The Covariance Extension Equation Revisited

Christopher I. Byrnes, Giovanna Fanizza and Anders Lindquist

***Abstract*— In this paper we study the steady state form of a discrete-time matrix Riccati-type equation, connected to the rational covariance extension problem and to the partial stochastic realization problem. This equation, however, is non-standard in that it lacks the usual kind of definiteness properties which underlie the solvability of the standard Riccati equation. Nonetheless, we prove the existence and uniqueness of a positive semidefinite solution. We also show that this equation has the proper geometric attributes to be solvable by homotopy continuation methods, which we illustrate in several examples.**

I. Introduction

Let

$$c = (c_0, c_1, \dots, c_n) \tag{1}$$

a sequence (for simplicity, taken to be real) that is *positive* in the sense that

$$T_n = \begin{bmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_0 & \cdots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & \cdots & c_0 \end{bmatrix} > 0.$$

Given a positive sequence (1), the rational covariance extension problem – or the covariance extension problem with degree constraint – amounts to finding a pair (a, b) of Schur polynomials[1]

$$a(z) = z^n + a_1 z^{n-1} + \cdots + a_n \tag{2a}$$

$$b(z) = z^n + b_1 z^{n-1} + \cdots + b_n \tag{2b}$$

satisfying the interpolation condition

$$\frac{b(z)}{a(z)} = \frac{1}{2}c_0 + c_1 z^{-1} + \cdots + c_n z^{-n} + O(z^{-n-1}) \tag{3}$$

and the positivity condition

$$\frac{1}{2}\left[a(z)b(z^{-1}) + b(z)a(z^{-1})\right] > 0 \quad \text{on } \mathbb{T}, \tag{4}$$

$\mathbb{T}$ being the unit circle. Then there is a Schur polynomial

$$\sigma(z) = z^n + \sigma_1 z^{n-1} + \cdots + \sigma_n \tag{5}$$

such that

$$\frac{1}{2}\left[a(z)b(z^{-1}) + b(z)a(z^{-1})\right] = \rho^2 \sigma(z)\sigma(z^{-1}) \tag{6}$$

for some positive normalizing coefficient ρ, and

$$\text{Re}\left\{\frac{b(e^{i\theta})}{a(e^{i\theta})}\right\} = \left|\rho \frac{\sigma(e^{i\theta})}{a(e^{i\theta})}\right|^2. \tag{7}$$

Georgiou [12], [13] raised the question whether there exists a solution for each choice of σ and answered this question in the affirmative. He also conjectured that this assignment is unique. This conjecture was proven in [6] in a more general context of well-posedness.

The question of actually computing the unique solution to the covariance extension problem with degree constraint was first addressed in a constructive way in [7] (also, see [8]) in the context of convex optimization.

This optimization approach completely superseded a first attempt, proposed in [5], to set up a paradigm for computation. In fact, in [5] we introduced a nonstandard matrix Riccati equation – the Covariance Extension Equation (CEE) – the positive semidefinite solutions of which parameterize the solution set of the rational covariance extension problem. In [1] we provided an algorithm for solving this equation based on homotopy continuation. The purpose of this paper is to revisit this topic along the lines of [1]. Although the CEE approach does not seem to offer any computational advantage to, e.g., [11], it does provide some additional insights into such issues as positive degree [5], [16], [10] and model reduction, since the rank of the solution matrix coincides with the degree of the interpolant.

II. The covariance extension equation

For simplicity, we normalize by taking $c_0 = 1$. Motivated by the rational covariance extension problem, we form the following n vectors

$$\sigma = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n \end{bmatrix}, \quad h = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{8a}$$

and $n \times n$ matrix

$$\Gamma = \begin{bmatrix} -\sigma_1 & 1 & 0 & \cdots & 0 \\ -\sigma_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sigma_{n-1} & 0 & 0 & \cdots & 1 \\ -\sigma_n & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{8b}$$

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[1] A *Schur polynomial* is a (monic) polynomial with all its roots in the open unit disc.

Defining $u_1, u_2, \ldots, u_n$ via

$$\frac{z^n}{z^n + c_1 z^{n-1} + \cdots + c_n} = 1 - u_1 z^{-1} - u_2 z^{-2} - \ldots \quad (9)$$

we also form

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad U = \begin{bmatrix} 0 & & & & \\ u_1 & 0 & & & \\ u_2 & u_1 & & & \\ \vdots & \vdots & \ddots & & \\ u_{n-1} & u_{n-2} & \cdots & u_1 & 0 \end{bmatrix}. \quad (10)$$

We shall also need the function $g : \mathbb{R}^{n\times n} \to \mathbb{R}^n$ defined by

$$g(P) = u + U\sigma + U\Gamma P h. \quad (11)$$

From these quantities, in [5], we formed the Riccati-like matrix equation

$$P = \Gamma(P - Phh'P)\Gamma' + g(P)g(P)', \quad (12)$$

which we sought to solve in the space of positive semidefinite matrices satisfying the additional constraint

$$h'Ph < 1, \quad (13)$$

where $'$ denotes transposition. We refer to this equation as the covariance extension equation (CEE).

To this end, define the semialgebraic sets

$$X = \{(c, \sigma) \mid T_n > 0,\ \sigma(z) \text{ is a Schur polynomial}\}$$

and

$$Y = \{P \in \mathbb{R}^{n\times n} \mid P \geq 0,\ h'Ph < 1\}.$$

On $X \times Y$ we define the rational map

$$F(c, \sigma, P) = P - \Gamma(P - Phh'P)\Gamma' - g(P)g(P)'$$

Of course its zero locus

$$Z = F^{-1}(0) \subset X \times Y$$

is the solution set to the covariance extension equations. We are interested in the projection map restricted to Z

$$\pi_X(c, \sigma, P) = (c, \sigma).$$

For example, to say that π_X is surjective is to say that there is always a solution to CEE, and to say that π_X is injective is to say that solutions are unique. One of the main results of [1] is the following, which, in particular, implies that CEE has a unique solution $P \in Y$ for each $(c, \sigma) \in X$ [5, Theorem 2.1].

Theorem 1: The solution set Z is a smooth semialgebraic manifold of dimention $2n$. Moreover, π_X is a diffeomorphism between Z and X.

In particular the map π_X is smooth with no branch points and every smooth curve in X lifts to a curve in Z. These observations imply that the homotopy continuation method will apply to solving the covariance extension equation [2].

The proof of Theorem 1 is based on the following result, found in [5]. Here a and b are the n-vectors $a := (a_1, a_2, \ldots, a_n)$ and $b := (b_1, b_2, \ldots, b_n)$ defined via (2).

Theorem 2: There is a one-to-one correspondence between symmetric solutions P of the covariance extension equation (12) such that $h'Ph < 1$ and pairs of monic polynomials (2a)-(2b) satisfying the interpolation condition (3) and the positivity condition (4). Under this correspondence

$$a = (I - U)(\Gamma P h + \sigma) - u, \quad (14a)$$

$$b = (I + U)(\Gamma P h + \sigma) + u, \quad (14b)$$

$$\rho = (1 - h'Ph)^{1/2}, \quad (14c)$$

and P is the unique solution of the Lyapunov equation

$$P = JPJ' - \frac{1}{2}(ab' + ba') + \rho^2\sigma\sigma', \quad (15)$$

where

$$J = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad (16)$$

is the upward shift matrix. Moreover the following conditions are equivalent

1) $P \geq 0$
2) $a(z)$ is a Schur polynomial
3) $b(z)$ is a Schur polynomial

and, if they are fulfilled,

$$\deg v(z) = \text{rank } P. \quad (17)$$

We can now prove Theorem 1. Let $\mathcal{P}_n$ be the space of pairs (a, b) whose quotient is positive real. Clearly, the mapping

$$f : \mathcal{P}_n \to X,$$

sending (a, b) to the corresponding (c, σ), is smooth. Our main result in [6] asserts that f is actually a diffeomorphism. In particular, for each positive sequence (1) and each monic Schur polynomial (5), there is a unique pair of polynomials, (a, b), satisfying (3) and (4), and consequently (a, b) solves the rational covariance extension problem corresponding to (c, σ). Moreover, by Theorem 2, there is a unique corresponding solution to the covariance extension equation, which is positive semi-definite.

Since J is nilpotent, the Lyapunov equation (15) has a unique solution, P, for each right hand side of equation (15). Moreover, the right hand side is a smooth function on X and, using elementary methods from Lyapunov theory, we conclude that P is also smooth as a function on X. As the graph in $X \times Y$ of a smooth mapping defined on X, Z is a smooth manifold of dimension $2n = \dim X$. Moreover, this mapping has the smooth mapping π_X as its inverse. Therefore, π_X is a diffeomorphism.

Remark 3: Our proof, together with the results in [6], shows more. Namely, that Z is an analytic manifold and that π_X is an analytic diffeomorphism with an analytic inverse.

III. Rational covariance extension and the CEE

In [5] we showed that, for any $(c,\sigma) \in X$, CEE has a unique solution $P \in Y$ and that the unique solution corresponding to σ to the rational covariance extension problem is given by

$$a = (I-U)(\Gamma P h + \sigma) - u, \tag{18a}$$

$$b = (I+U)(\Gamma P h + \sigma) + u. \tag{18b}$$

Clearly the interpolation condition (3) can be written

$$b = 2c + (2C_n - I)a, \tag{19}$$

where

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \quad C_n = \begin{bmatrix} 1 & & & & \\ c_1 & 1 & & & \\ c_2 & c_1 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots & 1 \end{bmatrix}.$$

Using the fact that $C_n u = c$ and $C_n(I-U) = I$, it was shown in [5] that (19) can be written

$$a = \frac{1}{2}(I-U)(a+b) - u. \tag{20}$$

For a fixed $(c,\sigma) \in X$, let $H : Y \to \mathbb{R}^{n\times n}$ be the map sending P to $F(c,\sigma,P)$, and let

$$dH(P;Q) := \lim_{t\to 0} \frac{H(P+tQ) - H(P)}{t}$$

be the derivative in the direction $Q = Q'$. A key property needed in the homotopy continuation solution of the CEE is the fact that this derivative is full rank.

Proposition 4: Given $(c,\sigma) \in X$, let $P \in Y$ be the corresponding solution of CEE. Then, if $dH(P;Q) = 0$, $Q = 0$.

Proof: Suppose that $dH(P;Q) = 0$ for some Q. Then

$$H(P) + \lambda dH(P;Q) = 0$$

for any $\lambda \in \mathbb{R}$. Since

$$\begin{aligned} dH(P;Q) \;=\; & Q - \Gamma Q \Gamma' + \Gamma P h h' Q \Gamma' + \Gamma Q h h' P \Gamma' \\ & - g(P) h' Q \Gamma' U' - U \Gamma Q h g(P)', \end{aligned} \tag{21}$$

this can be written

$$H(P_\lambda) = \lambda^2 R(Q), \tag{22}$$

where $P_\lambda := P + \lambda Q$ and

$$R(Q) := 2\Gamma Q h h' Q \Gamma' - 2U\Gamma Q h h' Q \Gamma' U'.$$

Proceeding as in the proof of Lemma 4.6 in [5], (22) can be written

$$P_\lambda = J P_\lambda J' - \frac{1}{2}(a_\lambda b_\lambda' + b_\lambda a_\lambda') + \rho_\lambda^2 \sigma\sigma' - \lambda^2 R(Q), \tag{23}$$

where

$$a_\lambda = (I-U)(\Gamma P_\lambda h + \sigma) - u, \tag{24a}$$

$$b_\lambda = (I+U)(\Gamma P_\lambda h + \sigma) + u, \tag{24b}$$

$$\rho_\lambda = (1 - h' P_\lambda h)^{1/2}. \tag{24c}$$

Observe that

$$a_\lambda = \frac{1}{2}(I-U)(a_\lambda + b_\lambda) - u, \tag{25}$$

and hence (a_λ, b_λ) satisfies the interpolation condition (20), or, equivalently, (3), for all $\lambda \in \mathbb{R}$.

Multipying (23) by $z^{j-i} = z^{n-i} z^{-(n-j)}$ and summing over all $i,j = 1,2,\dots,n$, we obtain

$$\begin{aligned} & \frac{1}{2}\left[a_\lambda(z) b_\lambda(z^{-1}) + b_\lambda(z) a_\lambda(z^{-1})\right] \\ & = \rho_\lambda^2 \sigma(z)\sigma(z^{-1}) - \lambda^2 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} R_{ij}(Q) z^{j-i} \end{aligned}$$

again along the calculations of the proof of Lemma 4.6 in [5]. Since $\sigma(z)\sigma(z^{-1}) > 0$ on $\mathbb{T}$,

$$\rho_\lambda^2 \sigma(z)\sigma(z^{-1}) - \lambda^2 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} R_{ij}(Q) z^{j-i} > 0 \quad \text{on } \mathbb{T}$$

for $|\lambda|$ sufficiently small. Then there is a Schur polynomial σ_λ and a positive constant $\hat{\rho}_\lambda$ such that

$$\hat{\rho}_\lambda^2 \sigma_\lambda(z)\sigma_\lambda(z^{-1}) = \rho_\lambda^2 \sigma(z)\sigma(z^{-1}) - \lambda^2 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} R_{ij}(Q) z^{j-i}.$$

Therefore,

$$\frac{1}{2}\left[a_\lambda(z) b_\lambda(z^{-1}) + b_\lambda(z) a_\lambda(z^{-1})\right] = \hat{\rho}_\lambda^2 \sigma_\lambda(z)\sigma_\lambda(z^{-1}) \tag{26}$$

for $|\lambda|$ sufficiently small.

Now recall that $a_0 = a$ and $b_0 = b$ are Schur polynomials and that the Schur region is open in $\mathbb{R}^n$. Hence there is an $\varepsilon > 0$ such that $a_\varepsilon(z), a_{-\varepsilon}(z), b_\varepsilon(z)$ and $b_{-\varepsilon}(z)$ are also Schur polynomials and (26) holds for $\lambda = \pm\varepsilon$.

Consequently, $(a_\varepsilon, b_\varepsilon)$ and $(a_{-\varepsilon}, b_{-\varepsilon})$ both satisfy the interpolation condition (3) and the positivity condition (6) corresponding to the same $\sigma := \sigma_\varepsilon = \sigma_{-\varepsilon}$. Therefore, since the solution to the rational covariance extension problem corresponding to σ is unique, we must have $a_\varepsilon = a_{-\varepsilon}$ and $b_\varepsilon = b_{-\varepsilon}$, and hence in view of (23), $P_\varepsilon = P_{-\varepsilon}$; i.e, $Q = 0$, as claimed. ∎

IV. Reformulation of the Covariance Extension Equation

Solving the covariance extension equation (12) amounts to solving $\frac{1}{2}n(n-1)$ nonlinear scalar equations, which number grows rapidly with increasing n. As in the theory of fast filtering algorithms [14], [15], we may replace these equations by a system of only n equations. In fact, setting

$$p = Ph \tag{27}$$

the covariance extension equation can be written

$$P - \Gamma P \Gamma' = -\Gamma p p' \Gamma' + (u + U\sigma + U\Gamma p)(u + U\sigma + U\Gamma p)' \tag{28}$$

If we could first determine p, P could be obtained from (28), regarded as a Lyapunov equation. We proceed to doing precisely this.

It follows from Theorem 2 that (28) may also be written

$$P = JPJ' - \frac{1}{2}(ab' + ba') + \rho^2\sigma\sigma', \tag{29}$$

with a, b and ρ given by (14). Multiplying (29) by $z^{j-i} = z^{n-i}z^{-(n-j)}$ and summing over all $i, j = 1, 2, \dots, n$, we obtain precisely (4), which in matrix form becomes

$$S(a)\begin{bmatrix}1\\ b\end{bmatrix} = 2\rho^2\begin{bmatrix}d\\ \sigma_n\end{bmatrix} \tag{30}$$

or, symmetrically,

$$S(b)\begin{bmatrix}1\\ a\end{bmatrix} = 2\rho^2\begin{bmatrix}d\\ \sigma_n\end{bmatrix}, \tag{31}$$

where $a \mapsto S(a)$ is the matrix function

$$\begin{bmatrix}1 & \dots & a_{n-1} & a_n\\ a_1 & \dots & a_n & \\ \vdots & \iddots & & \\ a_n & & & \end{bmatrix} + \begin{bmatrix} & a_1 & \dots & a_n\\ & 1 & \dots & a_{n-1}\\ & & \ddots & \vdots\\ & & & 1\end{bmatrix}$$

and

$$d = \begin{bmatrix}1 + \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2\\ \sigma_1 + \sigma_1\sigma_2 + \sigma_{n-1}\sigma_n\\ \sigma_2 + \sigma_1\sigma_3 + \sigma_{n-2}\sigma_n\\ \vdots\\ \sigma_{n-1} + \sigma_1\sigma_n\end{bmatrix}. \tag{32}$$

Inserting (14) and (27) in (30) yields

$$S(a(p))\begin{bmatrix}1\\ b(p)\end{bmatrix} = 2(1 - h'p)\begin{bmatrix}d\\ \sigma_n\end{bmatrix}, \tag{33}$$

where

$$a(p) = (I - U)(\Gamma p + \sigma) - u, \tag{34a}$$

$$b(p) = (I + U)(\Gamma p + \sigma) + u \tag{34b}$$

are functions of p. More precisely, (33) are $n + 1$ equations in the n unknown p. However, from (14) we have

$$\frac{1}{2}(a_n + b_n) = \rho^2\sigma_n,$$

which is precisely the last equation in (30). Hence (33) is redundant and can be deleted to yield

$$ES(a(p))\begin{bmatrix}1\\ b(p)\end{bmatrix} = 2(1 - h'p)d, \tag{35}$$

where E is the $n \times (n + 1)$ matrix

$$E = \begin{bmatrix}I_n & 0\end{bmatrix}. \tag{36}$$

These n equations in n unkowns $p_1, p_2, \dots, p_n$ clearly has a unique solution $\hat{p}$, for CEE has one.

V. Homotopy continuation

Suppose that $(c, \sigma) \in X$. To solve the corresponding covariance extension equation

$$P = \Gamma(P - Phh'P)\Gamma' + g(P)g(P)' \tag{37}$$

for its unique solution $\hat{P}$, we first observe that the solution is particularly simple if $c = c_0 = 0$. Then $u = 0$, $U = 0$ and (37) reduces to

$$P = \Gamma(P - Phh'P)\Gamma' \tag{38}$$

having the unique solution $P = 0$ in Y. Consider the deformation

$$c(\nu) = \nu c, \quad \nu \in [0, 1].$$

Clearly, $(c(\nu), \sigma) \in X$, and consequently the equation

$$H(P, \nu) := P - \Gamma(P - Phh'P)\Gamma' - g(P, \nu)g(P, \nu)' = 0, \tag{39}$$

where

$$g(P, \nu) = u(\nu) + U(\nu)\sigma + U(\nu)\Gamma Ph$$

with

$$u(\nu) = \begin{bmatrix}1 & & & & \\ \nu c_1 & 1 & & & \\ \nu c_2 & \nu c_1 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \nu c_{n-1} & \nu c_{n-2} & \nu c_{n-3} & \dots & 1\end{bmatrix}^{-1}\begin{bmatrix}\nu c_1\\ \nu c_2\\ \vdots\\ \nu c_n\end{bmatrix}$$

and

$$U(\nu) = \begin{bmatrix}0 & & & & \\ u_1(\nu) & 0 & & & \\ u_2(\nu) & u_1(\nu) & & & \\ \vdots & \vdots & \ddots & & \\ u_{n-1}(\nu) & u_{n-2}(\nu) & \cdots & u_1(\nu) & 0\end{bmatrix},$$

has a unique solution $\hat{P}(\nu)$ in Y.

The function $H : Y \times [0, 1] \to \mathbb{R}^{n\times n}$ is a homotopy between (37) and (38). In view of Theorem 1, the trajectory $\{\hat{P}(\nu)\}_{\nu=0}^{1}$ is continuously differentiable and has no turning points or bifurcations. Consequently, homotopy continuation can be used to obtain a computational procedure. However, the corresponding ODE will be of dimension $O(n^2)$. Therefore, it is better to work with the reduced equation (35), which yields an ODE of order n.

To this end, setting

$$V := \{p \in \mathbb{R}^n \mid p = Ph, P \in Y\},$$

consider instead the homotopy $G : V \times [0, 1] \to \mathbb{R}^n$ defined by

$$G(p, \nu) := ES(a(p))\begin{bmatrix}1\\ b(p)\end{bmatrix} - 2(1 - h'p)d,$$

where $a(p)$ and $b(p)$ are given by (34). *A fortiori* the corresponding trajectory $\{\hat{p}(\nu)\}_{\nu=0}^{1}$ is continuously differentiable and has no turning points or bifurcations. Differentiating

$$G(p, \nu) = 0$$

with respect to ν yields

$$ES(a)\begin{bmatrix}0\\ \dot{b}\end{bmatrix} + ES(b)\begin{bmatrix}0\\ \dot{a}\end{bmatrix} + 2h'\dot{p}d = 0,$$

where dot denotes derivative and

$$\dot{a} = (I-U)\Gamma\dot{p} - \dot{U}(\Gamma p+\sigma) - \dot{u}, \tag{40a}$$

$$\dot{b} = (I+U)\Gamma\dot{p} + \dot{U}(\Gamma p+\sigma) + \dot{u}, \tag{40b}$$

or, which is the same,

$$ES\left(\frac{a+b}{2}\right)\begin{bmatrix}0\\ \Gamma\dot{p}\end{bmatrix} - ES\left(\frac{b-a}{2}\right)\begin{bmatrix}0\\ U\Gamma\dot{p}\end{bmatrix} + dh'\dot{p} = \\ = ES\left(\frac{b-a}{2}\right)\begin{bmatrix}0\\ \dot{U}\Gamma p + \dot{U}\sigma + \dot{u}\end{bmatrix} .$$

In view of (34), this may be written

$$\left[\hat{S}(\Gamma p+\sigma) - \hat{S}(U\Gamma p + U\sigma + u) + dh'\right]\dot{p} \\ = \hat{S}(U\Gamma p + U\sigma + u)(\dot{U}\Gamma p + \dot{U}\sigma + \dot{u}),$$

where $\hat{S}(a)$ is the $n\times n$ matrix obtained by deleting the first column and the last row in $S(a)$. Hence we have proven the following theorem.

Theorem 5: The differential equation

$$\dot{p} = \left[\hat{S}(\Gamma p+\sigma) - \hat{S}(U(\nu)\Gamma p + U(\nu)\sigma + u(\nu)) + dh'\right]^{-1}\times \\ \hat{S}(U(\nu)\Gamma p + U(\nu)\sigma + u(\nu))(\dot{U}(\nu)\Gamma p + \dot{U}(\nu)\sigma + \dot{u}(\nu)),$$
$$p(0) = 0$$

has a unique solution $\{\hat{p}(\nu);\, 0\le\nu\le 1\}$. Moreover, the unique solution of the Lyapunov equation

$$P - \Gamma P\Gamma' = \\ -\Gamma\hat{p}(1)\hat{p}(1)'\Gamma' + (u + U\sigma + U\Gamma\hat{p}(1))(u+U\sigma+U\Gamma\hat{p}(1))',$$

where $U = U(1)$ and $u = u(1)$, is also the unique solution of the covariance extension equation (12).

The differential equation can be solved by methods akin to those in [4].

VI. Simulations

We illustrate the method described above by two examples, in which we use covariance data generated in the following way. Pass white noise through a given stable filter

$$\text{white noise} \xrightarrow{w} \boxed{\ w(z)\ } \xrightarrow{y}$$

with a rational transfer function

$$w(z) = \frac{\hat{\sigma}(z)}{\hat{a}(z)}$$

of degree $\hat{n}$, where $\hat{\sigma}(z)$ is a (monic) Schur polynomial. This generates a time series

$$y_0, y_1, y_2, y_3, \dots, y_N, \tag{41}$$

from which a covariance sequence is computed via the biased estimator

$$\hat{c}_k = \frac{1}{N}\sum_{t=k+1}^{N} y_t y_{t-k}, \tag{42}$$

n=2	n=3	n=4	n=5	n=6
.4287	.4289	.4292	.4297	.4300
.2532	.2537	.2539	.2541	.2543
	$3.041\cdot10^{-6}$	$2.504\cdot10^{-5}$	$2.105\cdot10^{-4}$	$4.348\cdot10^{-4}$
		$2.656\cdot10^{-7}$	$1.603\cdot10^{-6}$	$1.009\cdot10^{-4}$
			$3.602\cdot10^{-7}$	$9.163\cdot10^{-7}$
				$1.888\cdot10^{-7}$

TABLE I

Singular values of solution P of the CEE

which actually provides a sequences with positive Toepliz matrices. By setting $c_k := \hat{c}_k/\hat{c}_0$ we obtained a normalized covariance sequence

$$1, c_1, c_2, \dots, c_n, \quad n\ge\hat{n}. \tag{43}$$

Example 1: Detecting the positive degree

Given a transfer function $w(z)$ of degree $\hat{n}=2$ with zeros at $0.37e^{\pm i}$ and poles at $0.82e^{\pm 1.32i}$, estimate the covariance sequence (43) for $n=2,3,4,5$ and 6. Given these covariance sequences, we apply the algorithm of this paper to compute the $n\times n$ matrix P, using the zero polynomial $\sigma(z) = z^{n-\hat{n}}\hat{\sigma}(z)$, thus keeping the trigonometric polynomial $|\sigma(e^{i\theta})|^2$ constant. For each value of n 100 Monte Carlo simulations are performed, and the average of the singular values of P are computed and shown in Table 1.

For each $n>2$, the first two singular values are considerably larger than the others. Indeed, for all practical purposes, the singular values below the line in Table 1 are zero. Therefore, as the dimension of P increases, its rank remains close to 2. This is to say that the positive degree [5] of the covariance sequence (43) is approximately 2 for all n. In Fig. 2 the spectral density for $n=2$ is plotted together with those obtained by taking $n>2$, showing no major difference.

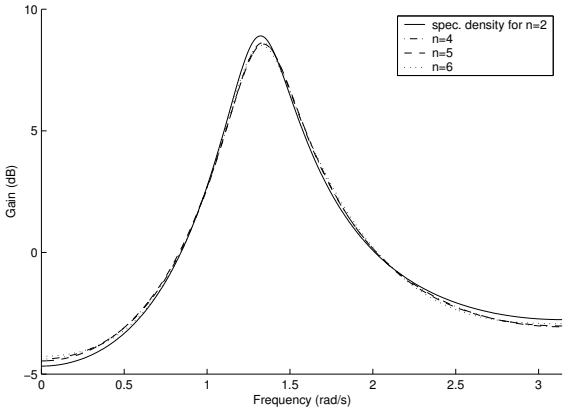

Fig. 1.
The given spectral density ($n=2$) and the estimated one for $n=4,5,6$.

Next, for $n=4$, we compute the solution of the CEE with

$$\sigma(z) = \hat{\sigma}(z)(z-0.6e^{1.78i})(z-0.6e^{-1.78i}).$$

As expected, the rank of the 4×4 matrix solution P of the CEE, is approximately 2, and, as seen in Fig. 3 , $a(z)$ has roots that are very close to cancelling the zeros $0.6e^{\pm 1.78i}$ of $\sigma(z)$.

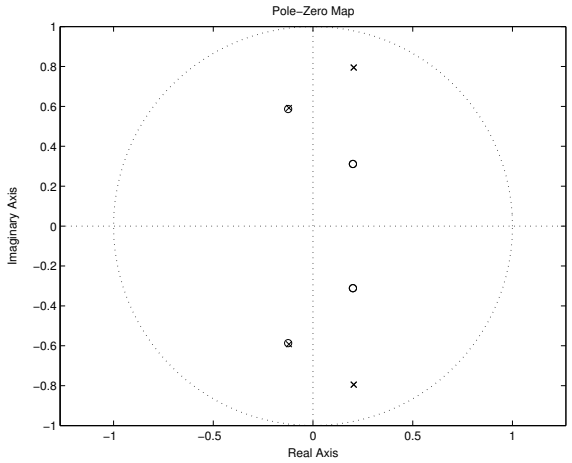

Fig. 2.
Spectral zeros (o) and the corresponding poles (x) for $n = 4$.

Example 2: Model reduction

Next, given a transfer function $w(z)$ of degree 10 with zeros $.99e^{\pm 1.78i}$, $.6e^{\pm 0.44i}$, $.55e^{\pm 2i}$, $.98e^{\pm i}$, $.97e^{\pm 2.7i}$ and poles $.8e^{\pm 2.6i}$, $.74e^{\pm 0.23i}$, $.8e^{\pm 2.09i}$, $.82e^{\pm 1.32i}$, $.77e^{\pm 0.83i}$, as in Fig. 3, we generate data (41) and a corresponding covariance sequence (43). Clearly, there is no zero-pole cancellation.

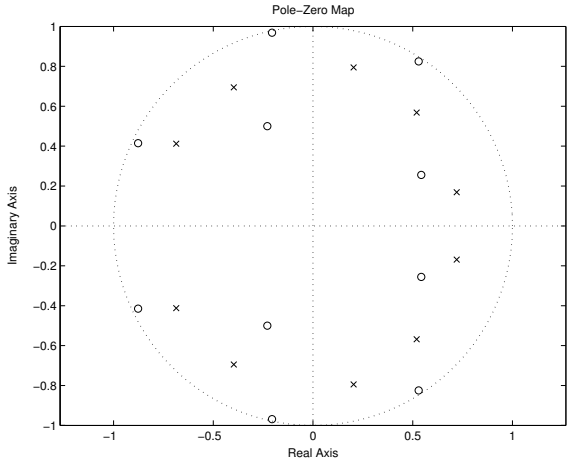

Fig. 3.
Zeros (o) and the corresponding poles (x) of $w(z)$.

Nevertheless, the rank of the 10×10 matrix solution P of CEE is close to 6. In fact, its singular values are equal to

$$1.1911 \quad 0.1079 \quad 0.0693 \quad 0.0627 \quad 0.0578 \quad 0.0434$$
$$0.0018 \quad 0.0012 \quad 0.0009 \quad 0.0008$$

The last four singular values are quite small, establishing an approximate rank of 6. The estimated spectral density ($n = 10$) is depicted in Fig. 4 together with the theoretical spectral density.

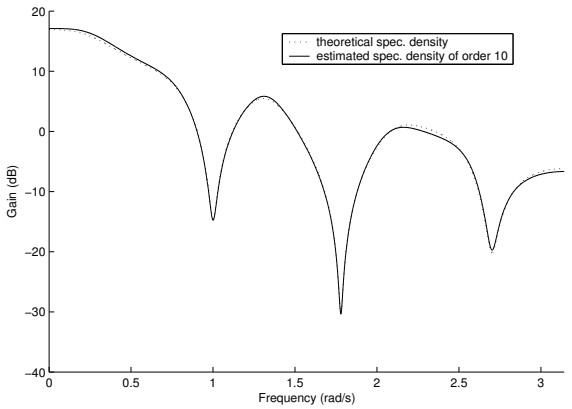

Fig. 4.
$n = 10$ estimate of spectral density together with the true spectral density.

Clearly six zeros are dominant, namely

$$0.98e^{\pm i}, \quad 0.99e^{\pm 1.78i}, \quad 0.97e^{\pm 2.7i},$$

and these can be determined from the estimated spectral density in Fig. 4. Therefore applying our algorithm to the reduced covariance sequence $1, c_1, \ldots, c_6$ using the six dominant zeros to form $\sigma(z)$, we obtain a 6×6 matrix solution P of CEE and a corresponding reduced order system with poles and zeros as in Fig. 5. Comparing with Fig. 3, we see that the

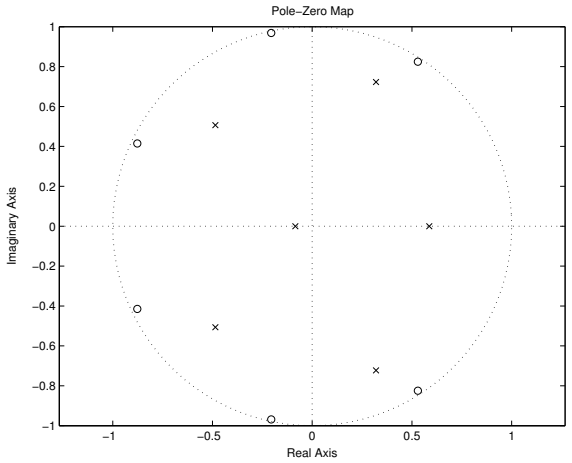

Fig. 5.
Zeros (o) and poles (x) of the reduced-order system.

poles are located in quite different locations. Nevertheless, the corresponding reduced-order spectral estimate, depicted in Fig. 6, is quite accurate.

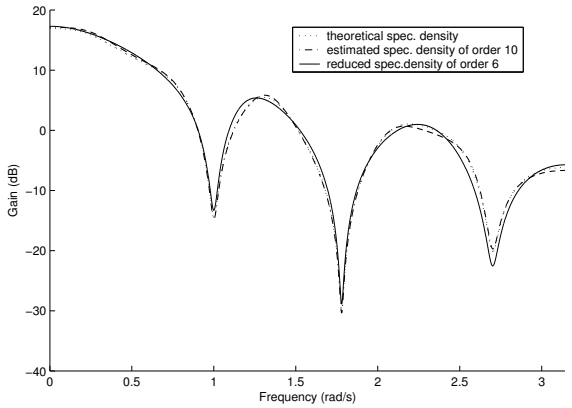

Fig. 6.
Reduced-order estimate of spectral density ($n = 6$) together with that of $n = 10$ and the true spectral density.

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