ISOTRIVIALITY AND THE SPACE OF MORPHISMS ON PROJECTIVE VARIETIES

ANUPAM BHATNAGAR AND ALON LEVY

Abstract. Let $K = k(C)$ be the function field of a smooth projective curve C over an infinite field k, let X be a projective variety over k. We prove two results. First, we show with some conditions that a K-morphism $\phi : X_K \to X_K$ of degree at least two is isotrivial if and only if ϕ has potential good reduction at all places v of K. Second, let $(X, \phi), (Y, \psi)$ be dynamical systems where X, Y are defined over k and $g : X_K \to Y_K$ a dominant K-morphism, such that $g \circ \phi = \psi \circ g$. We show under certain conditions that if ϕ is defined over k, then ψ is defined over a finite extension of k.

1. Introduction

Let C be a smooth projective curve over an infinite field k and let $K = k(C)$ denote the function field of C. In [12] the authors prove a criterion for isotriviality of endomorphisms on $\mathbb{P}^n_K$ in terms of good reduction at every place of K. In this paper we generalize this criterion to projective K-varieties. To state our result we need a few definitions.

Definition 1.1. A projective K-variety X is *trivial* if it is defined over the constant field k and X is *isotrivial* if there exists a finite extension K'/K such that $X_{K'} := X \times_{\operatorname{Spec}(K)} \operatorname{Spec}(K')$ is trivial. Now let X be a projective k-variety. A morphism $\phi : X_K \to X_K$ is *trivial* if it is defined over the constant field k and it is *isotrivial* if there exists a finite extension K'/K such that the induced morphism $\phi' : X_{K'} \to X_{K'}$ is defined over k', the algebraic closure of k in K'.

Let $\mathcal{X}$ be a projective model of X_K over C and for some non-empty open subset U of C, let $\mathcal{X}_U := \mathcal{X} \times_C U$. Let M_K denote the places of K and identify the places of K with the closed points of C.

Definition 1.2. The morphism $\phi : X_K \to X_K$ has *good reduction* at v if there exists an open subset $U \subset C$ containing v such that ϕ extends to a morphism on $\mathcal{X}_U$. We say ϕ has *potential good reduction* at v if there exists a finite extension K' of K and a place v' of K' over v such that ϕ' has good reduction at v'.

Definition 1.3. Let X be a projective variety and $\phi : X \to X$ a morphism. The dynamical system (X, ϕ) is said to be *polarized* if for some ample line bundle $\mathcal{L}$ on X, we have $\phi^*\mathcal{L} \cong \mathcal{L}^{\otimes q}$ with $q > 1$.

In §2 we prove our first result, which states:

Corollary 2.6. *Let $K = k(C)$ be the function field of a smooth projective curve C defined over an infinite field k, X a connected projective k-variety, $\mathcal{L}$ an ample line bundle on X, $\phi : X_K \to X_K$ a polarized K-morphism. Assume that the subgroup of automorphisms preserving the line bundle class*

$$\operatorname{Aut}(X, \mathcal{L}) := \{\tau \in \operatorname{Aut}(X) \mid \tau^*\mathcal{L} \cong \mathcal{L}\}$$

is reductive. Then ϕ is isotrivial if and only if ϕ has potential good reduction at all places v of K.

The proof of the theorem uses geometric invariant theory(GIT) and our approach is similar to [12] (Thm. 1). The condition that $\operatorname{Aut}(X, \mathcal{L})$ is reductive is nontrivial; there exist varieties for which $\operatorname{Aut}(X, \mathcal{L})$ is non-reductive that nonetheless admit polarized endomorphisms. The example with principally polarized abelian varieties in §1 of [12] shows that one cannot have a result similar to Corollary 2.6 for an arbitrary K-variety.

In §3 we prove our next result which arose from the following questions we learned from L. Szpiro and T. Tucker. To proceed we need some definitions.

Definition 1.4. Let $(X, \phi), (Y, \psi)$ be dynamical systems polarized with respect to $\mathcal{L}, \mathcal{M}$ respectively. We say (X, ϕ) *dominates* (Y, ψ) if there exists a dominant morphism $g : X \to Y$ such that:

(1) $g \circ \phi = \psi \circ g$, and
(2) $g^*\mathcal{M} = \mathcal{L}$

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{\psi} & Y \end{array}$$

Szpiro and Tucker asked the following:

(1) Let X, Y be projective K-varieties and $g : X \to Y$ a K-morphism. If X is isotrivial, does it imply that Y is isotrivial?
(2) Let $(X, \phi), (Y, \psi)$ be dynamical systems where the projective K-varieties X, Y are isotrivial and $g : X \to Y$ a dominant (or surjective) K-morphism such that $g \circ \phi = \psi \circ g$. If ϕ is isotrivial, does it imply that ψ is isotrivial?

In [2] the first author answered the first question affirmatively whenever $g_*\mathcal{O}_X = \mathcal{O}_Y$. Our strategy to answer the second question in certain cases is to show that at each place of K the reduction of ψ has degree $d^{\dim Y}$. This forces reduction of ψ to be well defined at every point. Let us denote reduction of some model of ψ at the place v by $\tilde{\psi}_v$. If ψ has potential good reduction at every place of K there is nothing to show. Otherwise, we first prove:

Theorem 3.2. *Let $(X, \phi), (Y, \psi)$ be dynamical systems over K polarized with respect to $\mathcal{L}, \mathcal{M}$ respectively, where X, Y are connected. Assume that X, Y, ϕ are isotrivial, $\mathcal{L}, \mathcal{M}$ are defined over k, (X, ϕ) dominates (Y, ψ) and one of the following conditions holds:*

(1) $\dim X = \dim \tilde{g}_v(X)$ *and $\tilde{g}_v$ is dominant.*
(2) *For some n there exists a real number $d > 0$ such that the action of $(\phi^n)^*$ on* $\operatorname{Pic} X$ *is multiplication by d^n.*
(3) $Y = \mathbb{P}^1_K$.

Then the rational map $\tilde{\psi}_v |_{\tilde{g}_v(X)} : \tilde{g}_v(X) \to \tilde{g}_v(X)$ is a morphism.

Now we have the following corollaries:

Corollary 3.4. *Let $(X, \phi), (Y, \psi)$ be dynamical systems as in the previous theorem. Suppose one of the conditions of Theorem 3.2 holds, $\tilde{g}_v$ is dominant at every place of K, and* $\operatorname{Aut}(Y, \mathcal{M})$ *is reductive. Then ψ is isotrivial.*

Definition 1.5. We say a non-empty proper subset $E \subset X$ is an *exceptional set for ϕ* if $\phi^{-1}(E) = E = \phi(E)$.

Corollary 3.7. *Let $(X, \phi), (Y, \psi)$ be dynamical systems as in the previous theorem. Suppose one of the conditions of Theorem 3.2 holds,* $\operatorname{Aut}(Y, \mathcal{M})$ *is reductive and ϕ has no non-empty proper exceptional subvarieties. Then ψ is isotrivial.*

In the case $Y = \mathbb{P}^1_K$ we prove a stronger result using geometric invariant theory. More precisely we prove,

Corollary 3.12. *Let $(X, \phi), (\mathbb{P}^1_K, \psi)$ be dynamical systems polarized w.r.t. $\mathcal{L}, \mathcal{O}(1)$ respectively. Assume that X is connected and (X, ϕ) dominates $(\mathbb{P}^1, \psi)$. Then ψ is isotrivial.*

2. Moduli Space of Polarized Dynamical Systems

Let k be an infinite field and $\mathrm{Hom}_{n,d}$ be the space of endomorphisms of $\mathbb{P}^n_k$ of degree d^n. By §3.2 of [12], $\mathrm{Hom}_{n,d}$ is an affine open subvariety of $\mathbb{P}(\mathrm{Sym}^d((k^{n+1})^{n+1}))$, where $\mathrm{Sym}^d((k^{n+1})^{n+1})$ is the space of homogeneous maps $k^{n+1} \to k^{n+1}$ of degree d.

Let X be a connected, projective variety, and let (X, ϕ) be dynamical system over k polarized w.r.t. $\mathcal{L}$. By a result of Fakhruddin [5] (Thm. 2.1), any dynamical system (X, ϕ) polarized w.r.t. $\mathcal{L}$ extends to a dynamical system on projective space $(\mathbb{P}^n_k, \psi)$ polarized w.r.t. $\mathcal{O}(1)$. More precisely, we can find some $s \geq 1$ such that there exists a morphism ψ on $\mathbb{P}^n$ making the following diagram commute

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle i_{\mathcal{L}^{\otimes s}}} & & \downarrow{\scriptstyle i_{\mathcal{L}^{\otimes s}}} \\ \mathbb{P}^n & \xrightarrow{\psi} & \mathbb{P}^n \end{array} \tag{1}$$

In general, ψ will not be unique. So we define the space of endomorphisms of X as the quotient:

$$\mathrm{Hom}_d(X, \mathcal{L}) := \{\psi \in \mathrm{Hom}_{n,d} : \psi(X) = X\} / \{\psi \sim \psi' \iff \psi|_X = \psi'|_X\}$$

as the analog of $\mathrm{Hom}_{n,d}$ for $\mathbb{P}^n$.

Remark 2.1. The construction of $\mathrm{Hom}_d(X, \mathcal{L})$ is similar to the construction of the space of polynomials in one dimension: the automorphism group we need to mod out by is the upper triangular group Γ, but we can instead embed the space of polynomials inside the space of rational maps, denoted by Rat_d and then study it as a subspace of the quotient Rat_d / Γ. Of course we cannot always mod out by a non-reductive group and expect a nice quotient, but the space of polynomials in one variable is sufficiently well-behaved: namely, we can consider *centered* polynomials i.e. those whose z^{d-1} coefficient is zero, and are acted on by a finite group of automorphisms. There is a natural bijection of sets between polynomials modulo the upper triangular group and centered polynomials modulo the diagonal group.

We also obtain (*a priori* merely set-theoretic) quotients by the conjugation action of automorphism group of X associated to the line bundle class $\mathcal{L}$. Observe that the full group $\mathrm{Aut}(X)$ does not act on $\mathrm{Hom}_d(X, \mathcal{L})$, because an element of $\mathrm{Aut}(X)$ may map the line bundle class $\mathcal{L}$ to a different class; e.g. if X is an elliptic curve, then the translation group does not fix any ample class. Thus we only quotient by the subgroup of automorphisms preserving the line bundle class i.e.

$$\mathrm{Aut}(X, \mathcal{L}) := \{\tau \in \mathrm{Aut}(X) \mid \tau^* \mathcal{L} \cong \mathcal{L}\}$$

To extend the GIT results in [8], [12] to projective varieties, we need to show that

$$\mathrm{M}_d(X, \mathcal{L}) := \mathrm{Hom}_d(X, \mathcal{L}) / \mathrm{Aut}(X, \mathcal{L})$$

is an affine geometric quotient (in the sense of [10]). We prove this when $\mathrm{Aut}(X, \mathcal{L})$ is reductive. To see that we can use GIT tools, we begin by proving the following result.

Lemma 2.2. The space of endomorphisms of a polarized dynamical system, denoted by $\mathrm{Hom}_d(X, \mathcal{L})$ is an affine scheme.

Proof. Identify X with its image in $\mathbb{P}^n$ under some representative $L \in [\mathcal{L}^{\otimes s}]$. We restrict our attention to embeddings in $\Gamma(X, \mathcal{L}^{\otimes s})$ such that the image of X is equal to its image under L. In other words, we are requiring the image of X to be a particular subvariety of $\mathbb{P}^n$ (which is isomorphic to X), which is a closed condition. Hence the space $\mathrm{Hom}_d(X, \mathcal{L})$ arises as a closed subvariety of the space $\Gamma(X, \mathcal{L}^{\otimes s})$ for some integer $s > 0$. Let X' be the Zariski closure of the pullback of X to $\mathbb{A}^{n+1} \smallsetminus \{0\}$ i.e.

$$X' = \overline{X \times_{\mathbb{P}^n} (\mathbb{A}^{n+1} \smallsetminus \{0\})}.$$

A homogeneous map from X' to $\mathbb{A}^{n+1}$ fails to induce a morphism from X to $\mathbb{P}^n$ if there is at least one line through the origin in X' that is mapped to the origin in $\mathbb{A}^{n+1}$, in which case we say it is *ill-defined* at the associated point of $\mathbb{P}^n$. Consider the space $\Gamma(X, \mathcal{L}^{\otimes s}) \times X'$, which parametrizes homogeneous maps from X' to $\mathbb{A}^{n+1} \smallsetminus \{0\}$ along with a point. Let V be the subvariety defined by

$$V = \{(\psi, x) : \psi(x) = 0\}$$

i.e. the parameter space of maps that are ill-defined at some point. Now V is closed. Moreover, for each $x \in X'$ there are $n + 1$ sections that we equate to 0. Within the space of rational maps from X to X the sections define a point on $\Gamma(X, \mathcal{L}^{\otimes s})$, so the condition of those sections being all zero is actually of pure codimension at most $\dim X' = \dim X + 1$.

Let π_1 be the projection from $\Gamma(X, \mathcal{L}^{\otimes s}) \times X'$ on the first factor. We now compute the codimension of V in $\Gamma(X, \mathcal{L}^{\otimes s}) \times X'$. Now, V is defined as the zero set of $\dim \Gamma(X, \mathcal{L}^{\otimes s})$ many polynomials. However, those polynomials are not independent conditions; we only have at most $\dim X'$ many independent conditions. This is because every $\psi \in \Gamma(X, \mathcal{L}^{\otimes s})$ maps X' to a subvariety of $\mathbb{P}^n$ that is isomorphic to a quotient of X'; within *every* subvariety of $\Gamma(X, \mathcal{L}^{\otimes s})$ defined by what the image of X' is (regardless of whether it is the same as in $\mathrm{Hom}_d(X, \mathcal{L})$ or not) and therefore it takes only $\dim X'$ many conditions on ψ to map a given $x \in X'$ to $0 \in \mathbb{A}^{n+1}$. A key fact is that $0 \in \mathbb{A}^{n+1}$ passes through all of these subvarieties, since ψ is by definition homogeneous. Moreover, every component of V has codimension at most $\dim X'$ since it arises as the intersection of $\dim X'$ many independent conditions. So $\dim V \geq \dim \Gamma(X, \mathcal{L}^{\otimes s})$, and this is true of every component of V.

Now, $\pi_1(V)$ has dimension at least $\Gamma(X, \mathcal{L}^{\otimes s}) - 1$, and this is again true of every component of $\pi_1(V)$. It is not possible for V to have any components on which a map is ill-defined at a locus U containing more than one point, because if $x \in U$ then maps that are ill-defined at x are generically only ill-defined at x. Therefore, the codimension of $\pi_1(V)$ in $\Gamma(X, \mathcal{L}^{\otimes s})$ is purely at most 1. If there exist morphisms from X to itself then this codimension is in fact 1 and is again pure. But now we have shown that $\mathrm{Hom}_d(X, \mathcal{L})$ is (a closed subvariety of) the complement of a closed codimension one subvariety of a projective variety, so it is affine.

□

Recall that the stabilizer of a point $\phi \in \mathrm{Hom}_d(X, \mathcal{L})$ under the action of $\mathrm{Aut}(X, \mathcal{L})$ is the subgroup $\mathrm{Stab}(\phi) = \{\gamma \in \mathrm{Aut}(X, \mathcal{L}) \mid \gamma^{-1}\phi\gamma = \phi\}$. Note that if $P \in X$ is ϕ-preperiodic for some integers $l > m \geq 1$, then $\gamma(P)$ is ϕ-preperiodic for l, m as well. We use this fact in the next lemma.

Lemma 2.3. For any $\phi \in \mathrm{Hom}_d(X, \mathcal{L})$ the stabilizer group, $\mathrm{Stab}(\phi)$ is finite under the action of $\mathrm{Aut}(X, \mathcal{L})$.

Proof. We follow the argument in [12] (Prop. 8). By [5] (Thm. 5.1), X has a dense set of preperiodic points for each ϕ. We say $x \in X$ has preperiod-(l, m) if $\phi^{l+m}(x) = \phi^m(x)$. By [12] (Prop. 2) the set of points of preperiod-(l, m) is finite. In particular, there exists a set S of $n + 2$ preperiodic points that span $\mathbb{P}^n$ having preperiod at most (l, m). Each $f \in \mathrm{Stab}(\phi)$ will act on the finite set of preperiod-(l, m) points, and moreover if f acts trivially then it fixes the spanning set S and is therefore the identity. Thus $\mathrm{Stab}(\phi)$ embeds into a finite group. □

Lemma 2.4. The action of $\mathrm{Aut}(X, \mathcal{L})$ on $\mathrm{Hom}_d(X, \mathcal{L})$ is closed.

Proof. Following the argument in [10] (p.10), if for each $\phi \in \mathrm{Hom}_d(X, \mathcal{L})$ there exists an open neighborhood U of ϕ where the dimension of the stabilizer is constant for all $\psi \in U$, then the action of $\mathrm{Aut}(X, \mathcal{L})$ on $\mathrm{Hom}_d(X, \mathcal{L})$ is closed. Since $\mathrm{Stab}(\phi)$ is finite for all $\phi \in \mathrm{Hom}_d(X, \mathcal{L})$, the action of $\mathrm{Aut}(X, \mathcal{L})$ on $\mathrm{Hom}_d(X, \mathcal{L})$ is closed. □

Theorem 2.5. Let X be a connected projective variety over an infinite field k and let $\mathcal{L}$ be an ample line bundle on X. Then the scheme $\mathrm{M}_d(X,\mathcal{L}) = \mathrm{Hom}_d(X,\mathcal{L})/\mathrm{Aut}(X,\mathcal{L})$ is an affine geometric quotient whenever $\mathrm{Aut}(X,\mathcal{L})$ is reductive.

Proof. By the previous lemma the action of $\mathrm{Aut}(X,\mathcal{L})$ on $\mathrm{Hom}_d(X,\mathcal{L})$ is closed and $\mathrm{Aut}(X,\mathcal{L})$ is reductive by assumption. By Amplification 1.3 of [10] (p.30), $\mathrm{Hom}_d(X,\mathcal{L})/\mathrm{Aut}(X,\mathcal{L})$ is an affine geometric quotient. Moreover it is equal to the spectrum of $\Gamma(\mathrm{Hom}_d(X,\mathcal{L}),\mathcal{O}_{\mathrm{Hom}_d(X,\mathcal{L})})^{\mathrm{Aut}(X,\mathcal{L})}$. □

As a corollary we obtain the following:

Corollary 2.6. Let $K = k(C)$ be the function field of a smooth projective curve C defined over an infinite field k, X a connected projective k-variety, $\mathcal{L}$ an ample line bundle on X, and $\phi : X_K \to X_K$ a polarized K-morphism. Assume that the subgroup of automorphisms preserving the line bundle class

$$\mathrm{Aut}(X,\mathcal{L}) := \{\tau \in \mathrm{Aut}(X) \mid \tau^*\mathcal{L} \cong \mathcal{L}\}$$

is reductive. Then ϕ is isotrivial if and only if ϕ has potential good reduction at all places v of K.

Proof. The only if direction is clear. For the if direction, we imitate the proof in [12] to create a morphism from the complete curve C to the affine variety $\mathrm{M}_d(X,\mathcal{L})$, which must be constant.

Specifically, assume that $\phi : X_K \to X_K$ has potential good reduction at every place of K. It already has good reduction at all but finitely many places of K, and at each of the remaining places, we only need to take a finite extension of K to obtain good reduction. Both the potential good reduction and the isotriviality conditions are stable under taking finite extensions of K, so we may assume that we have everywhere good reduction. Similarly, we may replace k with $\overline{k}$, since both conditions are stable under extending the base field.

Let v be any place of K i.e. $v \in C$ is a closed point. We can reduce ϕ mod the maximal ideal $\mathfrak{m}_v$. More specifically, let us consider $\Phi : X_{\mathcal{O}_K} \to X_{\mathcal{O}_K}$, an $\mathcal{O}_K$-integral model of ϕ. It has good reduction at all but finitely many places of K, and therefore, there are finitely many Φ_i, all of which are $\mathcal{O}_K$-integral models of ϕ, such that for every $v \in C$, we have at least one index i for which Φ_i has good reduction mod v. For each Φ_i, we have a reduction map $H_i : C \dashrightarrow \mathrm{Hom}_d(X,\mathcal{L})(k)$, $v \mapsto \Phi_{i,v}$, where H_i is a rational map defined for all places of K at which Φ_i has good reduction. We can moreover compose with the reduction map $\pi : \mathrm{Hom}_d(X,\mathcal{L}) \to \mathrm{M}_d(X,\mathcal{L})(k)$, and obtain finitely many maps $\pi \circ H_i$.

Now, the maps $\pi \circ H_i$ can be glued together. Suppose that Φ_i and Φ_j both have good reduction mod v, then there is some $A \in \mathrm{Aut}(X,\mathcal{L})(K)$ such that $A\Phi_i A^{-1} = \Phi_j$, and since both Φ_i and Φ_j have good reduction mod v, the automorphism A must also have good reduction mod v. This can be seen readily if $X = \mathbb{P}^n$ (see Proposition 6 of [12] for more detail); for general X, we may embed X into $\mathbb{P}^n$ and $\mathrm{Aut}(X,\mathcal{L})$ into PGL_{n+1}. We can now take reductions of both sides mod v and see that $A_v\Phi_{i,v}A_v^{-1} = \Phi_{j,v}$, or in other words $\Phi_{i,v}$ and $\Phi_{j,v}$ map to the same point in $\mathrm{M}_d(X,\mathcal{L})(k)$.

Since we can glue the maps $\pi \circ H_i$ on the intersections of their domains, we obtain a single morphism $H : C \to \mathrm{M}_d(X,\mathcal{L})$. This is a morphism from a complete variety to an affine variety, so its image is a point $[\psi] \in \mathrm{M}_d(X,\mathcal{L})(k)$, where $\psi \in \mathrm{Hom}_d(X,\mathcal{L})$. But now ϕ is K-conjugate to the base extension $\psi : X_K \to X_K$, i.e. ϕ is isotrivial. □

We now give an example which shows that the condition $\mathrm{Aut}(X,\mathcal{L})$ being reductive is non-trivial.

Example 2.7. Let X be the blowup of $\mathbb{P}^2$ at a single point x. Note that X is Fano (i.e. $-K_X$ is ample) and every automorphism of X fixes an ample class, namely the anticanonical class. Moreover there is a unique exceptional curve on X and every automorphism of X fixes it, thus every automorphism of X arises from an automorphism of $\mathbb{P}^2$ fixing x. Clearly the group of automorphisms of $\mathbb{P}^2$ fixing x is not reductive since the unipotent radical is non-trivial. In particular, the unipotent radical is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & * & 1 \end{pmatrix}$$

Remark 2.8. Let X be as above. We show that polarized endomorphisms of X exist. For example, any morphism of $\mathbb{P}^2$ which leaves x totally invariant. In fact, these are the only polarized morphisms of X: since the only pair of irreducible curves with negative intersection is the exceptional curve intersected with itself, the preimage of the exceptional curve must contain the exceptional curve, i.e. the forward image of the exceptional curve is itself. Now take a curve that does not intersect the exceptional curve; its preimage will also not intersect the exceptional curve, but will intersect any other curve on X, and so the preimage of the exceptional curve cannot contain anything except the exceptional curve. This means we can descend every morphism on X to a morphism on $\mathbb{P}^2$, and then we can construct the space of morphisms on X as the subvariety of morphisms on $\mathbb{P}^2$ leaving a point totally invariant.

In fact, the argument in Example 2.7 can be extended to many other varieties, as long as we allow iterates of ϕ.

Proposition 2.9. Let X be a connected, smooth projective variety, let U be the ample cone in $\mathrm{NS}(X) \otimes \mathbb{R}$. If $\overline{U}$ is polyhedral, then every polarized morphism has an iterate that acts on $\mathrm{NS}(X)$ as scalar multiplication; the minimal iterate we need to take depends only on U.

Proof. Let ϕ be a polarized morphism on X, and consider the action of ϕ^* on $\mathrm{NS}(X) \otimes \mathbb{R}$. The action is a linear map preserving U, hence also $\overline{U}$. If $\overline{U}$ is polyhedral then ϕ^* acts on the faces of $\overline{U}$; since there are finitely many faces, some iterate $(\phi^n)^*$ maps each such face to itself. Now we can intersect faces to obtain lines, which will also be preserved. Since the span of U is all of $\mathrm{NS}(X) \otimes \mathbb{R}$, and U is open, there exists a basis for $\mathrm{NS}(X) \otimes \mathbb{R}$ (since $\mathrm{NS}(X) \otimes \mathbb{R}$ is a finite dimensional vector space) with the property that each basis element lies on a ray in $\overline{U}$ that is an eigenvector for $(\phi^n)^*$, and also $u \in U$ only if u is a strictly positive linear combination of those elements. But now there exists a $u \in U$ that is an eigenvector for $(\phi^n)^*$, whence all of the basis eigenvectors have the same eigenvalue and the action is scalar. □

All Fano varieties have polyhedral ample cones [4], as do Mori dream spaces [3], such as blowups of $\mathbb{P}^n$ at sufficiently general points. There exist Fano varieties that have non-reductive automorphism groups, even without being blowups. For examples we refer the reader to [11] (Thm. 1.4, 1.5). On a more positive side, an arbitrary variety will not have any polarized morphisms. In particular, a compact complex manifold of general type (i.e. with maximal Kodaira dimension) does not admit any endomorphisms of degree greater than 1; see [7] (Prop. 10.10). As a result of the difficulty of finding endomorphisms of degree greater than 1 on arbitrary projective varieties, we do not have a counterexample of a variety X with a non-reductive automorphism group on which Lemma 2.3 or Corollary 2.6 is false.

In particular, if X is a blowup of $\mathbb{P}^n$ with polyhedral ample cone, then after choosing a suitable m, for every $\phi : X \to X$ the map ϕ^m fixes each exceptional divisor, and we can descend it to a morphism on $\mathbb{P}^n$ for which all of the blown up points are totally invariant.

Remark 2.10. The ample cone is not always polyhedral. For example, if E is an elliptic curve without complex multiplication, then the ample cone of $E \times E$ has a quadratic condition [1].

3. Descent of Morphisms on Projective Varieties

Throughout this section we have the following setup: Let C be a smooth projective curve defined over an infinite field k and let $K = k(C)$ be the function field of C. Let $(X, \phi), (Y, \psi)$ be dynamical systems over K polarized with respect to $\mathcal{L}, \mathcal{M}$ respectively, with X, Y connected. The varieties X, Y and the morphism ϕ are isotrivial, $\mathcal{L}, \mathcal{M}$ are defined over k, and (X, ϕ) dominates (Y, ψ).

Lemma 3.1. Let X be a smooth projective irreducible variety of dimension n, and let $\phi : X \dashrightarrow X$ be a rational map that is polarized with polarization degree d. In other words, suppose that $X \hookrightarrow \mathbb{P}^n$ and there exists $\psi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, defined by polynomials of degree d, such that $\psi(X) \subseteq X$ and $\psi|_X = \phi$. Suppose that the topological degree of ϕ is d^n i.e. a generic points has d^n pre-images. Then ϕ is a morphism.

Proof. We pass from space of maps to space of graphs of maps. This trick was used in private communication by R. Pandharipande in response to J. Silverman [13] to study spaces of maps on $\mathbb{P}^1$, and here we extend it to general X. We have a natural map Φ from $\mathrm{Hom}_d(X, \mathcal{L})$ to the space of dim X-cycles on $X \times X$, sending each morphism ϕ to its graph Γ_ϕ; Φ also extends to polarized rational maps, which occur as restrictions of rational maps on $\mathbb{P}^n$ (or, alternatively, as bad reductions of morphisms defined over function fields).

The image of Φ consists of cycles whose intersection number with the vertical divisor class $\{\text{point}\} \times X$ is 1, and whose intersection number with the horizontal divisor class $X \times \{\text{point}\}$ is d^n.

Observe that rational maps at the boundary of $\mathrm{Hom}_d(X, \mathcal{M})$, for any very ample divisor class $\mathcal{M}$ on X, also have graphs. Since they are not morphisms, the intersection with most but not all vertical divisors will be a single point.

Now, we have a simple test using graphs to check if a rational map is a morphism. The rational map ϕ is ill-defined at x, and cannot be extended to have a domain that includes x, if and only if Γ_ϕ has a vertical component at x. To see why, let us look at $\mathbb{P}^n$ and descend this to X using the fact that ϕ is polarized. In $\mathbb{P}^n$, if $\psi(x) = (\psi_0(x) : \ldots : \psi_n(x))$ and $\psi_i(x) = 0$ for all i, then for every $y \in \mathbb{P}^n$, the point (x, y) satisfies $(y_0 : \ldots : y_n) = (\psi_0(x) : \ldots : \psi_n(x))$ because this equation really means $y_i\psi_j(x) = y_j\psi_i(x)$, which is true because $\psi_i(x) = \psi_j(x) = 0$. Note also, again by going to $\mathbb{P}^n$, that the intersection degree with the horizontal class $X \times \{\text{point}\}$ remains d^n: this is because for y such that $y_i \neq 0$ for all i, we obtain n independent equations, of the form $\psi_i(x) = (y_i/y_{i+1})\psi_{i+1}(x)$, each of degree d. But the topological degree of ϕ is the intersection number of Γ_ϕ with the horizontal divisor class, *excluding* vertical components, since the intersection there would be at points of ill-definition. If there exists a vertical component, then every horizontal class will intersect with it. In particular, if the topological degree is d^n then the horizontal class intersects no vertical component, and therefore no vertical component exists. □

We denote Zariski closure of a set S by $\overline{S}$. For $v \in M_K$ and any morphism ρ we denote the reduction of some model of ρ at v by $\tilde{\rho}_v$.

Theorem 3.2. Let $(X, \phi), (Y, \psi)$ be dynamical systems polarized with respect to $\mathcal{L}, \mathcal{M}$ respectively, where X, Y are connected. Assume that X, Y, ϕ are isotrivial, $\mathcal{L}, \mathcal{M}$ are defined over k, (X, ϕ) dominates (Y, ψ) and one of the following conditions holds:

(1) $\dim X = \dim \tilde{g}_v(X)$ and $\tilde{g}_v$ is dominant.
(2) For some n there exists a $d > 0$ such that the action of $(\phi^n)^*$ on $\mathrm{Pic}\, X$ is multiplication by d^n.
(3) $Y = \mathbb{P}^1$.

Then the rational map $\tilde{\psi}_v |_{\tilde{g}_v(X)} : \tilde{g}_v(X) \to \tilde{g}_v(X)$ is a morphism.

Proof. Write $Z = \overline{\tilde{g}_v(X)}$. By restricting $\tilde{\psi}_v$ to $\tilde{g}_v(X)$, the morphism $\tilde{\psi}_v$ is polarized in the sense that the divisor class of the intersection $M \cdot Z$ pulls back to a d^{th} power of itself. It suffices to prove that the map has degree $d^{\dim Z}$ on Z. Then it is a morphism on Z, in which case it is a morphism on its open (not necessarily proper) subvariety $\tilde{g}_v(X)$ also.

The degree of $\tilde{\psi}_v |_Z$ is at most $d^{\dim Z}$. Let $z \in \tilde{g}_v(X)$, and consider the fiber $\tilde{g}_v^{-1}(z)$. The preimage of $\tilde{g}_v^{-1}(z)$ under ϕ consists of some fibers $\tilde{g}_v^{-1}(z_i)$. There is a generic cycle class for fibers of $\tilde{g}_v$, of codimension $\dim \tilde{g}_v(X)$. If we can show that this cycle class C satisfies the equation $\phi^*(C) = d^{\mathrm{codim}\, C} C$ then we are done, since we could pick z to be suitably generic so that $\tilde{g}_v(z)$ would have exactly $d^{\dim \tilde{g}_v(X)}$ pre-image fibers.

If the first condition in the lemma holds, then $\dim C = 0$ and we know that each point has $d^{\dim X}$ preimages, counted with multiplicity (which is generically 1); see [12] (Prop. 2). If the second condition holds for $n = 1$, then ϕ acts on each codimension-i cycle group by scalar multiplication by d^i and we are also done; if it holds for some n, then clearly $\widetilde{(\psi^n)}_v\,|_Z$ has degree $d^{n \dim \tilde{g}_v(X)}$ and then $\tilde{\psi}_v\,|_Z$ has degree $d^{\dim \tilde{g}_v(X)}$. Finally, suppose that the third condition holds. We can set $\mathcal{M} = \mathcal{O}(1)$, and then $\tilde{g}_v^{-1}(z)$ is a section of $\mathcal{L}$ since we can enlarge $\tilde{g}_v$ so that its indeterminacy locus has codimension at least 2, and ϕ^* acts on $\mathcal{L}$ as scalar multiplication by d by definition. □

Remark 3.3. The second condition in Theorem 3.2 is automatically satisfied when X has polyhedral ample cone, e.g. whenever X is Fano.

We now have the following corollaries:

Corollary 3.4. Let $(X,\phi),(Y,\psi)$ be as above. Suppose one of the conditions of Theorem 3.2 holds, $\tilde{g}_v$ is dominant at every place of K, and $\mathrm{Aut}(Y,\mathcal{M})$ is reductive. Then ψ is isotrivial.

Proof. If $\tilde{g}_v$ is dominant, then $\dim \tilde{g}_v(X) = \dim Y$, then $\tilde{\psi}_v$ has degree $d^{\dim Y}$, making it a morphism on Y. In other words, ψ has good reduction at every place. By Corollary 2.6, ψ is isotrivial. □

Remark 3.5. In the previous corollary we require k to be infinite since we invoke Corollary 2.6.

Lemma 3.6. Let $(X,\phi),(Y,\psi)$ be as above. Suppose one of the conditions of Theorem 3.2 holds. Then the locus of indeterminacy of $\tilde{g}_v$ is an exceptional set for ϕ, i.e. it is its own preimage.

Proof. Let W be the indeterminacy locus of $\tilde{g}_v$ and Z be the indeterminacy locus of $\tilde{\psi}_v\,|_{\tilde{g}_v(X)}$. The indeterminacy locus of $\tilde{g}_v \circ \phi$ is then $\phi^{-1}(W)$ and the indeterminacy locus of $\tilde{\psi}_v \circ \tilde{g}_v$ is $W \cup \tilde{g}_v^{-1}(Z)$. Since $\tilde{\psi}_v \circ \tilde{g}_v = \tilde{g}_v \circ \phi$, we obtain that $W \cup \tilde{g}_v^{-1}(Z) = \phi^{-1}(W)$. Now we apply Theorem 3.2 and obtain that Z is empty, so that $W = \phi^{-1}(W)$. □

In general, there are no exceptional subvarieties for a morphism on $\mathbb{P}^n$. Conjecturally ([6], Conjecture 2.6) all such subvarieties are linear, and in the linear case, a large number of monomials of $\phi = (\phi_0 : \ldots : \phi_n)$ has to be zero for ϕ to have an exceptional subvariety. This motivates our following corollary:

Corollary 3.7. Let $(X,\phi),(Y,\psi)$ be as above. Suppose one of the conditions of Theorem 3.2 holds, $\mathrm{Aut}(Y,\mathcal{M})$ is reductive and ϕ has no non-empty proper exceptional subvarieties. Then ψ is isotrivial.

Proof. By Corollary 3.6, $\tilde{g}_v$ is a morphism from X to Y. If we can show that $\tilde{g}_v$ is dominant, then by Corollary 3.4 we are done.

If $X = Y$ and g is polarized then by Corollary 2.6, g is isotrivial, and $\tilde{g}_v = g$ is dominant (in fact surjective). Otherwise, we do not have an equivalent of Corollary 2.6, but we do have something close enough. A rational map from X to Y may be viewed as a subvariety of $X \times Y$ (i.e. the graph of g, denoted Γ_g) that intersects the cycle class $\{x\} \times Y$ in exactly one point with multiplicity 1; it is a morphism if and only if $\Gamma_{\tilde{g}_v}$ intersects each subvariety $\{x\} \times Y$ in one point with multiplicity 1, rather than just the generic cycle class. Now consider the intersection $\Gamma_g \cdot (X \times \{y\})$. Since g is surjective, the intersection $\Gamma_g \cdot (X \times \{y\})$ is a nonzero cycle class of dimension $\dim X - \dim Y$. Finally, the intersection $\Gamma_{\tilde{g}_v} \cdot (X \times \{y\})$ is still a nonzero cycle class; this means that for the generic $y \in Y$, $(X \times \{y\}) \cdot \Gamma_{\tilde{g}_v}$ is nonempty, and since $\tilde{g}_v$ is a morphism, we have $(x,y) \in \Gamma_{\tilde{g}_v}$ if and only if $y = \tilde{g}_v(x) \in \tilde{g}_v(X)$. Thus we get that $\overline{\tilde{g}_v(X)} = Y$ (in fact, $\tilde{g}_v(X) = Y$) and $\tilde{g}_v$ is dominant (in fact, surjective). □

In the case $Y = \mathbb{P}^1_K$, we give another proof to show that ψ is defined over some finite extension k. We use the notion of semistability from geometric invariant theory to prove this.

Definition 3.8. (Hilbert-Mumford Criterion) Let G be a geometrically reductive group acting on a projective variety X, with $\mathcal{L}$ a G-invariant line bundle on X. By embedding X into $\mathbb{P}^n$ via $\mathcal{L}$, we obtain an action of G on $\mathbb{P}^n$, so we may assume that $X = \mathbb{P}^n$. For each one-parameter subgroup $\mathbb{G}_m \subseteq G$, pick a coordinate system for $\mathbb{P}^n$ with respect to which its action is diagonal, and consider the weight t_i with which the subgroup acts on each coordinate x_i. A point $x \in X$ is said to be *unstable* (respectively *not stable*) if there exists a one-parameter subgroup such that, after the appropriate coordinate change, we have $x_i = 0$ whenever $t_i \leq 0$ (resp. $t_i < 0$); otherwise, it is said to be *semistable* (resp. *stable*).

Recall from geometric invariant theory the following theorem:

Theorem 3.9. Let G be a geometrically reductive group acting on a projective variety X whose stable and semistable points are denoted by X^s and X^{ss} respectively. Let R be a discrete valuation ring with fraction field K, and let $x_K \in X^s_K$. Then for some finite extension K' of K, with R' the integral closure of R in K', x_K has an integral model over R' with semistable reduction modulo the maximal ideal. In other words, we can find some $A \in G(\overline{K})$ such that $A \cdot x_K$ has semistable reduction. If $x_K \in X^{ss}_K$, then the same result is true, except that $x_{R'}$ could be an integral model for some $x'_{K'}$ mapping to the same point of $X^{ss}//G$ such that $x'_{K'} \notin G \cdot x_K$.

Proof. [9] (Theorem 2.11). ☐

Definition 3.10. Let K be a complete non-archimedean field with ring of integers R, and let $\mathrm{Hom}(X_K, Y_K)$ be the space of K-morphisms from X_K to Y_K. We say $\phi_R \in \mathrm{Hom}(X_R, Y_R)$ is an *integral model* of $\phi_K \in \mathrm{Hom}(X_K, Y_K)$ if ϕ_R is generically equal to ϕ_K.

Lemma 3.11. Let X be a projective variety over k and $g : X_K \to \mathbb{P}^n_K$ be a K-morphism, where K is a complete non-archimedean field. Then g has an integral model whose reduction is non-constant.

Proof. By Theorem 3.9, it suffices to show that constant maps are unstable with respect to some action. Even though $\mathrm{Aut}\,\mathbb{P}^n = \mathrm{PGL}_{n+1}$; for the purposes of the Hilbert-Mumford criterion, SL_{n+1} is the correct group to use. Note that we are allowed to take finite extensions, and SL_{n+1} maps finite-to-one onto PGL_{n+1}. We choose the left action of SL_{n+1} on the space of maps from X to $\mathbb{P}^n$ i.e. for $h \in \mathrm{SL}_{n+1}, g : X \to \mathbb{P}^n$, we have $h \cdot g(x) = h \circ g$.

Now, let T be a one-parameter subgroup of SL_{n+1}, say with diagonal entries $t^{a_0}, \ldots, t^{a_n}$, with

$$a_0 \geq \ldots \geq a_n \quad \text{and} \quad \sum a_i = 0$$

A morphism from X to $\mathbb{P}^n$ is given by an $(n+1)$-tuple of sections of some line bundle on X, say $g_0, \ldots, g_n$, and the action of T has weight a_i on each g_i. Now if the morphism is constant, or even has image contained in a hyperplane, we assume the image is contained in $x_n = 0$, so that $g_n = 0$. If $a_0 = \ldots = a_{n-1} = 1$ and $a_n = -n$ then the coordinates of g are zero whenever the action of T has negative weights and thus g is unstable.

Finally, observe that, with respect to the $\mathcal{O}(1)$-bundle on $\mathbb{P}^n$, maps whose images are not contained in a hyperplane are not just semistable but also stable. This is because T acts on each g_i with weight depending only on i; therefore, if for some weight all coordinates of g are zero then we have $g_i = 0$ and then g is unstable. Thus the only group that can act with no negative weights but with some non-negative weights is the group all of whose weights are zero, i.e. the trivial group. This means that g is stable, and thus it has an integral model with semistable (in fact, stable) reduction. ☐

Corollary 3.12. Let $(X, \phi), (\mathbb{P}^1_K, \psi)$ be dynamical systems polarized w.r.t. $\mathcal{L}, \mathcal{O}(1)$ respectively. Assume that X is connected and (X, ϕ) dominates $(\mathbb{P}^1, \psi)$. Then ψ is isotrivial.

Proof. A morphism from a connected variety X to $\mathbb{P}^1_K$ is either constant or dominant. After conjugation we can force $\tilde{g}_v$ to be non-constant by Lemma 3.11. Now apply Corollary 3.4. ☐

4. Acknowledgements

This research project was initiated while the authors were postdoctoral fellows at ICERM, Brown University. We would like to thank Joseph Silverman and the staff at ICERM for their hospitality, and for providing an environment conducive for our research. The first author was partially supported by PSC-CUNY grant TRADA-44-179.

References

[1] T. Bauer, C. Schulz. *Seshadri constants on the self-product of an elliptic curve*. J. Algebra 320 (2008), 2981–3005.
[2] A. Bhatnagar. *Projective Varieties Covered by Isotrival Families.* Proc. of Amer. Math. Soc. 142 (2014), no. 5, 1561-1566.
[3] S. Cacciola, M. Dontent-Bury, O. Dumitrescu, A. Lo Guidice, J. Park. *Cones of divisors of blow-ups of projective spaces* Le Matematiche, Italy, 66, Dec. 2011. Available at: http://www.dmi.unict.it/ojs/index.php/lematematiche/article/view/903.
[4] I. Coskun, A. Prendergast-Smith. *Fano manifolds of index n-1 and the cone conjecture.* To appear in Int. Math. Res. Not. Available at http://arxiv.org/abs/1207.4046
[5] N. Fakhruddin. *Questions on Self Maps of Algebraic Varieties.* J. Ramanujan Math. Soc. 18 (2003), no. 2, 109–122.
[6] Y. Fujimoto, N. Nakayama. *Complex Projective Manifolds which admit non-isomorphic surjective endomorphisms.* Higher dimensional algebraic varieties and vector bundles, RIMS Kôkyûroku Bessatsu, B9, Res. Inst. Math. Sci., Kyoto, (2008), 51–79.
[7] S. Iitaka. *Algebraic Geometry: An Introduction to Birational Geometry of Algebraic Varieties.* Graduate Texts in Mathematics, No. 76, Springer, 1981.
[8] A. Levy. *The Space of Morphisms on Projective Space.* Acta Arith. 146 (2011), no. 1, 13–31.
[9] A. Levy. *The Semistable Reduction Problem for the Space of Morphisms on* $\mathbb{P}^n$. Algebra Number Theory 6 (2012), no. 7, 1483–1501.
[10] D. Mumford, J. Fogarty, F. Kirwan. *Geometric Invariant Theory, Third Edition.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 34, Springer-Verlag, Berlin, 1994.
[11] B. Nill. *Complete toric varieties with reductive automorphism group.* Mathematische Zeitschrift, 252 (2006), no. 4, 767–786.
[12] C. Petsche, L. Szpiro, M. Tepper. *Isotriviality is equivalent to potential good reduction for endomorphisms of* $\mathbb{P}^n$ *over function fields.* J. Algebra 322 (2009), no. 9, 3345–3365.
[13] J. Silverman. *The space of rational maps on* $\mathbb{P}^1$. Duke Math. J. 94 (1998), no. 1, 41–77.

Department of Mathematics, Borough of Manhattan Community College, The City University of New York; 199 Chambers Street, New York, NY 10007 U.S.A.
E-mail address: anupambhatnagar@gmail.com

Department of Mathematics, The University of British Columbia; 1984 Mathematics Road Vancouver, B.C. Canada V6T 1Z2
E-mail address: levy@math.ubc.ca