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IDENTIFIABILITY AND WELL-POSEDNESS OF SHAPING-FILTER PARAMETERIZATIONS: A GLOBAL ANALYSIS APPROACH*

CHRISTOPHER I. BYRNES†, PER ENQVIST‡, and ANDERS LINDQUIST‡

Abstract. In this paper, we study the well-posedness of the problems of determining shaping filters from combinations of finite windows of cepstral coefficients, covariance lags, or Markov parameters. For example, we determine whether there exists a shaping filter with a prescribed window of Markov parameters and a prescribed window of covariance lags. We show that several such problems are well-posed in the sense of Hadamard; that is, one can prove existence, uniqueness (identifiability), and continuous dependence of the model on the measurements. Our starting point is the global analysis of linear systems, where one studies an entire class of systems or models as a whole, and where one views measurements, such as covariance lags and cepstral coefficients or Markov parameters, from data as functions on the entire class. This enables one to pose such problems in a way that tools from calculus, optimization, geometry, and modern nonlinear analysis can be used to give a rigorous answer to such problems in an algorithm-independent fashion. In this language, we prove that a window of cepstral coefficients and a window of covariance coefficients yield a bona fide coordinate system on the space of shaping filters, thereby establishing existence, uniqueness, and smooth dependence of the model parameters on the measurements from data.

Key words. identifiability, parameterization, well-posedness, foliations, Carathéodory extension, spectral estimation, cepstrum

AMS subject classifications. 30E05, 42A15, 58C99, 93B29, 93E12

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1. Introduction.

It is common to model a (real, zero-mean) stationary process $\{y(t) \mid t \in \mathbb{Z}\}$ as a convolution

$$y(t) = \sum_{k=-\infty}^{t} w_{t-k} u_k$$

of an excitation signal $\{u(t) \mid t \in \mathbb{Z}\}$, which is a white noise, i.e., $\mathrm{E}\{u(t)u(s)\} = \delta_{ts}$, where δ_{ts} is one if $t = s$ and zero otherwise. In the language of systems and control, under suitable finiteness conditions this amounts to passing the white noise u through a linear filter with the transfer function $w(z)$ having the Laurent expansion

$$w(z) = \sum_{k=0}^{\infty} w_k z^{-k} \tag{1.1}$$

for all $z \geq 1$, thus obtaining the process y as the output, as depicted in Figure 1. In addition, we assume that $w_0 \neq 0$ and that $w(z)$ is a rational function, the latter assumption being the finiteness condition required in systems and control theory. Such a filter will be called a *shaping filter*, and the coefficients $w_0, w_1, w_2, \dots$ will be called the *Markov parameters*.

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†Department of Systems Science and Mathematics, Washington University, St. Louis, MO 63130 (chrisbyrnes@seas.wustl.edu).

‡Division of Optimization and Systems Theory, Department of Mathematics, Royal Institute of Technology, 100 44 Stockholm, Sweden (pere@math.kth.se, alq@math.kth.se).

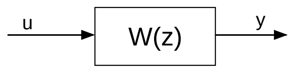

FIG. 1. *Representing a signal as the output of a black box.*

Clearly, any shaping filter must be *stable* in the sense that $w(z)$ has all of its poles in the open unit disc. To begin, we also assume that all zeros are located in the open unit disc. Such a shaping filter will be called a *minimum-phase shaping filter*.

Then the stationary stochastic process y has a rational spectral density

$$\Phi(e^{i\theta}) = |w(e^{i\theta})|^2,$$

which is positive for all θ. It is well known that the spectral density has a Fourier expansion

$$\Phi(e^{i\theta}) = r_0 + 2\sum_{k=1}^{\infty} r_k \cos k\theta,$$

where the Fourier coefficients

$$r_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\Phi(e^{i\theta})d\theta \tag{1.2}$$

are the covariance lags $r_k = \mathrm{E}\{y(t+k)y(t)\}$.

The spectral density $\Phi(z)$ is analytic in an annulus containing the unit circle and has there the representation

$$\Phi(z) = f(z) + f(z^{-1}),$$

where f is a rational function with all of its poles and zeros in the open unit disc. Hence, in particular, f is analytic outside the unit disc, and

$$f(z) = \frac{1}{2}r_0 + r_1 z^{-1} + r_2 z^{-2} + r_3 z^{-3} + \cdots . \tag{1.3}$$

Moreover,

$$\Phi(e^{i\theta}) = 2\mathrm{Re}\{f(e^{i\theta})\} > 0$$

for all θ, and, therefore, f is a real function which maps $\{|z| \geq 0\}$ into the right half-plane $\{\mathrm{Re}\, z > 0\}$; such a function is called *positive real*. For this to hold, the Toeplitz matrices

$$T_k = \begin{bmatrix} r_0 & r_1 & \cdots & r_k \\ r_1 & r_0 & \cdots & r_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ r_k & r_{k-1} & \cdots & r_0 \end{bmatrix} \tag{1.4}$$

must be positive definite for $k = 0, 1, 2, \ldots$.

Another way of representing the distribution of the stationary process is via the so-called *cepstrum*

$$\log \Phi(e^{i\theta}) = c_0 + 2\sum_{k=1}^{\infty} c_k \cos k\theta. \tag{1.5}$$

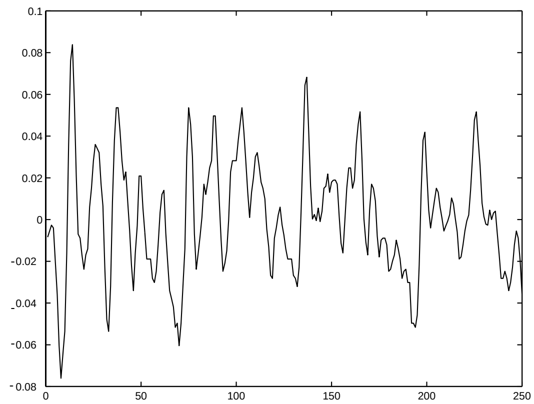

FIG. 2. *A frame of speech for the voiced nasal phoneme [ng].*

The Fourier coefficients

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\log\Phi(e^{i\theta})d\theta \tag{1.6}$$

are known as the *cepstral coefficients*.

Finite windows of covariance lags and cepstral coefficients can be estimated from an observed data record

$$y_0, y_1, y_2, \dots, y_N$$

of the process $\{y(t) \mid t \in \mathbb{Z}\}$. In fact, a limited number of covariance lags can be estimated via some ergodic estimate

$$r_k = \frac{1}{N+1-n}\sum_{t=0}^{N-n} y_{t+k}y_k. \tag{1.7}$$

However, we can only estimate

$$r_0, r_1, \dots, r_n, \tag{1.8}$$

where $n << N$, with some precision. A complementary set of observables are given by the window

$$c_0, c_1, \dots, c_n \tag{1.9}$$

of cepstral coefficients. One topic considered in this paper is to investigate the conditions under which these estimated coefficients can be used to determine minimum-phase shaping filters, i.e., to determine the identifiability of such shaping filters from covariance and cepstral windows.

As an example, to which we shall return several times in this paper, let us consider a 30 ms frame of speech from the voiced nasal phoneme [ng], depicted in Figure 2. Here $N = 250$, a typical sample length for a mobile telephone.

Figure 3 depicts a periodogram of this signal, i.e., a spectral estimate obtained by fast Fourier transform. This spectral estimate can be modeled as a smooth spectral envelope perturbed by contributions from an excitation signal. The spectral envelope

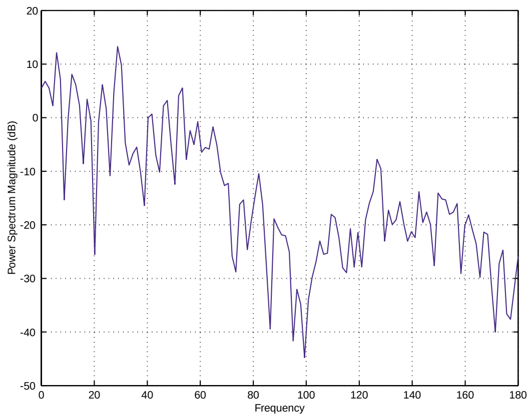

FIG. 3. *Periodogram for the voiced nasal phoneme [ng].*

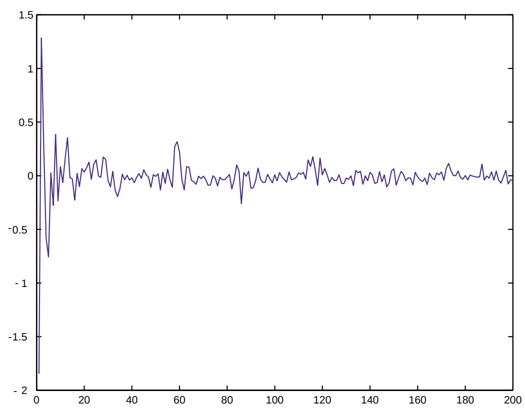

FIG. 4. *Cepstrum of voice speech signal.*

corresponds to the shaping of the vocal tract, which is described by the minimum-phase shaping filter.

As the Fourier transform of a convolution, the contributions of the shaping filter and the excitation signal to the spectral estimate are multiplicative. If we consider the logarithm of the spectral density Φ, the cepstrum, instead of Φ itself, the contribution of the excitation signal is additively superimposed on the that of the shaping filter.

Figure 4 shows the estimated cepstral coefficients of a frame of voiced speech. A contribution of the excitation signal is seen as spikes at multiples of the pitch period, corresponding to approximately $n_0 = 57$ in Figure 4. The spectral envelope can be estimated from a finite window

$$c_0, c_1, \dots, c_n \tag{1.10}$$

of cepstral coefficients, where $n < n_0$.

For minimum-phase shaping filters, the cepstral coefficients used in signal processing are closely related to the Markov parameters $w_0, w_1, w_2, \dots$ defined by (1.1). In more general systems problems, the minimum-phase requirement is relaxed to allow

σ to be an arbitrary (monic) polynomial. In this case, a record

$$w_0, w_1, \ldots, w_n \tag{1.11}$$

of Markov parameters are typically determined from the impulse response of an underlying system and not from data such as a finite time series, and for this reason Markov parameters can be quite useful in model reduction problems, starting from an underlying system. Nonetheless, for minimum-phase shaping filters, the cepstral coefficients used in signal processing are closely related to the Markov parameters of the shaping filter $w(z)$. Indeed, in section 6, we shall see that there is a one-to-one correspondence between windows of cepstral and Markov parameters of the same length.

In this paper, we are interested in the mathematical nature of the transformation of measurements, such as covariance lags and cepstral coefficients or Markov parameters, from data into the parameters of systems which produce such data. Our starting point will be the global analysis of linear systems, where one studies an entire class of systems or models as a whole and where one views measurements from data or model parameters as functions on the entire class. This point of view has been pioneered in [2, 4, 27, 16, 24]; see [5] for a survey. The central issue is whether the transformation from a set of measurements, viewed as functions, to a set of model parameters is well-posed, for example, in the sense of Hadamard. To be more precise, suppose the class of models is the class of (minimum-phase) shaping filters of bounded degree. This class can be viewed as a smooth manifold, for which any such shaping filter may be viewed as a point, and on which the coefficients of the numerator and denominator polynomials are a bona fide system of smooth coordinates on the global geometrization of this class of shaping filters. Matters being so, one can now ask, for example, whether a window of cepstral coefficients and a window of covariance coefficients also yield a bona fide coordinate system, so that, for example, the change of coordinates is a transformation which is smooth, one-to-one, onto, and with a smooth inverse. That is, the problem of passing from such data to models is indeed well-posed. Global analysis enables one to pose such problems in a way that tools from calculus, optimization, geometry, and modern nonlinear analysis can be used to give a rigorous answer to such problems.

In the next section, we shall review some of the basic spaces of systems we will use in our global analysis of certain transformations from data to models. In section 3, we will state our principal results, which we then prove in the following sections. These results focus on the identifiability of the models from collections of partial windows of covariance lags, cepstral coefficients, and Markov parameters and the questions of whether these parameters can be used to smoothly coordinatize spaces of shaping filters. For example, in section 4, a partial window of covariance lags and a partial window of cepstral coefficients are shown to jointly provide a system of local coordinates for shaping filters in the context of the geometry of certain foliations on the space of positive real functions.

In section 5, we prove that these are global coordinates, using methods from convex optimization theory. These schemes begin with an extension of the maximum entropy method, from the classical case of maximizing the zeroth cepstral gain to the problem of maximizing a "positive" linear combination of the entire partial cepstral window. This gives a new primal problem whose dual solves the rational covariance extension problem. In section 6, we provide a fairly complete local and global analysis of the use of a partial window of covariance lags and a partial window of cepstral coefficients. In lieu of a convex optimization argument, we used an extension of the solution to the rational covariance extension problem and the Lefschetz fixed point theorem as a generalization of the Brower fixed point theorem for the spaces of Schur polynomials. We conclude the paper in section 7 with a discussion and illustrations of the applications of some of these constructions to speech synthesis.

2. Some geometric representations of classes of models. Suppose the positive real function f is given by

$$f(z) = \frac{1}{2}\frac{a(z)}{b(z)}, \tag{2.1}$$

where

$$a(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n,$$
$$b(z) = b_0 z^n + b_1 z^{n-1} + \cdots + b_n$$

are (real) polynomials of degree n. Clearly, a_0 and b_0 must have the same sign. We assume that they are both positive. Then, since

$$f(z) + f(z^{-1}) = w(z)w(z^{-1}),$$

we must have

$$w(z) = \frac{\sigma(z)}{a(z)}, \tag{2.2}$$

where

$$\sigma(z) = \sigma_0 z^n + \sigma_1 z^{n-1} + \cdots + \sigma_n$$

is the unique polynomial with all roots in the open unit disc satisfying

$$\sigma(z)\sigma(z^{-1}) = \frac{1}{2}[a(z)b(z^{-1}) + a(z^{-1})b(z)] \tag{2.3}$$

and $\sigma_0 > 0$. We shall denote the class of such polynomials by $\hat{\mathcal{S}}_n$, and we shall denote the n-dimensional submanifold of monic (Schur) polynomials in $\hat{\mathcal{S}}_n$ by $\mathcal{S}_n$. Now, in order for f to be positive real, the pseudopolynomial

$$a(z)b(z^{-1}) + a(z^{-1})b(z)$$

must be positive on the unit circle, and $a(z)$ must belong to $\hat{\mathcal{S}}_n$. Then $b(z)$ also must belong to $\hat{\mathcal{S}}_n$.

Clearly, it is no restriction to take $a \in \mathcal{S}_n$ in (2.3). For each such $a(z)$, let

$$S(a)v = a(z)v(z^{-1}) + a(z^{-1})v(z)$$

define an operator $S(a) : V_n \to \mathcal{Z}_n$ from the vector space V_n of polynomials having degree less than or equal to n into the vector space $\mathcal{Z}_n$ of pseudopolynomials of degree at most n. Then (2.3) may be written as

$$S(a)b = 2\sigma\sigma^*, \tag{2.4}$$

where $\sigma^*(z) := \sigma(z^{-1})$. Now it is well known that $S(a)$ is bijective when $a \in \mathcal{S}_n$ (see, e.g., [8, p. 760]), and hence (2.4) establishes a one-to-one correspondence between f and w. We may normalize this relation by taking either $b(z)$ or $\sigma(z)$, but not both, in $\mathcal{S}_n$.

The normalization $b_0 = 1$ corresponds to taking $r_0 = 1$ in (1.3). We denote by $\mathcal{P}_n$ the set of all $(a, b) \in \mathcal{S}_n \times \mathcal{S}_n$ such that (2.1) is positive real. We know [13] that $\mathcal{P}_n$ is a smooth, connected, real manifold of dimension $2n$ and that it is diffeomorphic to $\mathbb{R}^{2n}$.

Choosing instead the normalization $\sigma \in \mathcal{S}_n$, corresponding to setting $w_0 = 1$ in (2.2) and $c_0 = 0$ in (1.5), we obtain an alternative coordinatization of $\mathcal{P}_n$ in terms of (a, σ). In fact, for each $(a, \sigma) \in \mathcal{S}_n \times \mathcal{S}_n$, we obtain the corresponding $(a, b) \in \mathcal{P}_n$ by dividing $b = 2S(a)^{-1}(\sigma\sigma^*)$ by b_0, thus normalizing it to form a monic b. This is a diffeomorphism, establishing that $\mathcal{P}_n$ is diffeomorphic to $\mathcal{S}_n \times \mathcal{S}_n$. In fact, the inverse of this coordinate transformation is the stable spectral factorization of $\frac{1}{2}S(a)b$ followed by the normalization of $\sigma(z)$. Since $\mathcal{S}_n$ is diffeomorphic to $\mathbb{R}^n$ (see Appendix A), spectral factorization gives an alternative method of exhibiting a diffeomorphism between $\mathcal{P}_n$ and $\mathbb{R}^{2n}$.

We shall generally use (a, σ)-coordinates to describe the geometry of $\mathcal{P}_n$. This normalizes the cepstral window (1.10) and the Markov window (1.11), fixing c_0 at zero and w_0 at one. However, a covariance window which is normalized in (a, b)-coordinates will not be normalized in (a, σ)-coordinates, and hence, to avoid increasing the dimension of the problem, we shall need to consider instead the *normalized covariance lags*

$$r_k = \frac{\int_{-\pi}^{\pi} e^{ik\theta}\Phi(e^{i\theta})d\theta}{\int_{-\pi}^{\pi}\Phi(e^{i\theta})d\theta}, \quad k = 1, 2, \ldots, n, \tag{2.5}$$

when working in (a, σ)-coordinates. In fact, in all of these descriptions, the polynomials $a(z)$, $b(z)$, and $\sigma(z)$ are monic. Working with unnormalized covariance lags (1.2), as we shall occasionally do, requires an extra parameter, bringing the number of coordinates to $2n + 1$.

There are several other spaces of models which we will need in this analysis. We denote by $\mathcal{P}_n^*$ the (dense) open subspace of $\mathcal{P}_n$ consisting of those pairs (a, σ) of polynomials which are coprime. Following the arguments in Appendix A, we see that $\mathcal{P}_n$ is diffeomorphic to the space of coprime pairs of real monic polynomials of degree n with poles and zeros in $\mathbb{C}$, first studied in [4] using the notation $\mathrm{Rat}(n)$. The space $\mathrm{Rat}(n)$ is a $2n$-dimensional manifold with $n + 1$ path-connected components, some of which have a rather complicated topology (see [4, 34, 37, 6]).

We shall also need to study the space Π_n of real, monic, degree n-polynomials, which is, of course, diffeomorphic to $\mathbb{R}^n$. Our interest in this space comes from the Markov expansion (1.11), where we take σ to be in Π_n and a to be in $\mathcal{S}_n$. Consequently, we allow (a, σ) to vary over the larger space

$$\mathcal{Q}_n := \mathcal{S}_n \times \Pi_n.$$

We shall also need to consider the space $\mathcal{Q}_n^*$, the (dense) open subspace of $\mathcal{Q}_n$ consisting of those pairs (a, σ) of polynomials which are coprime.

3. Main results. Our first results show that it is possible to parameterize minimum-phase shaping filters in terms of a window of cepstral coefficients and a window of covariance lags, both of which can be estimated from data. It is tempting, of course, to argue the plausibility of this result by counting parameters. This method typically works only when there is a rigorous way to compute the dimension of some geometric object—in this case, the smooth $2n$-dimensional manifold $\mathcal{P}_n$. In this setting, the implicit function theorem enables one to compute dimensions by computing the rank of certain Jacobian matrices or, equivalently, the linear independence of differentials. The following theorem is proved in section 4 (see Remark 4.7).

THEOREM 3.1. *The normalized covariance lags* $r_1, r_2, \dots, r_n$ *and the cepstral coefficients* $c_1, c_2, \dots, c_n$ *form a bona fide smooth coordinate system on the open subset* $\mathcal{P}_n^*$ *of* $\mathcal{P}_n$*; i.e., the map from* $\mathcal{P}_n^*$ *to* $\mathbb{R}^{2n}$ *with components* $(r_1, r_2, \dots, r_n, c_1, c_2, \dots, c_n)$ *has an everywhere invertible Jacobian matrix.*

Accordingly, when viewed as functions on $\mathcal{P}_n^*$, $(r_1, r_2, \dots, r_n, c_1, c_2, \dots, c_n)$ form local coordinates for the space $\mathcal{P}_n^*$ of pole-zero filters of degree n. At this point, one might hope to be able to use a global inverse function theorem, such as Hadamard's theorem, to show that these data define a global coordinate system. In part because of the complicated topology of $\mathcal{P}_n^*$, this is not possible, and instead we use a convex optimization scheme to conclude one of the important features of a global inverse function theorem. Indeed, the very nontrivial consequence of our next observation, to be proved in section 5, is that there is a one-to-one correspondence between the $2n$ coefficients $r_1, r_2, \dots, r_n, c_1, c_2, \dots, c_n$ of the minimum-phase shaping filter (2.2) and the $2n$ coefficients $a_1, a_2, \dots, a_n, \sigma_1, \sigma_2, \dots, \sigma_n$ of the denominator and numerator polynomials of (2.2), provided the degree of w is *exactly* n.

THEOREM 3.2. *Each shaping filter in* $\mathcal{P}_n^*$ *determines and is uniquely determined by its window* $r_1, r_2, \dots, r_n$ *of normalized covariance lags and its window* $c_1, c_2, \dots, c_n$ *of cepstral coefficients.*

As we have indicated, uniqueness follows from the remarkable fact that such a modeling filter arises as the minimum of a (strictly) convex optimization problem (see section 5). This optimization problem has, of course, antecedents in the literature, beginning with maximum entropy methods. Recall that linear predictive coding (LPC) is the most common method for determining shaping filters in signal processing. Given the window of (unnormalized) covariance data

$$r_0, r_1, \dots, r_n \tag{3.1}$$

with a positive definite Toeplitz matrix T_n, find the (unnormalized) shaping filter $w(z)$ and the corresponding spectral density

$$\Phi(e^{i\theta}) = |w(e^{i\theta})|^2,$$

which maximizes the entropy gain

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \Phi(e^{i\theta}) d\theta, \tag{3.2}$$

subject to the covariance-matching condition

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} \Phi(e^{i\theta}) d\theta = r_k, \quad k = 0, 1, \dots, n. \tag{3.3}$$

For this reason, the LPC filter is often called the *maximum entropy filter*.

Now observe that the entropy gain (3.2) is precisely the zeroth cepstral coefficient

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \Phi(e^{i\theta}) d\theta.$$

However, in cepstral analysis, one is interested not only in c_0 but in a finite window

$$c_0, c_1, \dots, c_n \tag{3.4}$$

of cepstral coefficients. It is natural, therefore, to maximize instead some (positive) linear combination

$$p_0 c_0 + p_1 c_1 + \cdots + p_n c_n \tag{3.5}$$

of the cepstral coefficients in the window (3.4). In view of (1.6), this may be written as a generalized entropy gain

$$\mathbb{I}_P(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(e^{i\theta}) \log \Phi(e^{i\theta}) d\theta, \tag{3.6}$$

where P is the symmetric pseudopolynomial

$$P(z) = p_0 + \frac{1}{2}p_1(z + z^{-1}) + \cdots + \frac{1}{2}p_n(z^n + z^{-n}), \tag{3.7}$$

and f is the positive real part (1.3) of Φ. We shall say that $P \in \mathcal{D}$ if P is nonnegative on the unit circle and $P \in \mathcal{D}_+$ if it is positive there. We note that the covariance matching condition (3.3) becomes

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\Phi(e^{i\theta}) d\theta = r_k, \quad k = 0, 1, \dots, n, \tag{3.8}$$

in terms of $\Phi(e^{i\theta}) = |w(e^{i\theta})|^2$.

Indeed, in section 5, we show that the problem of maximizing (3.5) subject to (3.8) has a finite solution only if the pseudopolynomial (3.7) belongs to $\mathcal{D}$. Indeed, if $P \in \mathcal{D}_+$, there is a unique solution Φ, and this solution has the form

$$\Phi(z) = \frac{P(z)}{Q(z)},$$

where

$$Q(z) = q_0 + \tfrac{1}{2}q_1(z + z^{-1}) + \cdots + \tfrac{1}{2}q_n(z^n + z^{-n})$$

belongs to $\mathcal{D}_+$.

In particular, we see that if we take P to be

$$P(z) = \sigma(z)\sigma(z^{-1})$$

and let $a(z)$ be the unique stable polynomial satisfying

$$Q(z) = a(z)a(z^{-1}),$$

then we have also determined the unique shaping filter (2.2) that matches the covariance data (3.1). Hence we have an alternative proof of the following result, first appearing in [11].

THEOREM 3.3. *Let $r_0, r_1, \dots, r_n$ be a partial covariance sequence, i.e., real numbers such that the Toeplitz matrix* (1.4) *is positive definite. Then, to any stable polynomial*

$$\sigma(z) = z^n + \sigma_1 z^{n-1} + \cdots + \sigma_{n-1}z + \sigma_n$$

of degree n, there corresponds a unique real stable polynomial

$$a(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

of degree n such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\left|\frac{\sigma(e^{i\theta})}{a(e^{i\theta})}\right|^2 d\theta = r_k, \quad k = 0, 1, \dots, n. \tag{3.9}$$

Theorem 3.3 was conjectured by Georgiou [21] as a solution to the partial covariance extension problem posed by Kalman [25]. Georgiou had already established the existence part, but a complete proof of the conjecture was given much later in [11]. Similarly, in [11], we also showed the following theorem.

THEOREM 3.4. *The normalized covariance lags $r_1, r_2, \dots, r_n$ and the zero coefficients $\sigma_1, \sigma_2, \dots, \sigma_n$ form a bona fide smooth coordinate system on the open manifold $\mathcal{P}_n$; i.e., the map from $\mathcal{P}_n$ to $\mathbb{R}^{2n}$ with components $(r_1, r_2, \dots, r_n, \sigma_1, \sigma_2, \dots, \sigma_n)$ has an everywhere invertible Jacobian matrix.*

In section 6, we derive the following results for coordinatization by covariance data and Markov parameters.

THEOREM 3.5. *The normalized covariance lags $r_1, r_2, \dots, r_n$ and the normalized Markov parameters $w_1, w_2, \cdots, w_n$ form a bona fide smooth coordinate system on $\mathcal{Q}_n^*$; i.e., the map from $\mathcal{Q}_n^*$ to $\mathbb{R}^{2n}$ with components $(r_1, r_2, \dots, r_n, w_1, w_2, \dots, w_n)$ has an everywhere invertible Jacobian matrix. For each choice of a covariance window and a Markov window, there exists exactly one shaping filter matching these windows.*

The last statement of this theorem is related to a class of results found in the literature on Q-Markov covers (see, e.g., [31, 29, 1]). Allowing windows of Markov parameters for which $w_0 = 0$, as in the literature cited above, would only add filters $w(z)$, which can be recovered from those of Theorem 3.5 by multiplying $w(z)$ by some power of z^{-1}.

4. Global analysis on $\mathcal{P}_n$. We choose to represent minimum-phase shaping filters (2.2) by a pair $(a, \sigma) \in \mathcal{S}_n \times \mathcal{S}_n$. This imposes the normalization discussed in section 2. There is a geometric manifestation of the fact that (a, σ) are smooth coordinates on $\mathcal{P}_n$, which we will use to show that the cepstral and covariance windows also form bona fide coordinate systems. First note that tangent vectors to $\mathcal{P}_n$ at (a, σ) may be represented as a perturbation $(a + \epsilon u, \sigma + \epsilon v)$, where u, v are polynomials of degree less than or equal to $n - 1$. If, as before, we denote the real vector space of polynomials of degree less than or equal to d by V_d, then the tangent space to $\mathcal{P}_n$ at a point (a, σ) is canonically isomorphic to $V_{n-1} \times V_{n-1}$.

Now, for $a \in \mathcal{S}_n$, define $\mathcal{P}_n(a)$ to be the space of all points in $\mathcal{P}_n$ with the polynomial a fixed. If we define $\mathcal{P}_n(\sigma)$ analogously, then $\mathcal{P}_n(a)$ and $\mathcal{P}_n(\sigma)$ are real, smooth, connected n-manifolds. In fact, both are clearly diffeomorphic to $\mathcal{S}_n$ and hence to $\mathbb{R}^n$ [7] (see also Appendix A). The tangent space to the submanifold $\mathcal{P}_n(a)$ at a point (a, σ) is, therefore,

$$T_{(a,\sigma)}\mathcal{P}_n(a) = \{(u, v) \in V_{n-1} \times V_{n-1} \mid u = 0\}.$$

Similarly, the tangent space to $\mathcal{P}_n(\sigma)$ is given by

$$T_{(a,\sigma)}\mathcal{P}_n(\sigma) = \{(u, v) \in V_{n-1} \times V_{n-1} \mid v = 0\}.$$

Now the n-manifolds $\{\mathcal{P}_n(a) \mid a \in \mathcal{S}_n\}$ form the leaves of a foliation of $\mathcal{P}_n$, as do the n-manifolds $\{\mathcal{P}_n(\sigma) \mid \sigma \in \mathcal{S}_n\}$. Moreover, these two foliations are complementary in the sense that if a leaf of one intersects a leaf of the other, the tangent spaces intersect in just $(0, 0)$. This transversality property is equivalent to the fact that the functions (a, σ) form a local system of coordinates.

We now turn to the cepstral functions and the covariance functions. Let $g : \mathcal{P}_n \to \mathbb{R}^n$ be the map which sends (a, σ) to the vector $c \in \mathbb{R}^n$ with components

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} \log |w(e^{i\theta})|^2 d\theta, \quad k = 1, 2, \ldots, n, \tag{4.1}$$

and let $\mathcal{C}_n := g(\mathcal{P}_n)$. Moreover, for each $c \in \mathcal{C}_n$, define the subset

$$\mathcal{P}_n(c) = g^{-1}(c).$$

We wish to show that $\mathcal{P}_n(c)$ is a smooth submanifold of dimension n. To this end, we will need to compute the Jacobian matrix of g, evaluated at tangent vectors to a point $(a, \sigma) \in \mathcal{P}_n$.

Thus, for each component

$$g_k(a, \sigma) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} \log \left| \frac{\sigma(e^{i\theta})}{a(e^{i\theta})} \right|^2 d\theta$$

of g, we form the directional derivative

$$D_{(u,v)} g_k(a, \sigma) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[g_k(a + \epsilon u, \sigma + \epsilon v) - g_k(a, \sigma) \right]$$

in the direction $(u, v) \in V_{n-1} \times V_{n-1}$. A straightforward calculation yields

$$D_{(u,v)} g_k(a, \sigma) = \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\mathrm{Re} \left\{ \frac{v(e^{i\theta})}{\sigma(e^{i\theta})} - \frac{u(e^{i\theta})}{a(e^{i\theta})} \right\} e^{ik\theta} d\theta \tag{4.2}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{S(\sigma)v}{\sigma\sigma^*} - \frac{S(a)u}{aa^*} \right] e^{ik\theta} d\theta. \tag{4.3}$$

Now, for any $\varphi \in \mathcal{S}_n$, define the linear map $G_\varphi : V_{n-1} \to \mathbb{R}^n$ by

$$G_\varphi u = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{S(\varphi)u}{\varphi\varphi^*} \begin{bmatrix} e^{i\theta} \\ e^{i2\theta} \\ \vdots \\ e^{in\theta} \end{bmatrix} d\theta.$$

Then the kernel of the Jacobian of g at (a, σ) is given by

$$\ker \mathrm{Jac}(g)|_{(a,\sigma)} = \{(u, v) \mid G_\sigma v = G_a u\}. \tag{4.4}$$

Lemma 4.1. *The linear map G_φ is a bijection.*

Proof. Suppose that $G_\varphi u = 0$. Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{S(\varphi)u}{\varphi\varphi^*} e^{ik\theta} d\theta = 0 \tag{4.5}$$

for $k = 1, 2, \ldots, n$. By symmetry this also holds for $k = -1, -2, \ldots, -n$. Moreover, since

$$\frac{S(\varphi)u}{\varphi\varphi^*}(z) = \frac{u(z)}{\varphi(z)} + \frac{u(z^{-1})}{\varphi(z^{-1})}$$

and $\frac{u(z)}{\varphi(z)}$ is strictly proper and analytic for $|z| \geq 1$, (4.5) holds for $k = 0$ also so that integration against $\frac{S(\varphi)u}{\varphi\varphi^*}$ annihilates all trigonometric pseudopolynomials of degree at most n. In particular, we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{S(\varphi)u}{\varphi}\right|^2 d\theta = 0,$$

which in turn yields $S(\varphi)u = 0$. But $S(\varphi)$ is nonsingular, and hence $u = 0$, establishing injectivity of G_φ. However, since the range and domain of G_φ are the same dimension, namely n, the map is also surjective. ∎

PROPOSITION 4.2. *For each $c \in \mathcal{C}_n$, the space $\mathcal{P}_n(c)$ is a smooth n-manifold. The tangent space $T_{(a,\sigma)}\mathcal{P}_n(c)$ at (a,σ) consists of precisely all $(u,v) \in V_{n-1} \times V_{n-1}$ such that*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{S(\sigma)v}{\sigma\sigma^*}e^{ik\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{S(a)u}{aa^*}e^{ik\theta}d\theta \tag{4.6}$$

for $k = 0, 1, \ldots, n$.

Proof. The tangent vectors of $\mathcal{P}_n(c)$ at (a,σ) are precisely the vectors in the null space of the Jacobian of g at (a,σ), as computed above. Consequently, by (4.4), (4.6) holds for $k = 1, 2, \ldots, n$. However, as pointed out in the proof of Lemma 4.1, (4.5) holds for $k = 0$, and hence (4.6) holds for $k = 0$ also. Moreover, by (4.4) and Lemma 4.1, the tangent space has dimension n. Therefore, the rank of $\mathrm{Jac}(g)|_{(a,\sigma)}$ is full, and the rest of the claim follows from the implicit function theorem. ∎

Because the rank of $\mathrm{Jac}(g)|_{(a,\sigma)}$ is everywhere n, the connected components of the submanifolds $\mathcal{P}_n(c)$ form the leaves of a foliation of $\mathcal{P}_n$. However, according to Lemma C.1, the submanifolds $\mathcal{P}_n(c)$ are themselves connected.

PROPOSITION 4.3. *The n-manifolds $\{\mathcal{P}_n(c) \mid c \in \mathcal{C}_n\}$ are connected and hence form the leaves of a foliation of $\mathcal{P}_n$.*

As an example of the more involved calculation we shall next undertake with the covariance window, we note a simple consequence of the results proven so far.

COROLLARY 4.4. *The foliations $\{\mathcal{P}_n(a) \mid a \in \mathcal{S}_n\}$ and $\{\mathcal{P}_n(c) \mid c \in \mathcal{C}_n\}$ are complementary; i.e., any intersecting pair of leaves, with one leaf from each foliation, intersects transversely. Moreover, any intersecting pair of leaves intersects in at most one point.*

Proof. Setting $u = 0$ in (4.4), we obtain $G_\sigma v = 0$. Hence, by Lemma 4.1, $v = 0$ so that the foliations are transverse. If a leaf $\mathcal{P}_n(a)$ intersects a leaf $\mathcal{P}_n(c)$ at a point (a,σ), then the a-coordinates, and hence the roots of a, are known. According to Appendix B, the value of the cepstral coefficients coincides with the difference of the Newton sums of the powers of the roots of a and the roots of σ. Therefore, the Newton sums of the powers of the roots of σ are known, and, therefore, by the Newton identities, so is σ. ∎

A similar statement for the foliation $\{\mathcal{P}_n(\sigma) \mid \sigma \in \mathcal{S}_n\}$ can be proved by the mirror image of this proof and will be omitted.

Next, let $f : \mathcal{P}_n \to \mathbb{R}^n$ be the map which sends (a,σ) to the vector $r \in \mathbb{R}^n$ of normalized covariance lags with components

$$r_k = \frac{\int_{-\pi}^{\pi} e^{ik\theta}\left|w(e^{i\theta})\right|^2 d\theta}{\int_{-\pi}^{\pi}\left|w(e^{i\theta})\right|^2 d\theta}, \quad k = 1, 2, \ldots, n, \tag{4.7}$$

and let $\mathcal{R}_n := f(\mathcal{P}_n)$. Of course, any $r \in \mathcal{R}_n$ satisfies the positivity condition

$$T_n = \begin{bmatrix} 1 & r_1 & \cdots & r_n \\ r_1 & 1 & \cdots & r_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ r_n & r_{n-1} & \cdots & 1 \end{bmatrix} > 0.$$

Now, for each $r \in \mathcal{R}_n$, we want to show that

$$\mathcal{P}_n(r) = f^{-1}(r)$$

is a smooth manifold of dimension n. To this end, note that the function $f : \mathcal{P}_n \to \mathbb{R}^n$ has the components

$$f_k(a, \sigma) = \frac{h_k(a, \sigma)}{h_0(a, \sigma)},$$

where

$$h_k(a, \sigma) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} \left| \frac{\sigma(e^{i\theta})}{a(e^{i\theta})} \right|^2 d\theta, \quad k = 0, 1, 2, \ldots, n.$$

Clearly, $h_0(a, \sigma) > 0$ for all $(a, \sigma) \in \mathcal{P}_n$.

A straightforward calculation shows that the directional derivative of f at $(a, \sigma) \in \mathcal{P}_n$ in the direction $(u, v) \in V_{n-1} \times V_{n-1}$ is

$$D_{(u,v)} f_k(a, \sigma) = \frac{1}{h_0(a, \sigma)} D_{(u,v)} h_k(a, \sigma) - \frac{h_k(a, \sigma)}{h_0(a, \sigma)^2} D_{(u,v)} h_0(a, \sigma), \tag{4.8}$$

where

$$D_{(u,v)} h_k(a, \sigma) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{S(\sigma)v}{aa^*} - \frac{S(a)u}{aa^*} \frac{\sigma\sigma^*}{aa^*} \right] e^{ik\theta} d\theta. \tag{4.9}$$

Therefore, defining

$$\varphi(a, \sigma; u, v) := D_{(u,v)} \log h_0(a, \sigma) = \frac{D_{(u,v)} h_0(a, \sigma)}{h_0(a, \sigma)},$$

the kernel of the Jacobian of f at (a, σ) consists of those $(u, v) \in V_{n-1} \times V_{n-1}$ for which

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{S(\sigma)v}{aa^*} e^{ik\theta} d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{S(a)u}{aa^*} \frac{\sigma\sigma^*}{aa^*} e^{ik\theta} d\theta + \varphi(a, \sigma; u, v) \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sigma\sigma^*}{aa^*} e^{ik\theta} d\theta$$

for $k = 0, 1, \ldots, n$. In fact, this equation holds trivially for $k = 0$, and so, to simplify the notation in what follows, we add this equation.

PROPOSITION 4.5. *The space $\mathcal{P}_n(r)$ is a smooth, connected, n-manifold, and its tangent space $T_{(a,\sigma)}\mathcal{P}_n(r)$ consists of those $(u, v) \in V_{n-1} \times V_{n-1}$ for which*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{S(\sigma)v}{aa^*} e^{ik\theta} d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{S(a)u}{aa^*} \frac{\sigma\sigma^*}{aa^*} e^{ik\theta} d\theta + \frac{\varphi}{2\pi} \int_{-\pi}^{\pi} \frac{\sigma\sigma^*}{aa^*} e^{ik\theta} d\theta \tag{4.10}$$

for $k = 0, 1, \dots, n$, *where*

$$\varphi = \frac{1}{h_0(a,\sigma)} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{S(\sigma)v}{aa^*} - \frac{S(a)u}{aa^*} \frac{\sigma\sigma^*}{aa^*} \right] d\theta. \tag{4.11}$$

The n*-manifolds* $\{\mathcal{P}_n(r) \mid r \in \mathcal{R}_n\}$ *form the leaves of a foliation of* $\mathcal{P}_n$.

Proof. The tangent space $T_{(a,\sigma)}\mathcal{P}_n(r)$ is the kernel of the Jacobian of f and is hence given by (4.10). Defining $p \in V_n$ as

$$p(z) := u(z) + \frac{1}{2}\varphi a(z),$$

these tangent equations may also be written as

$$Fp = Hv,$$

where $F : V_n \to \mathbb{R}^{n+1}$ and $H : V_{n-1} \to \mathbb{R}^{n+1}$ are the linear operators

$$Fp = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{S(a)p}{aa^*} \frac{\sigma\sigma^*}{aa^*} \begin{bmatrix} 1 \\ e^{i\theta} \\ \vdots \\ e^{in\theta} \end{bmatrix} d\theta, \qquad Hv = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{S(\sigma)v}{aa^*} \begin{bmatrix} 1 \\ e^{i\theta} \\ \vdots \\ e^{in\theta} \end{bmatrix} d\theta. \tag{4.12}$$

To see this, note that

$$\frac{1}{2} \frac{S(a)a}{aa^*} = 1.$$

Now the linear map F is nonsingular. In fact, supposing that $Fp = 0$ and, as in the proof of Lemma 4.1, taking the appropriate linear combination, we obtain

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|S(a)p|^2}{aa^*} \frac{\sigma\sigma^*}{aa^*} d\theta = 0,$$

which holds if and only if $S(a)p = 0$. But since a is a Schur polynomial, $S(a)$ is nonsingular, and hence $Fp = 0$ if and only if $p = 0$. Since the range and the domain of F have the same dimension, F is nonsingular, as claimed. Then, since the leading term of the n-polynomial

$$p = F^{-1}Hu$$

is precisely $\frac{1}{2}\varphi$, φ is a linear function of u. This defines a linear map $L : V_{n-1} \to V_{n-1}$, which sends v to $u := p - \frac{1}{2}\varphi a$ so that $T_{(a,\sigma)}\mathcal{P}_n(r)$ consists of those (u, v) such that $u = Lv$. This establishes that $T_{(a,\sigma)}\mathcal{P}_n(r)$ is n-dimensional and that $\mathcal{P}_n(r)$ is an n-manifold. Since the rank of $\mathrm{Jac}(f)|_{(a,\sigma)}$ is full, smoothness follows from the implicit function theorem. The connectedness of $\mathcal{P}_n(r)$ was proven in [7]. Since the rank of $\mathrm{Jac}(f)|_{(a,\sigma)}$ is everywhere n, the connected submanifolds $\mathcal{P}_n(r)$ form the leaves of a foliation of $\mathcal{P}_n$. □

The relation between the foliations $\{\mathcal{P}_n(r) \mid r \in \mathcal{R}_n\}$ and $\{\mathcal{P}_n(c) \mid c \in \mathcal{C}_n\}$ is indeed interesting.

THEOREM 4.6. *For each* $(a, \sigma) \in \mathcal{P}_n(r) \cap \mathcal{P}_n(c)$, *the dimension of*

$$\Theta := T_{(a,\sigma)}\mathcal{P}_n(r) \cap T_{(a,\sigma)}\mathcal{P}_n(c)$$

equals the degree of the greatest common divisor of the polynomials $a(z)$ and $\sigma(z)$.

Proof. Any $(u, v) \in \Theta$ satisfies both (4.6) and (4.10). Taking the linear combinations of these equations corresponding to the coefficients of $\sigma\sigma^*$ and aa^*, respectively, we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} S(\sigma)v d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(a)u\frac{\sigma\sigma^*}{aa^*}d\theta,$$
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} S(\sigma)v d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(a)u\frac{\sigma\sigma^*}{aa^*}d\theta + \varphi\|\sigma\|^2,$$

demonstrating that φ must be equal to zero. With $\varphi = 0$, (4.6) and (4.10) become

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{S(\sigma)v}{\sigma\sigma^*}e^{ik\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{S(a)u}{aa^*}e^{ik\theta}d\theta, \quad k = 0, 1, \dots, n, \tag{4.13}$$
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{S(\sigma)v}{aa^*}e^{ik\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{S(a)u}{aa^*}\frac{\sigma\sigma^*}{aa^*}e^{ik\theta}d\theta, \quad k = 0, 1, \dots, n. \tag{4.14}$$

Taking the appropriate linear combinations of (4.13) and (4.14), respectively, we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|S(\sigma)v|^2}{\sigma\sigma^*}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{[S(a)u][S(\sigma)v]}{aa^*}d\theta,$$
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{[S(\sigma)v][S(a)u]}{aa^*}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|S(a)u|^2}{aa^*}\frac{\sigma\sigma^*}{aa^*}d\theta.$$

Now, setting

$$f_1 := \frac{S(\sigma)v}{\sigma^*} \quad \text{and} \quad f_2 := \frac{\sigma S(a)u}{aa^*},$$

these equations can be written as

$$\|f_1\|^2 = \langle f_1, f_2\rangle \quad \text{and} \quad \langle f_1, f_2\rangle = \|f_2\|^2$$

in the inner product and norm of $L^2[-\pi, \pi]$. Using the parallelogram law yields

$$\|f_1 - f_2\|^2 = \|f_1\|^2 + \|f_2\|^2 - 2\langle f_1, f_2\rangle = 0,$$

which in turn implies that $f_1 = f_2$. Therefore,

$$\frac{S(\sigma)v}{\sigma\sigma^*} = \frac{S(a)u}{aa^*}$$

on the unit circle or, equivalently,

$$\operatorname{Re}\left\{\frac{v}{\sigma}\right\} = \operatorname{Re}\left\{\frac{u}{a}\right\}. \tag{4.15}$$

However, since these are harmonic functions, (4.15) must hold in the whole complex plane. In particular, as $a(z)$ and $\sigma(z)$ are real polynomials, this becomes

$$\frac{v}{\sigma} = \frac{u}{a} \tag{4.16}$$

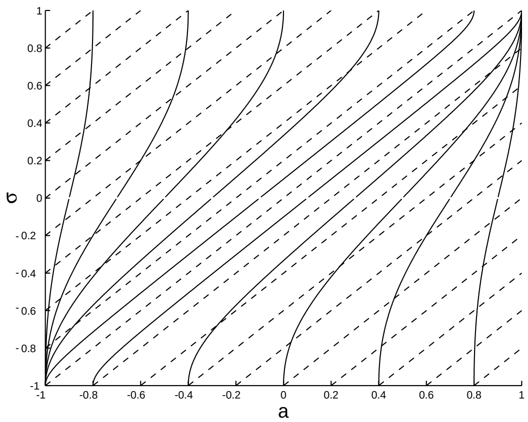

FIG. 5. *Cepstral (dotted line) and covariance (solid line) matching foliations of* $\mathcal{P}_1$.

on the real line. However, these functions are analytic outside the unit disc, and so, by the identity theorem, (4.16) is valid in the whole complex plane. Clearly, $u = v = 0$ satisfy (4.16), but let us see if there are nontrivial $u(z)$ and $v(z)$ of degree $n-1$. If so, (4.16) can also be written as

$$\frac{v}{u} = \frac{\sigma}{a},$$

which, of course, has no solution if $a(z)$ and $\sigma(z)$ are coprime. If $a(z)$ and $\sigma(z)$ have a greatest common factor of degree d, $u(z)$ and $v(z)$ could be polynomials of degree less than or equal to $n-1$ and have an arbitrary common factor of degree $d-1$, hence defining a vector space of dimension d, as claimed. $\square$

Remark 4.7. It follows from Theorem 4.6 that the foliations $\{\mathcal{P}_n(r) \mid r \in \mathcal{R}_n\}$ and $\{\mathcal{P}_n(c) \mid c \in \mathcal{C}_n\}$ are complementary at any point $(a,\sigma) \in \mathcal{P}_n$, where a and σ are coprime, as illustrated in Figure 5 for $n = 1$. From this it follows that the kernels of $\mathrm{Jac}(g)|_{(a,\sigma)}$ and $\mathrm{Jac}(f)|_{(a,\sigma)}$ are complementary at any point (a,σ) in $\mathcal{P}_n^*$. In particular, the Jacobian of the joint map $(a,\sigma) \to (r_1, r_2, \ldots, r_n, c_1, c_2, \ldots, c_n)$ has full rank, and, by the inverse function theorem, the joint map forms a smooth local coordinate system on $\mathcal{P}_n^*$. This proves Theorem 3.1.

5. Identifiability of shaping filters from cepstral and covariance windows. In this section, we shall show that the window of n cepstral coefficients and the window of n normalized covariance lags do indeed determine the (normalized) shaping filter which generates these data, provided the filter has degree n, thus proving Theorem 3.2. As a preliminary to this argument, however, we want to return to the generalization of the maximum entropy integral in terms of "positive" linear combinations of the entire cepstral window. Not only is this an appealing idea, but it also turns out to give a novel derivation of a result which is of independent interest in itself, a solution of the rational covariance extension problem. We now formalize our analysis of this generalized maximum entropy problem.

THEOREM 5.1. *If the pseudopolynomial* (3.7) *belongs to* $\mathcal{D}_+$, *the problem to maximize* (3.5) *subject to* (3.8) *has a unique solution* Φ, *and this solution has the form*

$$\Phi(z) = \frac{P(z)}{Q(z)}, \tag{5.1}$$

where

$$Q(z) = q_0 + \frac{1}{2}q_1(z + z^{-1}) + \cdots + \frac{1}{2}q_n(z^n + z^{-n}) \tag{5.2}$$

also belongs to $\mathcal{D}_+$.

It turns out that the algorithm needed to determine Q is precisely the convex optimization algorithm presented in [12]. In fact, the algorithm is based on the dual problem, in the sense of mathematical programming, of the problem to maximize (3.6) subject to (3.8). More precisely, let $\mathcal{F}_+$ be the set of bounded positive real functions

$$f(z) = \frac{1}{2}f_0 + f_1 z^{-1} + f_2 z^{-2} + \cdots$$

such that $\Phi(e^{i\theta}) := 2\mathrm{Re}\{f(e^{i\theta})\}$ is bounded away from zero, and consider the (primal) problem to maximize the generalized entropy (3.6) over $\mathcal{F}_+$, i.e.,

$$\max_{f\in\mathcal{F}_+} \mathbb{I}_P(f),$$

subject to (3.8). Then duality theory amounts to forming the Lagrangian

$$\begin{aligned} L(f,q) &= \mathbb{I}_P(f) + \sum_{k=0}^{n} q_k \left[r_k - \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\Phi(e^{i\theta})d\theta \right] \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P(e^{i\theta})\log\Phi(e^{i\theta})d\theta + r'q - \frac{1}{2\pi}\int_{-\pi}^{\pi} Q(e^{i\theta})\Phi(e^{i\theta})d\theta \end{aligned} \tag{5.3}$$

and determining the Lagrange multipliers $q \in \mathbb{R}^{n+1}$ by minimizing the dual functional

$$\psi(q) := \sup_{f\in\mathcal{F}_+} L(f,q).$$

Clearly, $\psi(q) < \infty$ only if both P and Q belong to $\mathcal{D}$. If the function $f \mapsto L(f,q)$ has a maximum in the open region $\mathcal{F}_+$, then

$$\frac{\partial L}{\partial f_k} = 0, \quad k = 0, 1, 2, \ldots,$$

in the maximizing point. This stationarity condition becomes

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\left[P(e^{i\theta})\Phi(e^{i\theta})^{-1} - Q(e^{i\theta})\right] d\theta = 0, \quad k = 0, 1, 2, \ldots,$$

which is satisfied if and only if (5.1) or, equivalently,

$$f_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\frac{P(e^{i\theta})}{Q(e^{i\theta})}d\theta \tag{5.4}$$

holds. Inserting this into (5.3) yields the dual functional

$$\psi(q) = \mathbb{J}_P(q) + \frac{1}{2\pi}\int_{-\pi}^{\pi} P(e^{i\theta})[\log P(e^{i\theta}) - 1]d\theta \tag{5.5}$$

for all $P, Q \in \mathcal{D}$, where

$$\mathbb{J}_P(q) = r_0 q_0 + r_1 q_1 + \cdots + r_n q_n - \frac{1}{2\pi}\int_{-\pi}^{\pi} P(e^{i\theta})\log Q(e^{i\theta})d\theta. \tag{5.6}$$

Since the last term in (5.5) does not depend on q, we shall call the optimization problem

$$\min_{Q \in \mathcal{D}} \mathbb{J}_P(Q) \tag{5.7}$$

the *dual problem.* The functional (5.6) is strictly convex, and, therefore, the minimum is unique, provided one exists. This is precisely the optimization problem considered in [12], where the following theorem was proven.

THEOREM 5.2. *The dual problem has a unique solution, and it belongs to* $\mathcal{D}_+$.

Since thus $\mathbb{J}_P$ takes its minimum in an interior point,

$$\frac{\partial \mathbb{J}_P}{\partial q_k} = r_k - \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} \frac{P(e^{i\theta})}{Q(e^{i\theta})} d\theta \tag{5.8}$$

equals zero there for $k = 0, 1, \dots, n$. This stationarity condition is precisely the covariance matching condition. The dual problem is easily solved by Newton's method [12, 14]. The statement of Theorem 5.2 is nontrivial. In fact, the proof [12] relies on the fact that the gradient (5.8) tends to infinity as Q tends to the boundary of $\mathcal{D}$.

Proof of Theorem 5.1. Let $\hat{Q} \in \mathcal{D}_+$ be the unique solution to the dual problem (5.7), let $\hat{q} \in \mathbb{R}^{n+1}$ be the corresponding vector of coefficients, and let

$$\hat{f}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} \frac{P(e^{i\theta})}{\hat{Q}(e^{i\theta})} d\theta.$$

Clearly, $\hat{f} \in \mathcal{F}_+$. Since the gradient (5.8) is zero for $Q = \hat{Q}$, the covariance matching condition (3.8) is fulfilled for $f = \hat{f}$, and, therefore, $\mathbb{I}_P(\hat{f}) = L(\hat{f}, \hat{q})$. But, by the construction above,

$$L(\hat{f}, \hat{q}) = \sup_{f \in \mathcal{F}_+} L(f, \hat{q}) \geq L(f, \hat{q})$$

for all $f \in \mathcal{F}_+$. Then, for any $f \in \mathcal{F}_+$ which satisfies the covariance matching condition (3.8),

$$\mathbb{I}_P(f) = L(f, \hat{q}) \leq \mathbb{I}_P(\hat{f}),$$

which establishes the optimality of $\hat{f}$. □

This analysis motivates the construction of a functional which will be the key in establishing uniqueness of minimum-phase shaping filters having prescribed windows $r_0, r_1, \dots, r_n$ and $c_1, c_2, \dots, c_n$ of covariance lags and cepstral coefficients, respectively. More precisely, consider the (primal) problem of finding a spectral density

$$\Phi(e^{i\theta}) = f_0 + 2 \sum_{k=1}^{\infty} f_k \cos k\theta,$$

which minimizes

$$\mathbb{I}(f) = \sum_{k=1}^{n} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} \log \Phi(e^{i\theta}) d\theta - c_k \right| - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \Phi(e^{i\theta}) d\theta \tag{5.9}$$

subject to the covariance-lag matching condition

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} \Phi(e^{i\theta}) d\theta = r_k, \quad k = 0, 1, \dots, n. \tag{5.10}$$

The objective function (5.9) is the (ℓ_1) "cepstral error" minus the entropy gain. As discussed in section 3, the entropy gain is precisely what is maximized in the LPC solution, and it is identical to the zeroth cepstral coefficient corresponding to Φ. This term compensates for the absence of a zeroth term in the cepstral error.

To obtain a suitable dual problem, we reformulate the *primal problem* to minimize

$$\sum_{k=0}^{n} \epsilon_k - \frac{1}{2\pi}\int_{-\pi}^{\pi} \log \Phi(e^{i\theta})d\theta$$

subject to the covariance matching condition (5.10) and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\log \Phi(e^{i\theta})d\theta - c_k - \epsilon_k \le 0, \quad k = 1, 2, \dots, n, \tag{5.11}$$

$$-\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\log \Phi(e^{i\theta})d\theta + c_k - \epsilon_k \le 0, \quad k = 1, 2, \dots, n. \tag{5.12}$$

Taking $q_0, q_1, \dots, q_n$ to be the Lagrange multipliers for the constraints (5.10) and $\lambda_1, \lambda_2, \dots, \lambda_n$ and $\mu_1, \mu_2, \dots, \mu_n$ to be nonnegative Lagrange multipliers for the sets of constraints (5.11) and (5.12), respectively, we obtain the Lagrangian

$$\begin{aligned} L(f, \epsilon, q, \lambda, \mu) = \sum_{k=1}^{n} \epsilon_k &- \frac{1}{2\pi}\int_{-\pi}^{\pi} \log \Phi(e^{i\theta})d\theta \\ &+ \sum_{k=0}^{n} q_k \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\Phi(e^{i\theta})d\theta - r_k\right] \\ &+ \sum_{k=1}^{n} \lambda_k \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\log \Phi(e^{i\theta})d\theta - c_k - \epsilon_k\right] \\ &- \sum_{k=1}^{n} \mu_k \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\log \Phi(e^{i\theta})d\theta - c_k + \epsilon_k\right]. \end{aligned}$$

Now, setting

$$p_0 = 1, \qquad p_k := \mu_k - \lambda_k, \quad k = 1, 2, \dots, n, \tag{5.13}$$

we can write this in the more compact form

$$\begin{aligned} L(f, \epsilon, q, \lambda, \mu) = &\sum_{k=1}^{n} (1 - \lambda_k - \mu_k)\epsilon_k \\ &+ c_1 p_1 + c_2 p_2 + \cdots + c_n p_n - r_0 q_0 - r_1 q_1 - \cdots - r_n q_n \\ &+ \frac{1}{2\pi}\int_{-\pi}^{\pi} Q(e^{i\theta})\Phi(e^{i\theta})d\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi} P(e^{i\theta})\log \Phi(e^{i\theta})d\theta, \end{aligned}$$

which clearly can have a finite minimum only for those values of the Lagrange multipliers for which both P and Q belong to $\mathcal{D}$ and $\lambda_k + \mu_k \le 1$ for $k = 1, 2, \dots, n$. For such Lagrange multipliers, if the function $(f, \epsilon) \to L(f, \epsilon, q, \lambda, \mu)$ has a minimum, then

$$\frac{\partial L}{\partial f_k} = 0, \quad k = 0, 1, 2, \dots, \tag{5.14}$$

and

$$(1-\lambda_k-\mu_k)\epsilon_k = 0, \quad k = 1,2,\dots,n, \tag{5.15}$$

in the minimizing point. The stationarity condition (5.14) becomes

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\left[P(e^{i\theta})\Phi(e^{i\theta})^{-1} - Q(e^{i\theta})\right]d\theta = 0, \quad k = 0,1,2,\dots,$$

or, equivalently,

$$\Phi(z) = \frac{P(z)}{Q(z)},$$

which, inserted together with (5.15) into the Lagrangian with P given by (5.13), yields the dual functional

$$\inf_{(f,\epsilon)\in\mathcal{F}_+\times\mathbb{R}^+} L(f,\epsilon,q,\lambda,\mu) = \mathbb{J}(P,Q)+1,$$

where the functional

$$\begin{aligned}\mathbb{J}(P,Q) = c_1p_1 + c_2p_2 + \cdots + c_np_n - r_0q_0 - r_1q_1 - \cdots - r_nq_n\\ - \frac{1}{2\pi}\int_{-\pi}^{\pi} P(e^{i\theta})\log\frac{P(e^{i\theta})}{Q(e^{i\theta})}d\theta\end{aligned} \tag{5.16}$$

is concave but not necessarily strictly concave.

THEOREM 5.3. *The dual problem to maximize $\mathbb{J}(P,Q)$ over all $(P,Q)\in\mathcal{D}\times\mathcal{D}$ such that $p_0 = 1$ has a solution $(\hat{P},\hat{Q})$, and, for any such solution, $\hat{Q}\in\mathcal{D}_+$, and*

$$\Phi(z) = \frac{\hat{P}(z)}{\hat{Q}(z)} \tag{5.17}$$

satisfies the covariance matching condition (5.10). *If, in addition, $\hat{P}\in\mathcal{D}_+$, then* (5.17) *is a solution of the primal problem with $\epsilon_1=\epsilon_2=\cdots=\epsilon_n=0$, i.e., there is both covariance matching and cepstral matching. A maximizing point $(\hat{P},\hat{Q})\in\mathcal{D}_+\times\mathcal{D}_+$ is unique if and only if $\hat{P}$ and $\hat{Q}$ are coprime.*

Proof. It can be shown along the same lines as in [12] that the functional $\mathbb{J}$ has compact sublevel sets in $\mathcal{D}\times\mathcal{D}$. Hence $\mathbb{J}$ has a maximal point $(\hat{P},\hat{Q})$ there. The boundary of $\mathcal{D}\times\mathcal{D}$ consists of those points where either $\hat{P}$ or $\hat{Q}$ or both have zeros on the unit circle. Now a straightforward calculation shows that

$$\frac{\partial\mathbb{J}}{\partial q_k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\frac{P(e^{i\theta})}{Q(e^{i\theta})}d\theta - r_k, \quad k = 0,1,\dots,n, \tag{5.18}$$

$$\frac{\partial\mathbb{J}}{\partial p_k} = c_k - \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\log\frac{P(e^{i\theta})}{Q(e^{i\theta})}d\theta, \quad k = 1,2,\dots,n. \tag{5.19}$$

From this and the argument in [12], it can be shown that the gradient (5.18) becomes infinite when Q lies on the boundary and hence that $\hat{Q}\in\mathcal{D}_+$. Therefore, since the functional $\mathbb{J}$ is concave, (5.18) must be zero at $(\hat{P},\hat{Q})$, and hence (5.17) satisfies the covariance matching condition (5.10).

Next suppose that $\hat{P} \in \mathcal{D}_+$. Then (5.19) must also be zero at $(\hat{P}, \hat{Q})$, and hence there is also cepstral matching. For any $f \in \mathcal{F}_+$ satisfying (5.10) and $\epsilon > 0$,

$$\mathbb{I}(f) \geq L(f, \epsilon, \hat{q}, \hat{\lambda}, \hat{\mu}) \geq \mathbb{J}(\hat{P}, \hat{Q}) + 1,$$

where $\hat{q}$, $\hat{\lambda}$, and $\hat{\mu}$ are Lagrange multipliers corresponding to $(\hat{P}, \hat{Q})$. On the other hand, if $\hat{f}$ is the positive-real part of (5.17), then

$$\mathbb{I}(\hat{f}) = L(\hat{f}, 0, \hat{q}, \hat{\lambda}, \hat{\mu}) = \mathbb{J}(\hat{P}, \hat{Q}) + 1,$$

and hence $\hat{f}$ minimizes $\mathbb{I}$, and $\hat{\epsilon} = 0$, as claimed.

Clearly, the maximizing solution $(\hat{P}, \hat{Q})$ cannot be unique if $\hat{P}$ and $\hat{Q}$ are not coprime. Therefore, the last statement of the theorem would follow if we could show that $\mathbb{J}$ is *strictly* concave over some neighborhood of $\mathcal{D}_+ \times \mathcal{D}_+$ if $\hat{P}$ and $\hat{Q}$ are coprime. To this end, we consider the Hessian. Let

$$\delta\mathbb{J}(P, Q; \delta P, \delta Q) = \lim_{\epsilon \to 0} \frac{\mathbb{J}(P + \delta P, Q + \epsilon\delta Q) - \mathbb{J}(P, Q)}{\epsilon}$$

denote the directional derivative in the direction $(\delta P, \delta Q)$. The admissible directions $(\delta P, \delta Q)$ are symmetric pseudopolynomials such that $(P + \epsilon\delta P, Q + \epsilon\delta Q) \in \mathcal{D}_+ \times \mathcal{D}_+$ for sufficiently small $\epsilon > 0$. Since $p_0 = 1$, we must also have $\delta p_0 = 0$. It is straightforward to see that

$$\begin{aligned}\delta\mathbb{J}(P, Q; \delta P, \delta Q) = c_1\delta p_1 + c_2\delta p_2 + \cdots + c_n\delta p_n - r_0\delta q_0 - r_1\delta q_1 - \cdots - r_n\delta q_n \\ + \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta Q(e^{i\theta})\frac{P(e^{i\theta})}{Q(e^{i\theta})}d\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta P(e^{i\theta}) \log\frac{P(e^{i\theta})}{Q(e^{i\theta})}d\theta,\end{aligned}$$

and hence second differentiation yields

$$\delta^2\mathbb{J}(P, Q; \delta P, \delta Q) = -\left\langle (P\delta Q - Q\delta P)^2, \frac{1}{PQ^2} \right\rangle \leq 0,$$

where equality holds if and only if $P\delta Q - Q\delta P = 0$, i.e., if and only if

$$\frac{\delta P}{\delta Q} = \frac{P}{Q}.$$

However, this is impossible if $\hat{P}$ and $\hat{Q}$ are to be coprime, since $p_0 = 1$ and $\delta p_0 = 0$. Consequently, $\mathbb{J}$ is strictly concave at $(\hat{P}, \hat{Q})$, as claimed. □

Now, given the minimizing pair of pseudopolynomials $(\hat{P}, \hat{Q})$ of Theorem 5.3, let $a(z)$ and $\sigma(z)$ be the normalized, polynomial spectral factors of $\hat{Q}$ and $\hat{P}$, respectively, i.e., the Schur polynomials satisfying

$$a(z)a(z^{-1}) = \frac{1}{a_0^2}\hat{Q}(z), \qquad \sigma(z)\sigma(z^{-1}) = \frac{1}{\sigma_0^2}\hat{P}(z),$$

where a_0^2 and σ_0^2 are the appropriate normalizing factors. Then Theorem 5.3 provides a procedure for determining, from a combined window $(r_0, r_1, \ldots, r_n, c_1, \ldots, c_n)$ of covariance lags and cepstral coefficients, a pair (a, σ), which is unique if and only if $a(z)$ and $\sigma(z)$ are coprime, i.e., $(a, \sigma) \in \mathcal{P}_n^*$, and a corresponding (unnormalized) shaping filter

$$w(z) = \frac{\sigma_0}{a_0}\frac{\sigma(z)}{a(z)}.$$

Therefore, in particular, we have proved Theorem 3.2. In fact, given any $(a, \sigma) \in \mathcal{P}_n^*$, a window $(r_1, \dots, r_n, c_1, \dots, c_n)$ is uniquely determined from (4.7) and (4.1). Conversely, given $(r_1, \dots, r_n, c_1, \dots, c_n)$, the optimization problem of Theorem 5.3 yields an $(a, \sigma) \in \mathcal{P}_n$, which matches this window and is unique if and only if $(a, \sigma) \in \mathcal{P}_n^*$.

6. The simultaneous partial realization problem. While the *stochastic realization problem* [25, 21, 26, 10, 30] amounts to determining shaping filters w having a fixed window of covariance lags $r_0, r_1, \dots, r_n$, the object of the *deterministic realization problem* (see, e.g., [3, 23]) is to find shaping filters w with a fixed window $w_0, w_1, \dots, w_n$ of Markov parameters (1.11). An important question is whether the two problems can be solved simultaneously so that both interpolation conditions are satisfied at the same time. This problem has been studied in the literature as the Q-Markov cover problem (see [31, 29, 1], where it has been used as a tool for performing model reduction).

This basic question will also be addressed in this section using geometric methods. Thus we would ask whether the two problems can be solved simultaneously and, if so, whether this solution is unique. We find a positive answer to the existence question in $\mathcal{Q}_n$ using fixed point methods. We also determine where these windows provide a bona fide set of smooth coordinates. Finally, we give a geometric proof of the uniqueness of the corresponding shaping filter, i.e., of identifiability of the shaping filter from covariance and Markov windows, providing an independent proof of a result which is basic to the existing theory of the Q-Markov cover problem. These results prove the assertions in Theorem 3.5. We also provide an independent proof of Theorem 3.4.

To address these issues, let $\psi : \mathcal{Q}_n \to \mathbb{R}^n$ be the map which sends (a, σ) to

$$w := \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix},$$

and let $\mathcal{W}_n := \psi(\mathcal{Q}_n)$. Given any $w \in \mathcal{W}_n$, define

$$\mathcal{Q}_n(w) := \psi^{-1}(w).$$

Now, multiplying (2.2) by $a(z)$ and identifying coefficients of nonnegative powers in z, we have

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} + \begin{bmatrix} 1 & & & \\ w_1 & 1 & & \\ \vdots & \vdots & \ddots & \\ w_{n-1} & w_{n-2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}. \tag{6.1}$$

Identifying coefficients in negative powers of z yields the appropriate Hankel system. From (6.1) we see first that $\mathcal{W}_n = \mathbb{R}^n$. Second, given w, a can be chosen arbitrarily in $\mathcal{S}_n$. Hence, $\mathcal{Q}_n(w)$ is completely parameterized by $a \in \mathcal{S}_n$, and hence it is a connected n-manifold, diffeomorphic to $\mathbb{R}^n$. Its boundary is characterized by a having a root on the unit circle. Clearly, the closure $\overline{\mathcal{Q}_n(w)}$ is the graph of a continuous function $\gamma : \overline{\mathcal{S}_n} \to \Pi_n$, defined by (6.1). Although the manifold $\mathcal{Q}_n$ is not bounded, $\mathcal{Q}_n(w)$ is. Moreover, $\overline{\mathcal{Q}_n(w)}$ is homeomorphic to $\overline{\mathcal{S}_n}$, which is compact with a contractible interior (see Appendix A).

THEOREM 6.1. *Any continuous map* $T : \overline{\mathcal{S}_n} \to \overline{\mathcal{S}_n}$ *has a fixed point.*

Proof. We first note that $\overline{\mathcal{S}_n}$ is contained in the (Euclidean) space of real monic polynomials with roots in the open disc of radius $1 + \epsilon$ for any positive ϵ. As in Appendices A and C, the continuous retraction $r : \mathbb{D}_{1+\epsilon} \to \mathbb{D}$, defined by

$$r(x) = \begin{cases} x \text{ if } \|x\| \leq 1, \\ \frac{x}{\|x\|} \text{ if } \|x\| \geq 1, \end{cases}$$

induces a continuous retraction of $\mathcal{P}_{\mathbb{D}_{1+\epsilon}}(n) \to \overline{\mathcal{S}_n}$. In particular, $\overline{\mathcal{S}_n}$ is a Euclidean neighborhood retract, and, therefore, the Lefschetz fixed point theorem applies to continuous maps of $\overline{\mathcal{S}_n}$ to itself [18, p. 209]. The Lefschetz fixed point theorem asserts that a continuous map f from a space X to itself has a fixed point provided its Lefschetz number is nonzero. More precisely, to define the Lefschetz number, we need to introduce the homology (real) vector spaces $H_i(X;\mathbb{R})$, defined for each $i = 0, 1, 2, \ldots$. If X is a compact Euclidean neighborhood retract in $\mathbb{R}^N$, then each $H_i(X;\mathbb{R})$ is finite-dimensional and vanishes for $i > N$. In this case, the Lefschetz number of f, $\mathrm{Lef}(f)$, is defined as

$$\mathrm{Lef}(f) = \sum_{i=0}^{n} \mathrm{tr}(f_{*i}),$$

where (f_{*i}) is the linear transformation

$$f_{*i} : H_i(X;\mathbb{R}) \to H_i(X;\mathbb{R})$$

introduced by f. For $X = \overline{\mathcal{S}_n}$, we have

$$H_i(\overline{\mathcal{S}_n}, \mathbb{R}) = \{0\} \quad \text{for } i \geq 1$$

since $\overline{\mathcal{S}_n}$ is contractable. Moreover, since $\overline{\mathcal{S}_n}$ is therefore connected,

$$H_0(\overline{\mathcal{S}_n}, \mathbb{R}) \sim \mathbb{R},$$

and the map f_{*i} is the identity. In summary, $\mathrm{Lef}(f) = 1$, and the Lefschetz fixed point theorem therefore implies that f has a fixed point. □

Remark 6.2. One might hope that the Brower fixed point theorem would apply directly to $\overline{\mathcal{S}_n}$. Even in the case when $n = 2$, this does not work. In fact, the space $\overline{\mathcal{S}_2}$ is represented by a triangle in the plane, and its interior is a manifold with corners and not a disc. While in this simple case the closure of the Schur region is homeomorphic to a disc, a proof in arbitrary dimensions has not yet been formulated, but the current standard methods of the Lefschetz fixed point theorem apply readily.

The tangent space of $\mathcal{Q}_n(w)$ at (a, σ) is given by the following proposition.

PROPOSITION 6.3. *For each* $w \in \mathcal{W}_n$, *the space* $\mathcal{Q}_n(w)$ *is a smooth, connected* n*-manifold with the tangent space*

$$T_{(a,\sigma)}\mathcal{Q}_n(w) = \{(u, v) \in V_{n-1} \times V_{n-1} \mid av - \sigma u = \rho;\ \deg \rho \leq n - 1\} \tag{6.2}$$

at $(a, \sigma) \in \mathcal{Q}_n(w)$. *The* n*-manifolds* $\{\mathcal{Q}_n(w) \mid w \in \mathcal{W}_n\}$ *form the leaves of a foliation of* $\mathcal{Q}_n$.

Proof. We have already established that $\mathcal{Q}_n(w)$ is a connected n-manifold, diffeomorphic to $\mathbb{R}^n$. To prove that $T_{(a,\sigma)}\mathcal{Q}_n(w)$ is given by (6.2), observe that the directional derivative

$$\begin{aligned} D_{(u,v)}\psi(a,\sigma) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\left[\frac{v}{a}-\frac{\sigma u}{a^2}\right]d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\frac{av-\sigma u}{a^2}d\theta \end{aligned}$$

is zero for $k=0,1,\dots,n$ if and only if the polynomial $\rho = av-\sigma u$ has degree at most $n-1$. In fact, $z^k\rho(z)/a(z)^2$ is analytic for $z \geq 1$ and strictly proper precisely when $\deg\rho < 2n-k$. Since the tangent space has dimension n, the rank of $\mathrm{Jac}(\psi)|_{(a,\sigma)}$ is everywhere n, and hence the connected submanifolds $\mathcal{Q}_n(w)$ form the leaves of a foliation of $\mathcal{Q}_n$. $\square$

As pointed out in the introduction, for *minimum-phase* shaping filters, there is a close relation between the cepstral coefficients and the Markov parameters of the corresponding shaping filter w. To establish these relations, make a Laurent expansion of

$$\log\Phi(z) = \log w(z) + \log w(z^{-1}) \tag{6.3}$$

on a subset Ω of the complex plane, where Ω is the intersection between an annulus containing the unit circle but none of the zeros of $w(z)$ or $w(z^{-1})$ and a sector containing the positive-real axis. The purpose of the sector is to avoid circling the origin. Then the Laurent expansion obtained from the series expansions on the corresponding segment of the real line of $\log w(z)$ and $\log w(z^{-1})$ extends to all of Ω and hence, in particular, to the arc on the unit circle contained in Ω. Then, however, the uniqueness of the Fourier transform ensures that the Laurent expansion also holds there. From this we see

$$\begin{aligned} c_0 &= 2\log w_0, \\ c_1 &= \frac{w_1}{w_0}, \\ c_2 &= \frac{w_2}{w_0} - \frac{1}{2}\left(\frac{w_1}{w_0}\right)^2, \\ c_3 &= \frac{w_3}{w_0} - \frac{1}{2}\left(2\frac{w_1}{w_0}\frac{w_2}{w_0}\right) + \frac{1}{3}\left(\frac{w_1}{w_0}\right)^3. \\ &\vdots \end{aligned}$$

Indeed, these equations form a triangular system, and hence the Markov parameters can also be obtained from the cepstral coefficients, and vice versa. Setting $w_0 = 1$, we obtain the usual normalization with $c_0 = 0$. Therefore, the nonempty submanifolds $\mathcal{Q}_n(w)\cap\mathcal{P}_n$ are precisely the leaves of the foliation $\{\mathcal{P}_n(c) \mid c \in \mathcal{C}_n\}$. In fact, let $\phi : \mathcal{P}_n \to \mathbb{R}^n$ be the restriction of ψ to $\mathcal{P}_n$, and define $\mathcal{P}_n(w) := \phi^{-1}(w)$ for each $w \in \mathcal{M}_n := \phi(\mathcal{P}_n)$. Then we have the following corollary.

Corollary 6.4. *The n-manifolds $\{\mathcal{P}_n(w) \mid w \in \mathcal{M}_n\}$ form the leaves of a foliation of $\mathcal{P}_n$, which is identical to $\{\mathcal{P}_n(c) \mid c \in \mathcal{C}_n\}$.*

In the present setting, however, we also consider nonminimum phase shaping filters, allowing σ to be an arbitrary real monic polynomial. Whereas in $\mathcal{P}_n$ there is a one-to-one correspondence between windows of cepstral coefficients and Markov parameters, this is no longer the case in $\mathcal{Q}_n$. The tangent vectors of $\mathcal{P}_n(w)$ at (a,σ)

do satisfy (4.6) of Proposition 4.2, but this does not extend to the situation where $\sigma(z)$ is no longer a Schur polynomial. Indeed, the first integral in (4.6) is not even defined when $\sigma(z)$ has a root on the unit circle. Nevertheless, we have the following lemma, which is all we need below.

LEMMA 6.5. *Any $(u,v) \in T_{(a,\sigma)}\mathcal{Q}_n(w)$ satisfies the equation*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} S(\sigma)v d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(a)u\frac{\sigma\sigma^*}{aa^*}d\theta. \tag{6.4}$$

Proof. By Proposition 6.3, the tangent space $T_{(a,\sigma)}\mathcal{Q}_n(w)$ consists of those (u,v) for which the polynomial $\rho := av - \sigma u$ has degree at most $n-1$. Since

$$v = \frac{\sigma u}{a} + \frac{\rho}{a},$$

we have

$$S(\sigma)v = S(a)u\frac{\sigma\sigma^*}{aa^*} + \sigma^*\frac{\rho}{a} + \sigma\left(\frac{\rho}{a}\right)^*. \tag{6.5}$$

However, ρ/a is strictly proper and analytic for $|z| \geq 1$, and hence it has a Laurent expansion

$$\frac{\rho}{a} = \alpha_1 z^{-1} + \alpha_2 z^{-2} + \cdots,$$

which is valid on the unit circle. Therefore,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sigma^*\frac{\rho}{a} + \sigma\left(\frac{\rho}{a}\right)^*\right]d\theta = 0, \tag{6.6}$$

and hence (6.4) follows. □

Next let $\phi : \mathcal{Q}_n \to \mathbb{R}^n$ be the map that sends (a,σ) to the vector $r \in \mathbb{R}^n$ of normalized covariance lags (4.7). Clearly, $\phi(\mathcal{Q}_n) = \mathcal{R}_n := f(\mathcal{P}_n)$. Given any $r \in \mathcal{R}_n$, define

$$\mathcal{Q}_n(r) := \phi^{-1}(r).$$

The following proposition is a $\mathcal{Q}_n$-version of Proposition 4.5, and the proof is the same mutatis mutandis.

PROPOSITION 6.6. *For each $r \in \mathcal{R}_n$, $\mathcal{Q}_n(r)$ is a smooth, connected manifold of dimension n. The tangent space $T_{(a,\sigma)}\mathcal{Q}_n(r)$ consists of those $(u,v) \in V_{n-1} \times V_{n-1}$ for which*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{S(\sigma)v}{aa^*}e^{ik\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{S(a)u}{aa^*}\frac{\sigma\sigma^*}{aa^*}e^{ik\theta}d\theta + \frac{\varphi}{2\pi}\int_{-\pi}^{\pi}\frac{\sigma\sigma^*}{aa^*}e^{ik\theta}d\theta \tag{6.7}$$

for $k = 0, 1, \ldots, n$, where

$$\varphi = \frac{1}{h_0(a,\sigma)}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{S(\sigma)v}{aa^*} - \frac{S(a)u}{aa^*}\frac{\sigma\sigma^*}{aa^*}\right]d\theta. \tag{6.8}$$

The n-manifolds $\{\mathcal{Q}_n(r) \mid r \in \mathcal{R}_n\}$ form the leaves of a foliation of $\mathcal{Q}_n$.

In the case in which $a(z)$ and $\sigma(z)$ are coprime, we can now show that if the tangent spaces $T_{(a,\sigma)}\mathcal{Q}_n(w)$ and $T_{(a,\sigma)}\mathcal{Q}_n(r)$ do intersect, they intersect transversely.

PROPOSITION 6.7. *Suppose that the polynomials $a(z)$ and $\sigma(z)$ are coprime. Then*

$$T_{(a,\sigma)}\mathcal{Q}_n(w) \cap T_{(a,\sigma)}\mathcal{Q}_n(r) = 0 \tag{6.9}$$

for any $(a, \sigma) \in \mathcal{Q}_n(w) \cap \mathcal{Q}_n(r)$.

Proof. Suppose that $(u, v) \in T_{(a,\sigma)}\mathcal{Q}_n(w) \cap T_{(a,\sigma)}\mathcal{Q}_n(r)$. Then (u, v) satisfies (6.7) for $k = 0, 1, \ldots, n$ and, by symmetry, also for $k = -1, -2, \ldots, -n$. Taking the linear combination corresponding to the coefficients of aa^*, we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} S(\sigma)v d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(a)u\frac{\sigma\sigma^*}{aa^*}d\theta + \varphi\|\sigma\|^2.$$

However, by Lemma 6.5, (u, v) also satisfies (6.4), and hence, since $\|\sigma\| > 0$, we must have $\varphi = 0$.

Consequently, $T_{(a,\sigma)}\mathcal{Q}_n(w) \cap T_{(a,\sigma)}\mathcal{Q}_n(r)$ consists of those $(u, v) \in V_{n-1} \times V_{n-1}$ which satisfy both

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{S(\sigma)v}{aa^*}e^{ik\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{S(a)u}{aa^*}\frac{\sigma\sigma^*}{aa^*}e^{ik\theta}d\theta, \quad k = 0, 1, \ldots, n, \tag{6.10}$$

and

$$av - \sigma u = \rho, \quad \deg \rho \le n - 1. \tag{6.11}$$

In view of (6.5), we have

$$\frac{S(\sigma)v}{aa^*} = \frac{S(a)u}{aa^*}\frac{\sigma\sigma^*}{aa^*} + \frac{S(a\sigma)\rho}{(aa^*)^2},$$

which, inserted into (6.10), yields

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{S(a\sigma)\rho}{(aa^*)^2}e^{ik\theta}d\theta = 0, \quad k = 0, 1, \ldots, n. \tag{6.12}$$

Clearly, there is a decomposition

$$\frac{S(a\sigma)\rho}{(aa^*)^2} = \frac{d}{a^2} + \frac{d^*}{(a^*)^2} = \frac{S(a^2)d}{(aa^*)^2}, \tag{6.13}$$

where $d(z)$ is a real polynomial of degree at most $2n$. Since $a(z)$ has all of its roots in the open unit disc, there is also a Laurent expansion

$$\frac{d(z)}{a(z)^2} = \frac{1}{2}\beta_0 + \sum_{j=1}^{\infty}\beta_j z^{-j}$$

valid on the unit circle, having real coefficients $\beta_0, \beta_1, \beta_2, \ldots$, in terms of which

$$\frac{S(a\sigma)\rho}{(aa^*)^2} = \sum_{j=-\infty}^{\infty}\beta_j e^{-ij\theta},$$

where $\beta_{-j} = \beta_j$ for all j. Inserting this into (6.12), we see that $\beta_0, \beta_1, \ldots, \beta_n = 0$, and hence the polynomial $d(z)$ has degree at most $n - 1$, precisely as $\rho(z)$ has.

Now from (6.13) we also have

$$S(a\sigma)\rho = S(a^2)d$$

or, equivalently,

$$a(\sigma^*\rho - a^*d)^* + a^*(\sigma^*\rho - a^*d) = 0.$$

Introducing the reversed polynomials $a_*(z) := z^n a(z^{-1})$ and $\sigma_*(z) := z^n\sigma(z^{-1})$, we may write this as

$$S(z^n a)(\sigma_*\rho - a_*d) = 0,$$

which is well defined since $\deg(\sigma_*\rho - a_*d) = 2n-1 < \deg(z^n a)$. Then, since the polynomial $z^n a$ has all of its roots in the open unit disc, $\ker S(z^n a) = 0$, and hence

$$\sigma_*\rho = a_*d. \tag{6.14}$$

Now, if $\rho \neq 0$,

$$\frac{d_*}{\rho_*} = \frac{\sigma}{a},$$

where $\rho_*(z) := z^{n-1}\rho(z^{-1})$ and $d_*(z) := z^{n-1}d(z^{-1})$. But this is impossible when $a(z)$ and $\sigma(z)$ are coprime because the left member is a proper rational function of degree at most $n-1$, while the right member has degree n. Hence only $\rho = d = 0$ satisfies (6.14). However, for $\rho = 0$, (6.11) has only the solution $u = v = 0$, as claimed. In fact, if $v \neq 0$,

$$\frac{v}{u} = \frac{\sigma}{a},$$

which has no solution if $a(z)$ and $\sigma(z)$ are coprime. □

Just as in Remark 4.7, this establishes that the Jacobian of the joint map $(a,\sigma) \to (r_1, r_2, \dots, r_n, w_1, w_2, \dots, w_n)$ has full rank, and, by the inverse function theorem, the joint map forms a smooth local coordinate system on $\mathcal{Q}_n^*$. This proves the first statement of Theorem 3.5.

Figure 6 illustrates the fact that the covariance foliation and the Markov foliation are everywhere transverse. Also note that the shaded region in Figure 6 is identical to Figure 5, thus illustrating Corollary 6.4.

Figure 6 also suggests that each leaf of the Markov foliation meets each leaf of the covariance matching foliation, a fact that we shall now establish in a slightly generalized form. As above, $\overline{\mathcal{Q}_n(r)}$ and $\overline{\mathcal{Q}_n(w)}$ denote the closures of the submanifolds $\mathcal{Q}_n(r)$ and $\mathcal{Q}_n(w)$, respectively.

Theorem 6.8. *The closure of every leaf of the Markov foliation intersects the closure of any leaf of the covariance matching foliation. Moreover, either the leaves themselves intersect, or every point of intersection is of the form (a,σ), where a has some roots on the unit circle and σ vanishes at each of these roots, while the ratio has the prescribed covariance and Markov windows.*

Proof. The basic space we work on is the product $\overline{\mathcal{S}_n} \times \Pi_n$. We have already seen that $\overline{\mathcal{Q}_n(w)}$ is the graph of a continuous function $\gamma : \overline{\mathcal{S}_n} \to \Pi_n$. We wish to exhibit $\overline{\mathcal{Q}_n(r)}$ as the graph of a continuous function $\delta : \Pi_n \to \overline{\mathcal{S}_n}$. Assuming this for the moment, we deduce from Theorem 6.1 that the continuous map

$$\delta \circ \gamma : \overline{\mathcal{S}_n} \to \overline{\mathcal{S}_n}$$

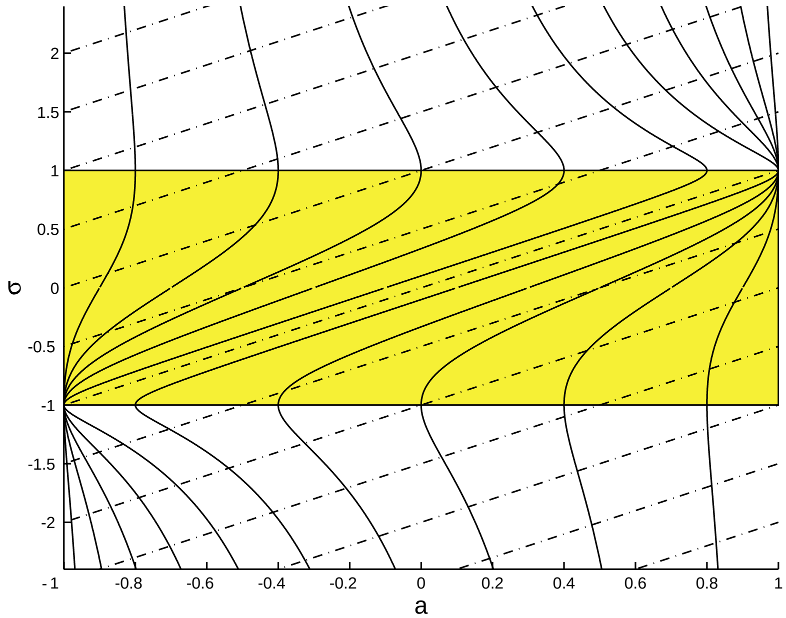

FIG. 6. *Markov (dotted line) and covariance (solid line) matching foliations of* $\mathcal{Q}_1$.

has a fixed point $\bar{a}$; i.e., $(\delta \circ \gamma)(\bar{a}) = \bar{a}$. If $\bar{\sigma} = \gamma(\bar{a})$, then $(\bar{a}, \bar{\sigma})$ is a point lying on both $\overline{\mathcal{Q}_n(w)}$ and $\overline{\mathcal{Q}_n(r)}$. To see this, note that $(\bar{a}, \bar{\sigma}) = (\bar{a}, \gamma(\bar{a}))$ by definition and that $(\bar{a}, \bar{\sigma}) = (\delta \circ \gamma(\bar{a}), \gamma(\bar{a}))$ by construction.

Therefore, it remains to construct δ. If σ is a Schur polynomial, then, according to Theorem 3.3, there exists a unique Schur polynomial a such that (a, σ) lies in $\mathcal{P}_n(r) \subset \mathcal{Q}_n(r)$. We shall write $\delta(\sigma) = a$. According to Theorem 3.4, δ is a smooth function on $\mathcal{S}_n$. Since this is crucial for what follows, we give an independent proof of Theorem 3.4, using the global analysis developed in section 4.

First, we note that the foliations $\{\mathcal{P}_n(r) \mid r \in \mathcal{R}_n\}$ and $\{\mathcal{P}_n(\sigma) \mid \sigma \in \mathcal{S}_n\}$ are complementary. To see this, we ask whether a tangent vector $(u, 0)$ to $\mathcal{P}_n(\sigma)$ at a point (a, σ) could also be tangent to the leaf $\mathcal{P}_n(r)$ through (a, σ). To this end, just as in the proof of Proposition 4.5, we first observe that (6.7) may be written as

$$Fp = Hv,$$

where $p(z) := u(z) + \frac{1}{2}\varphi a(z)$ and F, H are the linear maps (4.12). Then, substituting $(u, 0)$ into (6.7), we obtain $Fp = 0$. However, we also established in the proof of Proposition 4.5 that F is nonsingular, and hence $p = 0$, which, in turn, implies that $\varphi = 0$ and thus that $u = 0$.

Now consider the map $\eta : \mathcal{P}_n \to \mathcal{S}_n$ defined via $\eta(a, \sigma) = \sigma$. The kernel of the Jacobian of η at any point is the tangent space to $\mathcal{P}_n(\sigma)$ at that point. In particular, the kernel of the Jacobian of the map $\eta_r : \mathcal{P}_n(r) \to \mathcal{S}_n$ defined via $\eta_r(a, \sigma) = \sigma$ is zero at every point of $\mathcal{P}_n(r)$. According to Theorem 3.3, the map η_r has an inverse δ. Moreover, by the inverse function theorem, δ is smooth and hence continuous.

In [22], Georgiou proves that δ has a continuous extension to $\overline{\mathcal{S}_n}$ with a very interesting property. If σ has roots on the unit circle, $a = \delta(\sigma)$ may have roots on the unit circle, but σ must vanish at each of these roots, yielding a lower degree ratio having the prescribed covariance window. Of course, Theorem 3.3 and the constructions in [11, 22] start with the pseudopolynomial

$$d(z, z^{-1}) = \sigma(z)\sigma(z^{-1})$$

rather than σ itself. Since d is taken to be an arbitrary pseudopolynomial of degree less than or equal to zero and nonnegative on the unit circle, the continuity of δ on $\overline{\mathcal{S}_n}$ is equivalent to the continuity of δ on the larger space Π_n. This enables us to form the continuous function $\delta \circ \gamma$ on $\overline{\mathcal{S}_n}$ and apply the Lefschetz fixed point theorem, yielding the statement of the theorem. ◻

Since, according to Theorem 6.8, any intersection between $\overline{\mathcal{Q}_n(r)}$ and $\overline{\mathcal{Q}_n(w)}$ on the boundary of $\mathcal{Q}_n$ defines a pair (a, σ) of polynomials whose roots on the unit circle are common, after cancellation, $w(z) = \sigma(z)/a(z)$ has all of its poles in open unit disc and is thus a bona fide shaping filter. Consequently, Theorem 6.8 establishes the existence part of the last statement of Theorem 3.5. The uniqueness part follows from the following proposition.

PROPOSITION 6.9. *There is at most one shaping filter $w(z)$ having given windows $(1, w_1, \dots, w_n)$ and $(1, r_1, \dots, r_n)$ of normalized Markov parameters and normalized covariance lags, respectively.*

Proof. Let $w_1(z)$ and $w_2(z)$ be two shaping filters having the same window $(1, w_1, \dots, w_n)$ of normalized Markov parameters. Then, if

$$w_1(z) = \frac{\sigma_1(z)}{a_1(z)}, \qquad w_2(z) = \frac{\sigma_2(z)}{a_2(z)},$$

where (a_1, σ_1) and (a_2, σ_2) are coprime pairs of monic polynomials, the degree of the polynomial

$$\rho := \sigma_1 a_2 - \sigma_2 a_1$$

is at most $n-1$. In fact, the first n Markov parameters of

$$\frac{\sigma_1}{a_1} - \frac{\sigma_2}{a_2} = \frac{\rho}{a_1 a_2}$$

are zero.

Without restriction, we may order the shaping filters so that $\lambda_1 \geq \lambda_2$, where

$$\lambda_1 := \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sigma_1}{a_1} \right|^2 d\theta \right)^{-1}, \quad \lambda_2 := \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sigma_2}{a_2} \right|^2 d\theta \right)^{-1}.$$

Then, assuming that $w_1(z)$ and $w_2(z)$ also have the same normalized covariance lags $(1, r_1, \dots, r_n)$, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} \Psi(e^{i\theta}) d\theta = 0, \quad 0, 1, \dots, n,$$

where

$$\Psi := \lambda_1 \left| \frac{\sigma_1}{a_1} \right|^2 - \lambda_2 \left| \frac{\sigma_2}{a_2} \right|^2.$$

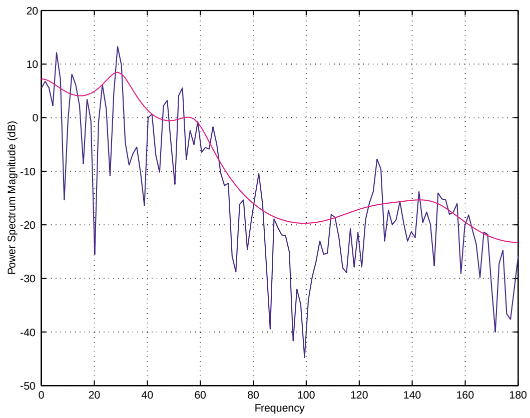

FIG. 7. *Spectral envelope of* 10*th order LPC filter.*

We want to show that $\rho = 0$. To this end, note that, in particular,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|a_2(e^{i\theta})|^2\Psi(e^{i\theta})d\theta = 0, \tag{6.15}$$

where

$$|a_2|^2\Psi = \lambda_1\frac{|\rho|^2}{|a_1|^2} + \lambda_1\frac{\sigma_2\rho^*}{a_1^*} + \lambda_1\frac{\sigma_2^*\rho}{a_1} + (\lambda_1 - \lambda_2)|\sigma_2|^2.$$

However, for the same reason as in (6.6),

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sigma_2\left(\frac{\rho}{a_1}\right)^* + \sigma_2^*\left(\frac{\rho}{a_1}\right)\right]d\theta = 0,$$

and hence (6.15) can be written as

$$\left\|\frac{\rho}{a_1}\right\|^2 + \left(1 - \frac{\lambda_2}{\lambda_1}\right)\|\sigma_2\|^2 = 0.$$

Since $\|\sigma_2\| > 0$ and $1 - \lambda_2/\lambda_1 > 0$, this implies that $\lambda_1 = \lambda_2$ and $\rho = 0$. Hence $w_1 = w_2$, as claimed. $\square$

7. Zero assignability vs. cepstral assignability. The theory derived in this paper was developed for dealing with problems encountered in applying Theorem 3.3 to the identification of speech segments. The maximum entropy solution described in section 3, often called the LPC method in the speech processing community, is a standard tool for representing the spectral envelope of speech signals [17]. Its popularity is mainly due to its low computation costs and nice matching of spectral peaks. The latter property is illustrated in Figure 7, which shows the periodogram of Figure 3 together with the spectral envelope determined by a tenth order LPC filter, based on ergodic estimates of $r_0, r_1, \dots, r_{10}$ from the data in Figure 2.

However, it is well known that this estimate of the spectral envelope may not reproduce the notches of the spectrum very well, especially for nasal sounds, where the spectra have a deep valley because of the dead end formed by the mouth. This "flatness" of the spectral envelope, illustrated by Figure 7, is one of the shortcomings of LPC filtering. It is due to the fact that the zeros of the modeling filter, being at the origin, are maximally removed from the unit circle, where the spectral density is evaluated. There is thus a need for introducing nontrivial zeros in the shaping filter.

By Theorem 3.3, to any Schur polynomial $\sigma(z)$, there is a unique shaping filter having $\sigma(z)$ as its numerator polynomial and matching the covariance window $r_0, r_1, \ldots, r_n$ in the same way as the LPC filter. In fact, there is even a convex optimization procedure, based on (5.7), to determine this shaping filter. However, this does leave us with the problem of how to choose the zeros.

It is generally agreed that a finite window (1.9) of cepstral coefficients contains more information about the zeros than does a finite window (1.8) of covariance lags. In fact, differentiate the expansion

$$\log \frac{\sigma(z)}{a(z)} = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k z^{-k},$$

obtained from (6.3), with respect to z to obtain

$$\frac{\sigma'(z)a(z) - \sigma(z)a'(z)}{\sigma(z)a(z)} = -\sum_{k=1}^{\infty} k c_k z^{-k-1}. \tag{7.1}$$

Consequently, $\{-kc_k\}$ are the Markov parameters of a filter whose poles are the original poles and zeros. Therefore, modulo deciding which are which, both the poles and the zeros can be determined from a finite number of *exact* cepstral coefficients by solving a Hankel system. In so-called homomorphic prediction, e.g., the method of Shanks [35], the zeros are estimated according to these principles once the poles have been determined using LPC analysis. Indeed, it is well known [32] that the LPC envelope has a nonuniform spectral weighting and that it matches the peaks much more accurately than the valleys, i.e., giving much better estimates of poles than zeros. While, in theory, these methods provide estimates of a shaping filter, and hence of a spectral envelope, they do not achieve covariance matching and may produce shaping filters that are neither stable nor minimum-phase. Therefore, these ad hoc methods do not as such provide an alternative to an algorithm based on Theorem 3.3, but they could provide the required zero estimates.

In this context, we suggest an alternative method for estimating the zeros: Given estimates of spectral values of a periodogram at equidistant points on the unit circle,

$$\Phi(e^{i\theta_k}), \quad k = 1, 2, \ldots, N, \tag{7.2}$$

find, by linear programming, pseudopolynomials P and Q which minimize

$$\max_k |Q(e^{i\theta_k})\hat{\Phi}(e^{i\theta_k}) - P(e^{i\theta_k})| \tag{7.3}$$

subject to the constraints that $|P(e^{i\theta_k}| \geq \epsilon$ and $|Q(e^{i\theta_k}| \geq \epsilon$ for some $\epsilon > 0$. Again, the shaping filter P/Q obtained in this way would have the same undesirable properties describe above, but we can use P as the pseudopolynomial required in the dual problem (5.7) to determine a new Q such that P/Q satisfies the covariance matching condition. In this procedure, the Q obtained via (7.3) can be used as an initial condition when applying Newton's method to solve the dual problem. For all the reasons described above, it is better to use a *cepstrally smoothed periodogram* in determining (7.2). Explicitly, the cepstral parameters are calculated from the data (1.9) using an

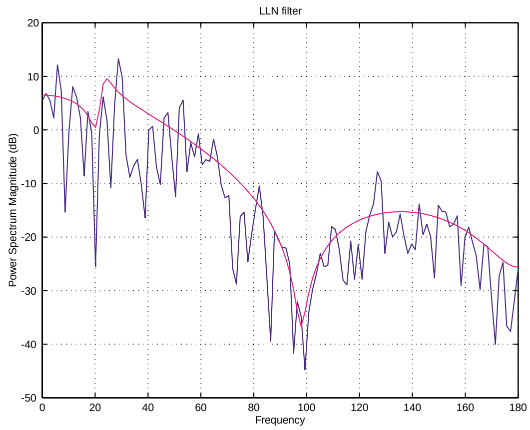

Fig. 8. *Spectral envelope of a* 6*th order LLN filter.*

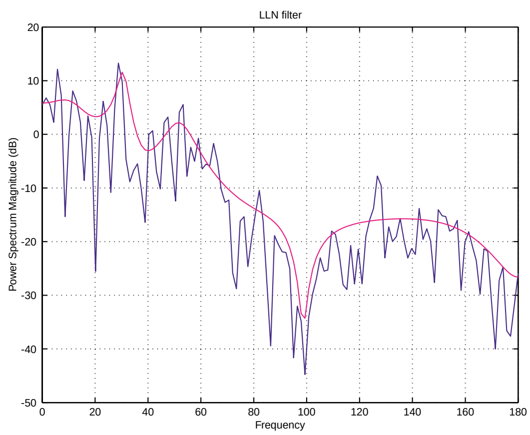

Fig. 9. *Spectral envelope of* 10*th order LLN filter.*

inverse discrete Fourier transform on the logarithm of the periodogram, after which the cepstral coefficients are windowed and inversely transformed [33, pp. 494–495]. As we have seen, the logarithm evens out the difference of energy in the valleys and the peaks and then treats valleys and peaks the same. In Figure 8, we show the spectral envelope of the signal in Figure 2 obtained from a sixth order shaping filter computed by this method. This spectral envelope should be compared with that of the tenth order LPC filter in Figure 7. Instead using a tenth order filter, we obtain the spectral envelope in Figure 9.

However, instead of matching covariance lags and zeros, we may match covariance lags and cepstral coefficients, thus applying an algorithm based on the dual problem to maximize (5.16) described in Theorem 5.3. The covariance and cepstrum interpolation problem is very appealing since both the covariances and the cepstral parameters can be estimated directly from data using ergodicity. Estimation of covariances is well analyzed (see e.g., the books [28, 36]), whereas the estimation of the cepstrum is a less studied problem. One method based on taking the discrete Fourier transform of the periodogram has been analyzed in, e.g., [20]. Using estimated covariance and cepstrum parameters, the filter depicted in Figure 10 was determined.

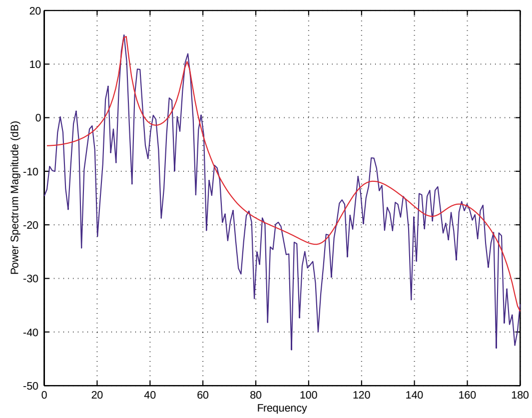

FIG. 10. *Spectral envelope of* 10*th order cepstral match filter.*

More specifically, Figure 10 shows the periodogram of a frame of speech for the phoneme [s] together with a tenth order spectral envelope produced by this method. In this case, $P \in \mathcal{D}_+$, so there is both covariance and cepstral matching. In general, however, this is not the case, as Theorem 5.3 states. This can be seen already in the case when $n = 1$. In Figure 5, the covariance matching foliation (straight lines) is depicted together with the cepstral matching foliation (curved). Clearly, a leaf in one foliation in general does not intersect all leaves in the other. Therefore, methods for determining approximate solutions in the interior $\mathcal{D}_+$ have been developed [19].

The problem that P may tend to the boundary of $\mathcal{D}$ led us to relax the stability constraint of the numerator polynomial σ and hence, in view of the bijection between cepstral and Markov parameters, prompted us to consider the simultaneous partial realization problem of section 7.

Appendix A. Divisors and polynomials. In global analysis, we shall also need to recognize spaces of real polynomials which are diffeomorphic to $\mathbb{R}^n$ as well as certain subsets of polynomials having certain properties, e.g., connectivity, in the relative topology. For this reason, we will adapt the standard treatment of divisors and elementary symmetric functions to the real case.

Let Ω be a self-conjugate, open subset of $\mathbb{C}$, which we take to be path-connected. For such an Ω we denote by $\mathcal{P}_\Omega(n)$ the space of real monic polynomials $p(z)$, of degree n, with all roots lying in Ω. Now the roots of any $p \in \mathcal{P}_\Omega(n)$ determine a self-conjugate, unordered n-tuple $(\lambda_1, \ldots, \lambda_n)$ of points $\lambda_i \in \mathbb{C}$, not necessarily distinct, known as a real *divisor* of degree n on Ω. We denote this divisor by D_p and refer to the space of such divisors as the *real symmetric product* $\Omega^{(n)}$ of Ω.

Alternatively, it is standard to construct the symmetric product $\Omega^{(n)}$ by letting the permutation group S_n on n-letters act on the ordinary Cartesian product Ω^n by permuting the coordinates of n-vectors with entries in Ω. The set of equivalence classes, or orbits of S_n, in the Cartesian product form the points in the symmetric product. In general, the real symmetric product $\Omega^{(n)}$ is always a smooth n-manifold; in fact, $\Omega^{(n)}$ is diffeomorphic to $\mathcal{P}_\Omega(n)$ using the identification

$$(\lambda_1, \ldots, \lambda_n) \to (p_1, \ldots, p_n),$$

where $p(z) = z^n + p_1 z^{n-1} + \cdots + p^n := \prod_{k=1}^n (z - \lambda_k)$. For example, we see that the real symmetric product $\Omega^{(n)}$ for $\Omega = \mathbb{C}$ is diffeomorphic to $\mathbb{R}^n$. For the unit disc,

$\mathbb{D}$, the real symmetric product is diffeomorphic to the space of real Schur polynomials, i.e., those real polynomials satisfying the Schur–Cohn conditions, while for the open left half-plane the real symmetric product is diffeomorphic to the space of those real monic polynomials satisfying the Routh–Hurwitz conditions. Each of these real symmetric products is in turn diffeomorphic with $\mathbb{R}^n$, although not via the standard correspondence given above. Indeed, if $\Omega \subset \mathbb{C}$ is a self-conjugate open subset of the Riemann sphere, with a simple, closed, rectifiable, orientable curve as boundary, then $\mathcal{P}_\Omega(\backslash)$ is diffeomorphic to $\mathbb{R}^n$. As noted in [7], this follows from the Riemann mapping theorem and the corresponding result for the open unit disc $\mathbb{D}$. For $\Omega = \mathbb{D}$ this may be explicitly represented using the real diffeomorphism T of $\mathbb{D}$ to $\mathbb{C}$, defined in polar coordinates via

$$T(r,\theta) = \left(\tan\frac{r\pi}{2}, \theta\right).$$

In general, the projection $P_n : \Omega^n \to \Omega^{(n)}$ is smooth, and any diffeomorphism $T : \Omega_1^{(n)} \to \Omega_2^{(n)}$ is induced by a unique S_n-invariant diffeomorphism $\tilde{T} : \Omega_1^n \to \Omega_2^n$. In particular, if $T : \Omega_1 \to \Omega_2$ is a diffeomorphism, then the induced map $\bar{T} : \Omega_1^{(n)} \to \Omega_2^{(n)}$ defined on divisors of degree n via

$$\bar{T}(\lambda_1, \dots, \lambda_n) = (T(\lambda_1), \dots, T(\lambda_n))$$

is a diffeomorphism. In particular, $\mathcal{P}_\mathbb{D}(n)$ is diffeomorphic with $\mathcal{P}_\mathbb{C}(n)$, which is diffeomorphic to $\mathbb{R}^n$.

Appendix B. Calculation of cepstral coefficients. Suppose

$$\Phi(e^{i\theta}) = \rho^2 \left|\frac{\sigma(e^{i\theta})}{a(e^{i\theta})}\right|^2,$$

where

$$a(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$

and

$$\sigma(z) = z^n + \sigma_1 z^{n-1} + \cdots + \sigma_n$$

are Schur polynomials, i.e., have all of their roots in the open unit disc, and ρ is a real number. Then the cepstral coefficients, i.e., the Fourier coefficients in the expansion (1.5), are given by

$$\begin{aligned} c_0 &= 2\log\rho, \\ c_k &= \frac{1}{k}\{s_k(a) - s_k(\sigma)\}, \qquad k = 1, 2, 3, \dots, \end{aligned}$$

where

$$\begin{aligned} s_k(a) &= p_1^k + p_1^k + \cdots + p_n^k, \\ s_k(\sigma) &= z_1^k + z_1^k + \cdots + z_n^k, \end{aligned}$$

in which $p_1, p_2, \dots, p_n$ are the roots of $a(z)$ and $z_1, z_2, \dots, z_n$ are the roots of $\sigma(z)$.

For the case of maximal entropy (or LPC) filters, we have $z_i = 0$, and the above formula is well known. For pole-zero models, this formula is, to the best of our knowledge, new but straightforward to derive using the basic algebraic properties of the logarithm.

Moreover, using Newton's identities [15, p. 5], one derives the following recursions:

$$s_k(a) = -ka_k - \sum_{j=1}^{k-1} a_{k-j} s_j(a),$$

$$s_k(\sigma) = -k\sigma_k - \sum_{j=1}^{k-1} \sigma_{k-j} s_j(\sigma).$$

These equations also hold for $k > n$ provided we set $a_k = 0$ and $\sigma_k = 0$ whenever $k > n$.

Appendix C. Connectivity of $\mathcal{P}_n(c)$. We also need to know about various coordinates on $\mathcal{P}_\Omega(n)$ and hence about C^∞ functions. If $f : \Omega^{(n)} \to \mathbb{R}$ is C^∞, then f lifts to a C^∞ function on Ω^n which is S_n-invariant, and, conversely, any C^∞ function on Ω^n which is S_n-invariant descends to a C^∞ function defined on $\Omega^{(n)}$. We denote the algebra of C^∞ functions on $\Omega^{(n)}$ by $\mathcal{C}^\infty[\Omega^{(n)}]$ and the algebra of S_n-invariant C^∞ functions on Ω^n by $\mathcal{C}^\infty[\Omega^n]^{S_n}$. In light of the remarks made above, $\mathcal{C}^\infty[\Omega^{(n)}]$ is canonically isomorphic to $\mathcal{C}^\infty[\Omega^n]^{S_n}$.

Whenever a real diffeomorphism M maps such a domain Ω_1 onto such a domain Ω_2, M commutes with the actions of S_n on Ω_1^n and on Ω_2^n, so that composition with M induces an isomorphism between $\mathcal{C}^\infty[\Omega_2^n]^{S_n}$ and $\mathcal{C}^\infty[\Omega_1^n]^{S_n}$ and hence between $\mathcal{C}^\infty[\Omega_2^{(n)}]$ and $\mathcal{C}^\infty[\Omega_1^{(n)}]$. Therefore, composition with M^{-1} will map generators of $\mathcal{C}^\infty[\Omega_1^n]^{S_n}$ to generators of $\mathcal{C}^\infty[\Omega_2^{(n)}]$.

As an example, consider $\Omega = \mathbb{C}$. Then the algebra of S_n-invariant real polynomials is generated by the coefficients p_i of the polynomials $p(z)$, treated as the points of the real symmetric product. We denote this by writing

$$\mathcal{C}^\infty[\mathbb{C}^{(n)}] = \mathcal{C}^\infty[p_1, \ldots, p_n].$$

Any diffeomorphism of $\mathbb{R}^n$ with itself will give another set of n generators, and, conversely, any other choice of n generators will define a diffeomorphism. Indeed, consider the self-conjugate polynomials in λ,

$$s_k(\lambda) = \lambda_1^k + \cdots + \lambda_n^k,$$

which are invariant under the action of S_n on the n-fold Cartesian product of $\mathbb{C}$. Each $s_k(\lambda)$ lies in $\mathcal{C}^\infty[\mathbb{C}^{(n)}]$ and is in fact a real polynomial in $(\lambda_1, \ldots, \lambda_n)$, as described by the Newton identities [15, p. 5]

$$s_k(\lambda) = -k\lambda_k - \sum_{j=1}^{k-1} \lambda_{k-j} s_j(\lambda),$$

where we set $a_k = 0$ and $\sigma_k = 0$ whenever $k > n$.

Conversely, the Newton identities also show that the λ_i are real polynomials in the s_k, and so we may write

$$\mathcal{C}^\infty[\mathbb{C}^{(n)}] = \mathcal{C}^\infty[\lambda_1, \ldots, \lambda_n].$$

To put this another way, the functions s_k form a system of smooth coordinates on the real Euclidean n-space, $\mathbb{C}^{(n)}$.

If

$$\tilde{c}_k(\tilde{T}(a), \tilde{T}(\sigma)) = c_k(\sigma, a),$$

then the functions $\tilde{c}_k$ form a set of generators for $\mathcal{C}^\infty[\mathbb{C}^{(n)}]$. In particular, in these coordinates, the sets are affine planes and are hence connected.

LEMMA C.1. *The submanifolds $\mathcal{P}_n(c)$ are connected.*

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