The McMullen Map In Positive Characteristic

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Abstract. McMullen proved the moduli space of complex rational maps can be parametrized by the spectrum of all periodic-point multipliers up to a finite amount of data, with the well-understood exception of Lattès maps. We generalize McMullen's method, which is complex-analytic, to large positive characteristic; a modification of the method using rigid analysis works over a function field over a finite field of characteristic larger than the degree of the map. Over a finite field with such characteristic it implies that, generically, rational maps can indeed be parametrized by their multiplier spectra up to a finite-to-one map. Moreover, the set of exceptions, that is positive-dimension varieties in moduli space with identical multipliers, maps to just a finite set of multiplier spectra. We also prove an application, generalizing a result of McMullen over the complex numbers: there is no generally convergent purely iterative root-finding algorithm over a non-archimedean field whose residue characteristic is larger than either the degree of the algorithm or the degree of the polynomial whose roots the algorithm finds.

1. Introduction

A rational function over a field F can be studied by associating invariants to its fixed and periodic points. At a finite fixed point z of a rational function φ, the derivative $\varphi'(z)$ can be interpreted as the map on tangent spaces, and is conjugation-invariant: that is, if A is any linear-fractional transformation, then $\varphi'(z) = (A\varphi A^{-1})'(z)$. We call this conjugation-invariant derivative the **multiplier** of φ at z. If z is the point at infinity, we can extend the definition of the multiplier by conjugating it to be any finite point. We can furthermore define the multiplier of a point of exact period n to be $(\varphi^n)'(z)$; here and in the sequel, powers of maps denote iteration rather than multiplication. The set of multipliers of period n, called the **multiplier spectrum**, depends only on the map φ, and because it is conjugation-invariant it really only depends on its conjugacy class.

Thus the symmetric functions in the multipliers are regular algebraic functions on the space of rational maps of degree d, Rat_d, which in fact descend to the quotient of Rat_d by PGL(2)-conjugation, M_d. Collecting the symmetric functions in the period-n multipliers for each n, we obtain regular algebraic maps Λ_n from M_d to large affine spaces $\mathbb{A}^{K_n}$ where K_n is the number of period-n points. Because there are infinitely such maps, one for each n, a priori we would expect there to be some large n for which the map

$$\Lambda_1 \times \ldots \times \Lambda_n : \mathrm{M}_d \to \mathbb{A}^{K_1+\ldots+K_n}$$

is injective. This would allow us to analyze the geometry of M_d purely in terms of multipliers. The map is not injective, but we do have a partial result:

Theorem 1.1. *Over a field of characteristic* 0 *or greater than* d*, for sufficiently large* n *the map*

$$\Lambda_1 \times \ldots \times \Lambda_n : \mathrm{M}_d \to \mathbb{A}^{K_1+\ldots+K_n}$$

is finite-to-one outside a proper closed subset of M_d*.*

Unfortunately, we only get generic finiteness. We say that a family C of rational maps in Rat_d or M_d is **isospectral** if, for all n, the map Λ_n is constant on C, and that an isospectral family is **trivial** if it reduces to a point in M_d and nontrivial otherwise.. Theorem 1.1 says that all nontrivial isospectral families are contained in a proper closed subset. This subset is nonempty, at least when d is a square, as shown in the following example:

Definition 1.2. The map $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ is called a **Lattès map** if there is an elliptic curve E, a finite morphism $\alpha : E \to E$, and a finite separable map π such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "\alpha"] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "\varphi"'] & \mathbb{P}^1 \end{tikzcd}$$

If we choose π to be the projection by $P \sim -P$, then α must be of the form $\alpha_{m,T} : P \mapsto mP + T$ where $T \in E[2]$.

It is well-known that for each integer m, the Lattés maps corresponding to $\alpha_{m,0}$, with π defined by $P \sim -P$, have multipliers that do not depend on the choice of the curve E, and therefore the family of such Lattés maps is a nontrivial isospectral family in M_{m^2}; for more details, see section 6.5 of [19].

McMullen has proven that,

Theorem 1.3. [15] *Excluding the Lattès maps, there are no nontrivial isospectral families over* $\mathbb{C}$*.*

The goal of this paper is to extend McMullen's result to positive characteristic as far as possible. McMullen's proof uses inherently complex-analytic methods. Although it is true over any characteristic-0 field by the Lefschetz principle, it does not give us any proof in positive characteristic, and to generalize it to obtain Theorem 1.1, we need to use a somewhat different theory of non-archimedean analysis. The first difference between characteristics 0 and p is that a naive generalization of Theorem 1.3 is false in characteristic p, as shown in,

Example 1.4. Let $\varphi(z) = \psi(z^p) + az$ where a is a constant and ψ is a family of rational functions. The family is isospectral (all period-n multipliers at finite points are a^n). However, if ψ is large enough, for example if $\dim \psi > 3$, then the family has no hope of reducing to a point in M_d since $\mathrm{M}_d = \mathrm{Rat}_d / \mathrm{PGL}_2$ where Rat_d is the space of rational functions of degree d and PGL_2 acts by coordinate change.

However, Example 1.4 involves wild ramification. Other than wildly ramified examples and the Lattés maps, we do not know of any nontrivial isospectral families. Let us now explain the theoretical reasons to believe that no such families, exist, or, in other words, that a positive-characteristic version of Theorem 1.3 is true for tamely ramified maps. Briefly, McMullen's theorem is intimately connected with Thurston's rigidity theorem. Specifically:

Definition 1.5. A point z is a **preperiodic point** of φ if for some n, $\varphi^n(z)$ is a periodic point; we say the minimal such n is the **tail length**. Observe that z is preperiodic if and only if it has finite forward orbit. If all of the critical points of φ are preperiodic, we say that φ is **postcritically finite**, or in short PCF.

There are $2d-2$ critical points counted with multiplicity when φ is tamely ramified, and so the condition that φ is PCF, at least if we fix the cycle and tail length of each critical point, is a set of $2d-2$ algebraic equations. Since $\dim \mathrm{M}_d = 2d-2$, there should be finitely many PCF maps for each degree, cycle length, and tail length. Over $\mathbb{C}$, Thurston has proven that this is more or less the case:

Theorem 1.6. [5, 7] *(Thurston's Rigidity) Excluding the Lattès maps, all of which are PCF, there are no positive-dimension PCF families.*

Remark 1.7. It is expected that the PCF equations meet transversally, i.e. fixing a cycle and tail length, the scheme of non-Lattès PCF maps is a finite set of reduced points; see Epstein's work in Section 5 of [8]. Favre and Gauthier [9] also show this explicitly for some relations involving PCF polynomials.

It is not a coincidence that the Lattès maps form the sole counterexample to both Thurston's rigidity and McMullen's theorem. McMullen's proof heavily uses rigidity. Briefly, the method in [15] is to assume an isospectral family exists, and then label an infinite set of so-called **repelling points**, that is periodic points whose multiplier λ satisfies $|\lambda| > 1$, accumulating at a point in the Julia set. A crucial fact about isospectral families is that their periodic points move without collision with one another or with critical points. This labeling cannot be done globally, but can be done locally, using Mañé-Sad-Sullivan stability theory. Using previous results showing that this infinite set moves holomorphically, McMullen shows that a critical point cannot pass through a point that is preperiodic to a repelling point; since there are infinitely many repelling points over $\mathbb{C}$, this implies that in an isospectral family all critical points are preperiodic, and then Thurston's rigidity shows that such a family is Lattès or trivial.

We will follow a similar program. Many of the methods of complex analysis have been successfully ported to the non-archimedean setting, and although a full version of Thurston's rigidity is beyond our reach, a sufficiently good approximation is proven in [1]. First, we have,

Definition 1.8. Let φ be defined over a complete algebraically closed non-archimedean field K. We say that φ is **tame** if the local degree at any open analytic disk is not divisible by the residue characteristic p of K. Equivalently, $\varphi = f/g$ is tame if and only if for each integral model of φ over $\mathcal{O}_K$, the reduction mod the maximal ideal is tamely ramified after clearing common factors of the reductions of f and g. If φ is defined over a global field, we say it is tame if it is tame when we regard it as a map over every completion.

Remark 1.9. Whenever $\deg \varphi < p$, or whenever φ is the composition of maps of degree less than p, φ is tame.

We have as a prior result [1],

Theorem 1.10. *Let $\varphi(z) \in F(z)$ where F is a global function field such that* $\operatorname{char} F = p$. *If φ is PCF and tame at every place of F, then the multipliers of φ are all defined over $\overline{\mathbb{F}_p}$.*

Remark 1.11. An immediate corollary of Theorem 1.10 is that if we can prove McMullen's theorem for some M_d, then we can prove Thurston's rigidity for it as long as the local degrees are not divisible by p. In [18] it is proven that Λ_1 is an isomorphism over $\mathbb{Z}$ from M_2 to a plane in $\mathbb{A}^3$, and thus Thurston's rigidity is also true for quadratic maps in characteristic 0 or $p \geq 3$.

Like McMullen, we will prove that in an isospectral family we can label a set of periodic points that accumulate at a point that is preperiodic to a repelling point, and that if a critical point passes through the preperiodic point it is equal to it for all maps in the family. We do not have a non-archimedean Mañé-Sad-Sullivan stability theory, but we do have a way to locally label an infinite set of repelling points, for which the key ingredient is Lemma 2.4. This will suffice to show that an isospectral map is PCF. This requires a repelling point in $\mathbb{P}^1_K$ where K is a complete algebraically closed characteristic-p field; unlike in the complex case, a repelling point is not guaranteed to exist. However, if we start from a global field F, if the multipliers are not all in $\overline{\mathbb{F}_p}$ then we can take the completion with respect to a valuation that occurs in the denominator of a multiplier.

We will prove,

Theorem 1.12. *Let F be a global function field of characteristic p, that is, an extension of $\mathbb{F}_p$ of transcendence degree 1. Suppose that C is a one-parameter family in* Rat_d *defined over F, such that every rational function (defined over F) corresponding to a point of C is a composition of maps of degree less than p. Suppose that all but finitely many of the multiplier spectra on C are constant, and suppose further that these constant multipliers are not all contained in $\overline{\mathbb{F}_p}$. Then C is trivial.*

Remark 1.13. The assumption that C consists of compositions of maps of degree less than p implies that C insists of tame maps, as noted in Remark 1.9. An extension of the Theorem to all tame maps seems plausible, but just out of reach for the method we use; we only use the stronger assumption that the maps are compositions of low-degree maps in one place, in the proof of Lemma 2.4.

Remark 1.14. The case where C is isospectral is a special case of Theorem 1.12; the more general statement is required for one specific corollary.

Remark 1.15. The family of Lattès has multipliers defined over $\overline{\mathbb{F}_p}$, and so is automatically excluded from the hypotheses of the theorem.

Let us now prove Theorem 1.1 assuming Theorem 1.12. We will prove the slightly stronger corollary:

Corollary 1.16. *Let k be an algebraically closed field of characteristic greater than d. For sufficiently large n, there are only finitely many spectra in $\mathbb{A}_k^{K_1+\ldots+K_n}$, including the Lattès multiplier spectra, for which there is a positive-dimension family of morphisms in* $\mathrm{M}_d(k)$ *with these spectra.*

Proof. Let X_n be the union of all positive-dimension varieties in $\mathbb{A}_k^{K_1+\ldots+K_n}$ on which the map $\Lambda_1 \times \ldots \times \Lambda_n$ has positive relative dimension. Note that if $m < n$, the map $\Lambda_1 \times \ldots \times \Lambda_m$ commutes with the composition of $\Lambda_1 \times \ldots \times \Lambda_n$ with the projection obtained by forgetting all period-higher-than-m multipliers. This gives us maps from X_n to X_m, commuting with the multiplier spectrum maps on M_d. We need to prove that for sufficiently large n, X_n is empty. Suppose to the contrary that X_n is nonempty for all n. For sufficiently large n, the generic relative dimension of $\Lambda_1 \times \ldots \times \Lambda_n$ on the preimage of X_n stabilizes; therefore, for sufficiently large n and m, with $m < n$, the map $X_n \to X_m$ is generically one-to-one. Informally, it means that for sufficiently large n, finding more multipliers will not give us any more information.

Now, label an infinite sequence of curves Y_i contained in X_i, where $i \geq n$, such that the curves are all birationally equivalent and, if $i > j$, Y_i maps to Y_j under the forgetful map on multiplier spectra. Let Z be a subvariety of the common preimage of all the Y_is in M_d that maps to each Y_i with relative dimension 1. We can consider Z to be a one-parameter family over a finite extension of $F = k(Y_n)$, which is isospectral, mapping to a single point of $\mathbb{A}_F^{K_1+\ldots+K_n}$. Now, we would like to apply Theorem 1.12. First, since Z is by assumption not the trivial family in $\mathbb{A}_k^{K_1+\ldots+K_n}$, as a point of $\mathbb{A}_F^{K_1+\ldots+K_n}$ it has multipliers not contained in k. If $k = \overline{\mathbb{F}_p}$ then we apply Theorem 1.12, which then says Z is a trivial family, a contradiction. If k is itself transcendental, we may apply a positive-characteristic analog of the Lefschetz principle: we are dealing with finitely many spectra, so Z can be defined over a finitely-generated extension of $\mathbb{F}_p$, and we can replace k with a finitely-generated subfield and moreover specialize to reduce it to an algebraic extension of $\mathbb{F}_p$. $\square$

Theorem 1.1 follows as soon as we exhibit two maps with two different multiplier spectra, proving that $\Lambda_1 \times \ldots \times \Lambda_n$ doesn't collapse all of M_d to a single point. Finding such a pair of maps is easy: for example, $z^d + 1$ has just one fixed point that is also critical whereas z^d has two, and fixed points are critical precisely when their multipliers are 0.

The result is not constructive: it does not tell us what n we need to choose to ensure that the map $\Lambda_1 \times \ldots \times \Lambda_n$ will be finite-to-one, nor does it tell us the degree of such a map. Unlike in McMullen's characteristic 0 case, Theorem 1.1 does not even tell us what the exceptions are to finiteness, i.e. what the finitely many spectra with positive-dimension preimages in Corollary 1.16 are. For some partial results in that direction, see [10, 18].

Effective versions of Theorem 1.1 would require different techniques. Unlike the techniques in [10], we are unable to list the finite set of exceptions in Corollary 1.16 (or any proper subset of M_d that contains all of them), although we conjecture that the Lattès family is the only one. Nor can we use these techniques to find or even bound the degree of the McMullen map, or the minimal period n such that $\Lambda_1 \times \ldots \times \Lambda_n$ is generically finite.

We spend Section 2 proving Theorem 1.12. This proof is easier than in the complex case, since non-archimedean analysis is more rigid than complex analysis, making it easier to show that a critical point cannot move very freely in an isospectral family. The main tool is Lemma 2.4, which we use to label an infinite sequence of repelling periodic points, which we then show implies a critical point cannot move freely.

We give an application of the result in Sections 3. McMullen originally proved Theorem 1.3 in order to study generally convergent purely iterative root-finding algorithms. Following McMullen, we make the following definition:

Definition 1.17. A **purely iterative root-finding algorithm** is a rational map $T : \mathrm{Poly}_r \dashrightarrow \mathrm{Rat}_d$, where $\mathrm{Poly}_r \subseteq \mathrm{Rat}_r$ is the space of degree-r polynomials. Let us denote the image of a polynomial $f(z)$ under T by $T_f(z)$. We say that T is **generally convergent** if, for an open dense set of $f \in \mathrm{Poly}_r$, the sequence $T_f^n(z)$ converges to a root of f on an open dense subset of $\mathbb{P}^1$.

The definition of generally convergent algorithms is made relative to a fixed topology. It is a question of Smale [20] whether such algorithms exist in the complex setting. McMullen proved that they do not when $r \geq 4$ in the complex setting, by showing that the image of a generally convergent algorithm, $T(\mathrm{Poly}_r) \subseteq \mathrm{Rat}_d$, is in fact an isospectral family, hence constant. We use the full strength of Theorem 1.12, i.e. allowing C to have finitely many nonconstant multipliers rather than assuming it is isospectral, to prove,

Theorem 1.18. *There is no generally convergent purely iterative root-finding algorithm $T : \mathrm{Poly}_r \dashrightarrow \mathrm{Rat}_d$ over a non-archimedean field of residue characteristic p if $p = 0$, $p > d$, or $p \geq r$.*

2. Proof

Let C be an irreducible curve as in the hypotheses of Theorem 1.12. Since the multipliers of C are not defined over $\overline{\mathbb{F}_p}$, there is a place of F such that one multiplier is repelling. Let K be an algebraically closed field, complete with respect to a nontrivial non-archimedean absolute value $|\cdot|$; we choose K to be an algebraically closed completion of F such that one of the constant multipliers of the family C is repelling. We have the following result:

Proposition 2.1. *Let $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ be a tamely ramified map over K, and let z be a repelling periodic point. Then φ has infinitely many repelling periodic points, and the Julia set of φ is the closure of the set of repelling points, and in particular if z' is preperiodic to a repelling point, then there is a set of repelling periodic points that accumulates at z'.*

Proof. We apply a result of Bézivin [3], who proves this when $\mathrm{char}\, K = 0$ (when $K = \mathbb{C}$, this is classical, proven in [12] by Julia). First, we may asume z is fixed, replacing φ by an iterate if necessary. Julia and Bézivin's technique is the linearization: if z is an attracting or repelling fixed point, with multiplier λ, then there exists a meromorphic function

$$g(z) = z + (\text{higher-order terms})$$

such that $g(\lambda z) = \varphi(g(z))$.

Using the linearization, Bézivin constructs a sequence of repelling points that accumulate at any preimage (to any order) of z. At only one point does the proof use the fact that $\mathrm{char}\, K = 0$: in the paper's Lemma 2, the computations depend on $|h|$, where h is the order of the zero of $g(\lambda z)$ at some point $z = c$.

Let us now show that this order cannot be divisible by p. Since φ is tamely ramified, the local degree of φ is prime to p at every point. Therefore, the p-part of the local degree of g is the same at λz and at z. In our case $|\lambda| > 1$, so we apply this repeatedly with λ^{-1}, so that the p-part of the local degree of g is the same at the sequence $z, \lambda^{-1}z, \lambda^{-2}z, \dots$ This sequence limits at 0, where the local degree of g is 1. So in fact g is always tamely ramified, and $|h| = 1$. $\square$

Moreover, all but finitely many repelling points move without collision: that is, if we mark any number of them, they will stay distinct, since a fixed point is repeated iff it has multiplier 1, and by assumption all but finitely many repelling points have constant multipliers. McMullen's proof in fact hinges on using this to show that the repelling points, and with them the entire Julia set, move rigidly, so that after a local analytic conjugation the Julia set is constant in the family.

We may mark special points, such as critical points, critical values, and periodic points, to obtain an **enhanced** moduli space $\mathrm{M}_d(\Gamma)$; by an enhanced moduli space, or (by abuse of terminology) an enhanced rational map, we are invoking the equivalent terminology for modular curves of elliptic curves, see for example Section 1.5 of [6]. Observe that we can enhance each $\mathrm{M}_d(\Gamma)$ further by including additional marked points. The natural projection $\mathrm{M}_d(\Gamma) \to \mathrm{M}_d$ is finite-to-one, and by abuse of notation we denote the multiplier map on $\mathrm{M}_d(\Gamma)$ by Λ_n, sending each enhanced φ to an ordered set of the multipliers of all the marked period-n points and the spectrum of the multipliers of the unmarked points.

Let us now replace C by $C' = C \times_{\mathrm{M}_d} \mathrm{M}_d(\Gamma)$, where Γ is obtained by marking three points, as follows. First, mark a repelling fixed point with constant multiplier. Note that the family parametrizing φ is trivial if and only if the family parametrizing φ^n is trivial for any (or all) n; this appears well-known to dynamicists, and is written down in Proposition 2.1 of [13]. Therefore, we may apply iteration and assume that such a point exists. We place this point at ∞, and mark one of its preimages, which we place at 0; observe that they will never collide, or else ∞ would be critical, violating the condition on its multiplier. Finally, we mark a third point that will never collide with either ∞ or 0, for example a repelling periodic point of constant multiplier, and place it at 1. This ensures that there are no nontrivial automorphisms of any enhanced $\varphi \in C'$.

Now, the repelling point ∞ and its preimage 0 are is in the Julia set, so there is a sequence of repelling periodic points converging to it for every map in the family C', and since all but finitely many periodic points have constant multipliers, there is in fact a sequence of repelling periodic points with constant multipliers converging to 0 for every map in C'. We need to make this sequence of repelling points uniform; this way, any critical point that collides with 0 for some $\varphi \in C'$ will have to remain at 0 for all points of C', since repelling points cannot collide with repelling periodic points. In turn, studying this sequence of repelling points requires us to be able to keep enhancing Γ. Our main step is then showing that we can uniformly mark infinitely many repelling points.

First, let us show what happens when we enhance Γ with just one additional repelling point:

Lemma 2.2. *Suppose that C' is chosen in such a way that there exist three marked points in Γ no two of which collide on C'. Let $C'' = C' \times_{\mathrm{M}_d(\Gamma)} \mathrm{M}_d(\Gamma')$ where Γ' is obtained from Γ by marking one additional repelling periodic point, of sufficiently high period such that all points of this period have constant multipliers. Then the natural projection $\pi : C'' \to C'$ is unramified.*

Proof. First, let n be the period of the additional point we mark in C''. There exists a morphism f, defined over Rat_d, mapping each φ to the set of its points of so-called formal period n; see [19], Section 4.1. In $\mathrm{Rat}_d(\Gamma)$, we have f again, and we can even factor out roots (i.e. periodic points) that are marked in Γ. Furthermore, we can turn f into a morphism defined on $\mathrm{M}_d(\Gamma)$ by placing our three non-colliding marked points at $(0, 1, \infty)$. Observe that for each $\varphi \in C'$, $f(\varphi)$ is a polynomial whose roots are the formal-period-n periodic points of φ. Now, the roots of $f(\varphi)$ corresponding to repelling points are simple, since repeated roots, i.e. repeated periodic points, have multiplier 1. $\square$

If C' is singular, we may replace it with its normalization, a curve that lies not in $\mathrm{M}_d(\Gamma)$ but in an appropriate blowup. This does not change anything, since we can still define functions such as f on the normalized curves. If $\pi : C'' \to C'$ is unramified, then the same is true of the induced map between the normalizations of C'' and C', essentially since the property of being unramified is stable under base change, where we change base from C' to its normalization. In the sequel, we will assume C' is nonsingular.

Recall now the following results from non-archimedean analysis:

Proposition 2.3. *Let C be an algebraic curve over K. Let $x \in C$, and let $D \ni x$ be a neighborhood of x in C that is analytically isomorphic to a disk in $\mathbb{P}^1$. Any algebraic function from C to a curve X over K is an analytic function on D. Moreover, if D' is also analytically isomorphic to a disk in $\mathbb{P}^1$, then every surjective analytic function $D' \to D$ that has local degree 1 has an analytic inverse.*

Proof. The first statement follows from the definition of a regular algebraic function between varieties as one that is defined by polynomial maps. The second statement is essentially the implicit function theorem. See, for example, Section 2.2 of [11]. The implicit function theorem as stated in [11] explicitly identifies D' and D with the open disk $D^-(0, 1)$ and $\pi : D' \to D$ with a power series in $\mathcal{O}_K[[z]]$, and requires that $\pi(0) = 0$ and that $\pi'(0)$ not be contained in the maximal ideal $\mathfrak{m}_K$. For our purposes, the assumptions are the same: after changing coordinates to identify both D' and D with $D^-(0, 1)$ and fix $\pi(0) = 0$, a surjection has local degree 1 precisely when $\pi'(0) \notin \mathfrak{m}_K$. $\square$

Observe also that if in a projection map from C'' to C' a point $x \in C'$ is unramified, there exists a neighborhood of x in C' with a preimage disk that maps onto it with degree 1, so that we can apply the second case of Proposition 2.3 and obtain a local inverse. We have,

Lemma 2.4. *Let C' be as in the hypotheses of Lemma 2.2, and let $x \in C'$. There exists a neighborhood $D \ni x$ in C' that is analytically isomorphic to a $\mathbb{P}^1$-disk, depending only on C', with the following property: for any C'' as in the hypotheses of Lemma 2.2, the preimage of D in C'' is a disjoint union of $\mathbb{P}^1$-disks each mapping into D with degree 1.*

Proof. Let D be any neighborhood of x in C' that is isomorphic to a $\mathbb{P}^1$-disk. In [2], Berkovich proves that if the map $\pi : C'' \to C'$ is Galois, unramified, and tame, and D is a disk in C', then $\pi^{-1}(D)$ is a disjoint union of disks mapping to D with degree 1; this is Theorem 6.3.2. More precisely, Theorem 6.3.2 assumes that the map $C'' \to C'$ is étale in the category of Berkovich spaces (which Berkovich calls analytic spaces), whereas we only know it is unramified (thus étale) as a map of algebraic curves, but as per Theorem 6.3.7 of [2], tame finite étale Galois covers of algebraic curves and of their associated Berkovich spaces form equivalent categories. Moreover, it is highly likely that a more general result holds for tame non-Galois covers; in personal correspondence, Benedetto sketched an argument that made good progress toward a proof.

Unfortunately, we cannot guarantee *a priori* that the map $C'' \to C'$ is tame: we know that φ, a self-map in the dynamical space $\mathbb{P}^1$, is tame, and we will use this in the remainder of the proof of this lemma, but we do not know that the map $\pi : C'' \to C'$ of curves in moduli space is tame. We could in principle find that there are p distinct periodic points of the same cycle length, producing a map π of degree p. We will in fact prove that this is not the case. To ensure that we can find a uniform disk D as required, we use a workaround, which is similar in flavor to infinite descent.

First, let us replace D with the subdisk D' containing x that is minimal with respect to the property that $\pi^{-1}(D')$ is not a disjoint union of disks mapping to D' with degree 1. This is based on the minimum distance between marked periodic points in C'', so D' is closed. We will derive a contradiction. Note that we are not shrinking D non-uniformly; as in standard number-theoretic infinite descent, we are still assuming that the preimage affinoids of D map with degree more than 1, and using D' to derive a contradiction.

Let us replace C'' with its Galois closure over C'. We can keep track of the locus of all periodic points of a given formal period, as explained in the construction of Γ; indeed, as in the proof of Lemma 2.2, we obtain Γ' by adjoining one root of a polynomial f whose roots are all the points of a specified formal period. So now, the condition that C'' is Galois over C' means C'' is obtained from C' by marking a complete set of points of specified formal period; note that there may be smaller Galois extensions, if the polynomial f defined in the proof of Lemma 2.2 is reducible, but we lose nothing by marking extra points. Let D'' be a connected affinoid preimage of D' in C'', mapping to D' with degree more than 1; here, we use the term **connected** in the sense of rigid geometry. Now, a point $c \in C'$ consists of φ and finitely many marked points, including all the critical points. A point of C'' is of the form $(c, \theta_1, \ldots, \theta_m)$, where the θ_i's are the new periodic points. This turns each θ_i into an algebraic map from C'' to $\mathbb{P}^1$, while the symmetric functions in the θ_is define algebraic maps from C' to $\mathbb{P}^1$.

Now, let us look at the image of θ_i on D''. An affinoid maps to another affinoid, or to all of $\mathbb{P}^1$. But mapping over all of $\mathbb{P}^1$ would require θ_i to collide with the marked fixed point ∞, which is impossible. Moreover, for each i and j the images of θ_i and θ_j on $\pi^{-1}(D')$ (of which D'' is one connected component) are the same, because we can compose $\theta_1 \times \ldots \times \theta_m : (\pi^{-1}(D'))^m \to (\mathbb{P}^1)^m$ with a Galois automorphism mapping θ_i to θ_j. This common image is an affinoid in $\mathbb{P}^1$, not necessarily connected.

The image of each θ_i on D'' is one of the connected affinoid components of $\theta_i(\pi^{-1}(D'))$. It is possible for different θ_is to map D' to different connected components of $\theta_i(\pi^{-1}(D'))$. However, if each θ_i maps to a separate component, then $\pi : D'' \to D'$ has degree 1, because then we have a way of analytically distinguishing the θ_is on D'. Let us now assume that $\theta_1, \ldots, \theta_q$ map D'' into the same component, with $q > 1$. Moreover, by the assumption that D' is minimal, all θ_is map D'' to the same closed disk, except there may be different open subdisks removed for different

is. Similarly, we may assume that $\theta_1, \dots, \theta_q$ all have the same constant repelling multiplier, again because we can analytically distinguish θ_is of different constant multipliers.

Since we are assuming D' is minimal, each of its residue class open subdisks E satisfies the property that $\pi^{-1}(E)$ is a disjoint union of disks mapping to E with degree 1. By Proposition 2.3, for any component E' of $\pi^{-1}(E)$, $\pi^{-1} : E \to E'$ is analytic, and therefore each θ_i is analytic, sending E to an open disk $\theta_i(E') \subseteq \theta_i(D'')$, with $\theta_1(E), \dots, \theta_q(E)$ disjoint. It is not difficult to see that this turns $\theta_i(D'')$ into a closed disk, say $D(a, r)$, possibly with finitely many residue class open subdisks removed (as opposed to open disks of potentially smaller radius); we will only ever use the $D(a, r)$ and $D^-(a, r)$ notations to refer to disks in the dynamical space $\mathbb{P}^1$, and abstract letters such as D to refer to disks in moduli space.

For each i, each residue class E, and each preimage E' of E, the open disk $\theta_i(E') = D^-(a_{i,E'}, r)$ contains the preimage (to some order) of some critical point under φ. This is because $\theta_i(E')$ is an open neighborhood of a repelling periodic point, i.e. a point in the Julia set of φ. Pick the minimum j such that there exists some $c \in D'$ such that $\varphi_c^j(D^-(a_{i,E'}, r))$ contains a critical point of φ_c, the map corresponding to $c \in D'$. Label the critical point in $\varphi_c^j(D^-(a_{i,E}, r))$ by γ. Since γ is a marked point, this behavior is persistent, and in fact $D^-(a_{i,E'}, r)$ contains a jth preimage of γ for any $c \in D'$ and for any $i = 1, \dots, q$. Moreover, the disk $D^-(a_{i,E'}, r)$ cannot have two points mapping to the same image unless it has a critical point, and this implies $\varphi^{-j}(\gamma)$ is unique within $D^-(a_{i,E'}, r)$, so we can label it as ξ_i.

Now, we can enhance C' and C'' with preimages of γ. We denote the curves enhanced with all jth preimages of γ by C'_γ and C''_γ. We also let D''_γ be a connected affinoid obtained from D'' by marking ξ_i in the same order as θ_i for $i = 1, \dots, q$, and D'_γ be its projection to C'_γ. We will now show that D''_γ maps to D'_γ with degree 1. Note that on each residue class E of D' and for each choice E' of component of $\pi^{-1}(E)$, the map θ_i ranges over $\theta_i(E')$, which is a residue class of the affinoid $\theta_i(D'')$. Now, on E', ξ_i is unique, so we regard it as a map $\xi_i : E' \to D^-(a_{i,E'}, r)$. We now have a map $\theta_i - \xi_i : E' \to D^-(0, r)$, which misses 0 since periodic points and preimages of critical points do not collide. Since E' is a disk, $(\theta_i - \xi_i)(E')$ is a disk of strictly smaller radius than r, say $D^-(b, s)$, where $s \le |b| < r$.

We extend the map $\theta_i - \xi_i$ to D''_γ, and also construct $\theta_i - \xi_j$ for $i \neq j$. Again consider each open residue class subdisk E' of D''; since $\theta_i - \xi_i$ maps E' to an open disk of radius s, the affinoid $(\theta_i - \xi_i)(D''_\gamma)$ is closed and of radius s and is contained in $D(b, s)$. Conversely, if $i \neq j$, then on each residue class open subdisk E', the disks $\theta_i(E')$ and $\xi_j(E')$ in $\mathbb{P}^1$ are distinct residue classes of the same closed disk $D(a, r)$, so $(\theta_i - \xi_j)(E')$ is a disk whose points all have absolute value r, and therefore cannot lie in $D(b, s)$.

The upshot is that if we mark the ξ_is, as we do in D'_γ, then we can distinguish θ_i from θ_j with $j \neq i$: $\theta_j - \xi_i$ maps to a different (thus disjoint) affinoid from $\theta_i - \xi_i$. Therefore, the maps θ_i are all defined on D'_γ, forcing the projection $D''_\gamma \to D'_\gamma$ to be of degree 1.

Finally, let us prove that the projection from D'_γ to D' is tame. This is the only step in the proof where we require φ to be the composition (or iteration) of maps of degree less than p, rather than more generally tame. The cover $D'_\gamma \to D'$ is the Galois closure of the cover defined by marking just one jth preimage of γ, say ξ_1.

Since φ is composed of maps of degree less than p, the Galois group of $\varphi^j(z) - \gamma$ over the field $k(C')$ has order not divisible by p; this is because the Galois group of the composition of rational functions is a subgroup of an iterated wreath product (see the introduction of [4] for some more details). This means the Galois subgroup fixing the affinoid component D'_γ has order not divisible by p as well, and thus the degree of the mapping cannot be divisible by p.

Putting this all together, we have a cover D''_γ of D', which is tame because it is the composition of a tame cover and a degree-1 cover. This means that $\pi : D'' \to D'$ is also tame. By definition, D'' is a Galois cover of D' that is tame and has degree more than 1, so it has a critical point, using Theorem 6.3.2 of [2]. This is a contradiction to the assumption the cover $C'' \to C'$ is unramified. □

Remark 2.5. The proof of Lemma 2.4 could be simplified, dropping the use of Galois closures, if an analog of Theorem 6.3.2 of [2] were proven. This would also allow us to replace the assumption that φ is the composition of low-degree maps with the weaker assumption that φ is tame.

Now, we join Lemma 2.4 and Proposition 2.3 to get what we need:

Corollary 2.6. *Suppose that* 0 *is persistently preperiodic to a repelling fixed point with constant multiplier, as in the construction of* Γ*. For each point* $\varphi \in C'$*, there exists an infinite set of repelling points* z_i *that converge to* 0*, such that for some neighborhood of* $D \ni \varphi$*, the functions* z_i *are all analytic on* D*. Moreover, the convergence of* z_i *is uniform on* D*.*

Proof. By Proposition 2.1, there is a sequence of repelling points z_i converging to 0. Since all but finitely many multipliers are constant, we can in fact choose our sequence such that all repelling points have constant multipliers. By Lemma 2.4 and Proposition 2.3 each repelling point z_i is an analytic map from D to a neighborhood D'' in C''. Now the map from C'' to $\mathbb{P}^1$ defined by z_i is rational, and again using Proposition 2.3 it is analytic, which forces z_i to be an analytic map from D to $\mathbb{P}^1$. Now, the point 0 cannot collide with any of the points z_i for any map in D. Thus, $z_i(D) \neq 0$ and so since z_i is analytic it maps D into a disk $D(a_i, r_i)$ where $r_i < |a_i|$ and since the absolute value is constant on each such disk we have $a_i \to 0$ and thus $z_i \to 0$ uniformly on D. □

An infinite labeled sequence of analytic functions converging to 0 is the best we can hope for. We use the non-collision of periodic points to further prove,

Lemma 2.7. *Suppose that a critical point* γ *is preperiodic to a repelling cycle of constant multiplier for some point on* C'*. Then it is preperiodic, with the same cycle and tail lengths, for all points on* C'*.*

Proof. After iteration, we may assume that $\varphi(z)$ is a fixed point. We will show that this remains true under infinitesimal perturbation; since the property of mapping to a fixed point is algebraic, cut out by the single equation $\varphi^2(z) = \varphi(z)$, it holds either for finitely many points on C' or on all of C'.

We assume that ∞ is repelling and that 0 maps to ∞ for all $\varphi \in C'$. Because we marked the critical points on C', we may regard γ as a rational function from C' to $\mathbb{P}^1$. The image of γ misses ∞, so it maps every disk in C' to a disk in $\mathbb{P}^1$ analytically. Suppose that there exists $c_0 \in C'$ such $\gamma = 0$. We label the map φ

associated to c_0 by φ_0, and take a disk $D \ni \varphi_0$ such that γ becomes an analytic function on D.

Let us denote points of D by c. By Corollary 2.6, there exists an infinite sequence of analytic functions $z_i : D \to \mathbb{P}^1$ such that $z_i(c)$ is a repelling periodic point for φ_c and $z_i(c) \to 0$ uniformly. From the proof of Corollary 2.6, $|z_i(c)| = |z_i(c_0)|$ for all i and for all $c \in D$.

Moreover, $z_i(c) - \gamma(c)$ is an analytic function on D that is never zero: the multiplier at z_i is not zero, which makes it impossible for it to collide with a critical point. Now $z_i(c) - \gamma(c)$ maps over a disk that includes $z_i(c_0)$ and excludes 0, so that $|z_i(c) - \gamma(c)| \leq |z_i(c_0)|$.

Finally, we obtain

$$|\gamma(c)| \leq \max\{|z_i(c_0)|, |z_i(c)|\} = |z_i(c_0)| \to_{i\to\infty} 0$$

and so, $\gamma(c) = 0$ on D, hence on all of C'. □

The next step is almost identical to the complex case:

Lemma 2.8. *All critical points of all maps in C are persistently preperiodic, with uniformly bounded cycle and tail lengths.*

Proof. We consider each critical point separately. If it is periodic, then it is persistently so because C' is isospectral, and the period is constant. If it is preperiodic to a repelling cycle, then the same is also true by Lemma 2.7. Let us now assume it is not preperiodic to a repelling cycle and derive a contradiction. There is a rational function γ from C' to $\mathbb{P}^1$ sending a map to a critical point, and then the functions $\varphi^i(\gamma)$ are rational for all i. Those functions all miss the repelling points, so if we fix any three repelling points, they become functions from C' to $\mathbb{P}^1$ minus three points. Recall that,

Lemma 2.9. *There are finitely many nonconstant separable maps from an affine algebraic curve to $\mathbb{P}^1$ minus three points.*

Proof. It suffices to pass to the normalization. Write the algebraic curve as X minus $x_1, \ldots, x_n$, where X is smooth. We need to show there are finitely many maps from X to $\mathbb{P}^1$ such that the preimage of $\{0, 1, \infty\}$ is contained in $\{x_1, \ldots, x_n\}$. For each degree d there are finitely many possible maps, determined by the preimages of $(0, 1, \infty)$ and their multiplicities; this is because knowing the zeros and poles (counted with multiplicity) of a rational function on an algebraic curve is enough to determine the map up to multiplication by a constant. It remains to bound d. Now let e_i be the multiplicity with which x_i maps. We have $\sum e_i \geq 3d$ and $\sum(e_i - 1) \leq 2d - 2 + 2g(X)$, which gives us $n \geq d + 2 - 2g(X)$, valid as long as the maps are separable, even if they are wildly ramified. □

Now, the function γ is separable; to see this, we need to parametrize C' by γ and its orbit. Knowing the value of φ at $2d + 1$ distinct points, counting multiplicity, is enough to determine φ. Therefore, on the open dense subset of C' on which the points $\gamma, \varphi(\gamma), \ldots, \varphi^{2d}(\gamma)$ are distinct, they fully specify φ (we have $2d$ distinct points and their images, with γ mapping to its image with multiplicity at least 2). But now the map from $(\gamma, \ldots, \varphi^{2d}(\gamma))$ to γ has local degree 1 at most points since φ is tamely ramified, which means it has to be separable. Furthermore, $\varphi^i(\gamma)$ is also separable, since φ is separable.

Note that if we view C as a morphism from an abstract curve into M_d then this does not work, because this morphism could be inseparable; an example given by the anonymous referee of this paper is $\varphi_a(z) = z^2 + a^p z$, and then $\gamma(a) = a^p$. However, once we make sure to parametrize C' as a subvariety of $\mathrm{M}_d(\Gamma)$ by a suitable critical orbit, we obtain $\varphi_\gamma(z) = z^2 + \gamma z$ and then we obtain a separable map.

We can now apply Lemma 2.9. Since the sequence of functions $\varphi^i(\gamma)$ comes from a finite list, two functions (say $i = m, n$) have to coincide, making x persistently preperiodic after all, and the tail and cycle lengths are again fixed by the choice of m and n. Alternatively, all but finitely many functions have to be constant, and then we know the values of φ at infinitely many given points; knowing the values of φ at $2d + 1$ distinct points is enough to determine φ, and therefore C is trivial, contradicting the assumption that it is nontrivial. □

The final step is to note that tame PCF maps over function fields have multipliers lying in $\overline{\mathbb{F}_p}$; this is Corollary 1.7 of [1]. Since by assumption this is not the case for C', we have a contradiction, and Theorem 1.12 is proved.

3. Iterative Root-Finding Algorithms

Iterative root-finding algorithms are techniques used to find the roots of a polynomial by using iteration; see Definition 1.17. To prove that they do not exist in most non-archimedean cases, we will use the full strength of Theorem 1.12, allowing C to have finitely many nonconstant multipliers.

In McMullen's original formulation, the families $C \subseteq \mathrm{M}_d$ that were proven to be constant or Lattès were slightly broader than isospectral ones, too: McMullen studied so-called stable families, that is, families for which the period of the longest attracting cycle is bounded. Of course, for cycle lengths beyond this uniform bound, the multipliers miss infinitely many values (all attracting ones), and therefore are constant in any algebraic family. Thus, McMullen's stable families meet the condition in Theorem 1.12 that all but finitely many multiplier spectra be constant. They trivially meet the other conditions: all functions over $\mathbb{C}$ are tame (and, for the proof of Lemma 2.4, so are all covers), and the condition that the multipliers not all be contained in $\overline{\mathbb{F}_p}$ really exists only to obtain a repelling cycle in some completion K_v of F, whereas by [17], all but at most $2d - 2$ cycles of any map in $\mathrm{M}_d(\mathbb{C})$ are repelling.

Proof of Theorem 1.18. If $p \geq r$ or $p = 0$, then the proof is not a corollary of the results in the remainder of this paper. However, if $p > d$ then it is a corollary of Theorem 1.12; we prove the theorem under either condition.

If $T_f^n(z)$ converges to a root of f on an open dense subset of $\mathbb{P}^1$, then it is impossible for T_f to have non-repelling cycles except those corresponding to the roots of f. To see why, observe that attracting cycles have open basins of attraction, and over a non-archimedean field indifferent cycles have open basins on which points are recurrent; see section 3 of [16]. Attracting fixed points that are not roots of f will then have an open set converging to them, and indifferent points and attracting or indifferent cycles of period more than 1 have basins on which points either are already periodic or have orbits that do not converge to anything.

Remark 3.1. Periodic points may occur near an indifferent fixed point z if its multiplier is a root of unity, and we work over a field K of positive characteristic. If $\operatorname{char} K = 0$ or the multiplier at z is not a root of unity, then it is proven in [16] that we can find an open set containing z without periodic points other than z; see also [14]. It is conjectured that we can also find such an open set if $\operatorname{char} K > 0$ and the multiplier at z is a root of unity. However, even if this conjecture is false, it is certainly true that near an indifferent fixed point z, points have orbits that do not converge to another attracting fixed point.

We will deal with the $p \geq r$ and $p > d$ cases separately. First, let us assume that $p \geq r$ or $p = 0$. When the fixed points are all distinct, that is, when none of their multipliers is 1, the following formula holds for their multipliers λ_i:

$$\sum_{i=1}^{d+1} \frac{1}{1-\lambda_i} = 1$$

Let us now look at the formula modulo the maximal ideal $\mathfrak{m}$ of the ring of integers of the field K. Whenever λ_i corresponds to an attracting point, we have $1/(1-\lambda_i) \equiv 1 \mod \mathfrak{m}$, and whenever it corresponds to a repelling point, $|1-\lambda_i| > 1$ and so $1/(1-\lambda_i) \equiv 0 \mod \mathfrak{m}$. Since we have r attracting points, we have $r \equiv 1 \mod \mathfrak{m}$. This implies $p|(r-1)$, which contradicts the assumption that $p \geq r$ or $p = 0$.

Let us now assume that $p < r$, but still $p > d$. Observe that this implies that $r \geq 4$. The image $T(\mathrm{Poly}_r) \subseteq \mathrm{Rat}_d$ is an algebraic family, and Λ_n applied to $T(\mathrm{Poly}_r)$ is also algebraic, so if it misses infinitely many values of each multiplier (namely, the attracting ones), it must be constant. This shows that the multiplier spectra on $T(\mathrm{Poly}_r)$ are constant except possibly for the finitely many corresponding to fixed points. Moreover, the attracting cycles of period more than 1 must be repelling, so clearly the multipliers are not in $\overline{\mathbb{F}_p}$.

We apply Theorem 1.12 and obtain that $T(\mathrm{Poly}_r)$ maps to a single point of M_d, which we call φ. An open dense set of $\mathbb{P}^1$ converges to a finite set of points, called the sinks of φ, which must be contained in the set of roots of every $f \in \mathrm{Poly}_r$. Moreover, the set of sinks of φ must coincide with the set of roots of f, because the coefficients of T_f depend algebraically on the coefficients of f, which are symmetric functions of the roots of f. This means that the existence of a generally convergent algorithm implies that an open dense set of $f \in \mathrm{Poly}_r$ have identical configurations of roots in $\mathbb{P}^1$. Since $r \geq 4$, this contradicts the existence of the cross-ratio. $\square$

The portion of the proof assuming $p \geq r$ is unique to the non-archimedean setting, while the portion assuming $p > d$ generalizes McMullen's work. As a result, McMullen only shows there do not exist generally convergent algorithms if $r \geq 4$. Over the complex numbers, there exist generally convergent algorithms for polynomials of degrees 2 and 3. In degree 2, Newton's method is such an algorithm. Over a non-archimedean field, Newton's method, equivalent to the rational function $\varphi(z) = z^2$, is no longer generally convergent. Over the complex numbers, the open dense set lying away from the unit circle has orbit converging to 0 and ∞, but in any non-archimedean field, the equivalent of the unit circle is the set of finite nonzero residue classes, which has nonempty interior.

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