{y}_t = h\hat{x}_t, \end{cases}$$

where k_t is determined from $Q_t(z)$ in the following way: If (F, g, h) is chosen so that $h = (1, 0, \ldots, 0)$, F has characteristic polynomial $\chi_F(z) = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_n$, and

$$Q_t(z) = r_t[z^n + q_1(t)z^{n-1} + \cdots + q_n(t)],$$

then the gain k_t is given by

$$k_t = q(t) - \alpha.$$

Moreover, Q_t/r_t is the characteristic polynomial of the feedback matrix

$$F - k_t h, \tag{A.6}$$

which hence is stable.

In the same way as in [28], a direct calculation using (A.4) yields

$$\begin{aligned} &Q_{t+1}(z)Q_{t+1}(z^{-1}) - P_{t+1}(z)P_{t+1}(z^{-1}) \\ &\qquad = (1 - \gamma_t^2)[Q_t(z)Q_t(z^{-1}) - P_t(z)P_t(z^{-1})] \end{aligned} \tag{A.7}$$

for $t = 0, 1, 2, \ldots$.

Now set

$$a_t(z) := \frac{Q_t(z) - P_t(z)}{2r_t\tau(z)}, \quad b_t(z) := \frac{Q_t(z) + P_t(z)}{2r_t\tau(z)}. \tag{A.8}$$

We first note that $a_0 = a$ and $b_0 = b$. Moreover, since $|z^{-1}| < 1$ in $\mathbb{D}^c$, $z^{-1}\varphi_{t+1}(z)$ is a Schur function, and, consequently,

$$f_t(z) := \frac{b_t(z)}{a_t(z)} = \frac{1 + z^{-1}\varphi_{t+1}(z)}{1 - z^{-1}\varphi_{t+1}(z)} \tag{A.9}$$

is strictly positive real for $t = 0, 1, 2, \ldots$ so that $(a_t, b_t) \in \mathcal{P}_n$. This verifies (4.3).

From (A.4) we readily obtain the recursion (4.1), Moreover, in view of (A.9),

$$\varphi_{t+1}(z) = z\frac{b_t(z) - a_t(z)}{b_t(z) + a_t(z)},$$

and hence (4.2) follows from (A.2). We also note that (4.4) is equivalent to (A.7). It now only remains to verify (4.5). To this end, we recall that, for rational positive real functions, the Schur parameters form an ℓ_2 sequence [18, p. 447], and hence $\gamma_t \to 0$ as $t \to \infty$. Consequently, r_t tends to some limit r_∞ as $t \to \infty$, and it follows from (A.4) that $Q_t(z)$ tends to a constant polynomial Q_∞, which is the characteristic polynomial of the steady-state feedback matrix (A.6) defined by the steady-state Kalman gain. Hence $r_\infty^{-1} Q_\infty \in \mathcal{S}_n$. It also follows that $P_t(z)$ tends to zero. Therefore, by (A.8), a_t and b_t tend to σ as $t \to \infty$, where

$$\sigma(z) = \frac{Q_\infty(z)}{r_\infty \tau(z)}.$$

Clearly, $\sigma \in \mathcal{R}$, as claimed.

This proves the claims made in Lemma 4.1. In [5], a much more refined analysis of the global phase portrait of the fast filtering algorithm is given, with the explicit derivation of the global stable manifolds which we employ in section 6. This analysis has many other consequences. For example, it can be shown [6] that for a rational strictly positive real function the sequence of Schur parameters decays to zero geometrically, generalizing previous results in the literature on conditional and absolute summability of the corresponding series of Schur parameters [21, 2, 18].

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