

UNIFORM BOUNDS FOR PRE-PERIODIC POINTS IN FAMILIES OF TWISTS

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ABSTRACT. Let ϕ be a morphism of \mathbb{P}^N defined over a number field K . We prove that there is a bound B depending only on ϕ such that every twist of ϕ has no more than B K -rational preperiodic points. (This result is analogous to a result of Silverman for abelian varieties [14].) For two specific families of quadratic rational maps over \mathbb{Q} , we find the bound B explicitly.

1. INTRODUCTION

Let K be a number field and $\phi : \mathbb{P}^N(K) \rightarrow \mathbb{P}^N(K)$ be a morphism on projective space of dimension N . (Note: When the ground field is clear we write simply \mathbb{P}^N .) We denote by ϕ^n the n^{th} iterate of ϕ under composition. A point $P \in \mathbb{P}^N$ is *periodic* (with period n) if there exists an integer $n > 0$ such that $\phi^n(P) = P$, and P has *primitive period* n if n is the smallest such number. Also, P is *preperiodic* if there exist integers $n > m \geq 0$ such that $\phi^n(P) = \phi^m(P)$. Let

$$\text{PrePer}(\phi, \mathbb{P}^N(K)) = \{P \in \mathbb{P}^N(K) : P \text{ is preperiodic under } \phi\}.$$

A motivating problem in the field of arithmetic dynamics is the uniform boundedness conjecture of Morton and Silverman [10].

Conjecture. *Let K/\mathbb{Q} be a number field of degree D , and let $\phi : \mathbb{P}^N \rightarrow \mathbb{P}^N$ be a morphism of degree $d \geq 2$ defined over K . There is a constant $\kappa(D, N, d)$ such that*

$$\#\text{PrePer}(\phi, \mathbb{P}^N(K)) \leq \kappa(D, N, d).$$

This is a deep and difficult problem. It implies, for example, uniform boundedness for torsion points on abelian varieties over number fields (see [1]). Even the special case $N = 1$ and $d = 4$ is enough to imply Merel's uniform boundedness of torsion points on elliptic curves [7]. Though much work has been done on this problem for nearly 20 years, to date only non-uniform bounds are known.

As a first step, one might ask if Conjecture 1 holds in interesting families of dynamical systems. For example, in [13] Poonen conjectures a precise bound for quadratic polynomials over \mathbb{Q} , which can be viewed as a one-parameter family $f_c(z) = z^2 + c$:

Conjecture. *If $z_0, c \in \mathbb{Q}$ such that z_0 has primitive period n for $f_c(z) = z^2 + c$, then $n \leq 3$.*

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Even this special of Conjecture 1 remains very much open. Morton [9] has shown that $n \neq 4$. Flynn, Poonen, Schaefer [2] showed that $n \neq 5$. And Stoll [16] proved that the Birch and Swinnerton-Dyer Conjecture implies $n \neq 6$. Given the difficulty of the question for quadratic polynomials, our Theorem 4.1 regarding a different one-parameter family of quadratic functions appears surprisingly strong.

Theorem. *The rational map $\phi_b(z) = z + \frac{b}{z}$ where $b \in \mathbb{Q}$ has no rational points with primitive period $n \geq 5$.*

It turns out that these two one-parameter families are different in a dynamically significant way. The quadratic functions ϕ_b considered above are all quadratic twists of the function $\phi(z) = z + 1/z$. This is not the case for the quadratic polynomials f_c considered by Poonen and others.

In Theorem 2.9, we show that Conjecture 1 holds for families of twists of rational maps. More precisely, for any twist ψ of a morphism of projective space ϕ defined over a number field K , the number of K -rational preperiodic points is uniformly bounded by a constant depending only on the map ϕ , but independent of the twist. This statement and its proof are similar to a result for abelian varieties in [14]. There, Silverman shows that given an abelian variety A , for all but finitely many twists A_ξ of A the set of K -rational torsion points $A_\xi(K)_{\text{tors}}$ is contained in

$$\{P \in A_\xi(\overline{K}) : \text{for some } f \in \text{Aut } A_\xi, f \neq \text{id}, f(P) = P\}.$$

We conclude with an outline of the paper and a survey of the techniques used.

Section 2: We provide background on Northcott's Theorem and twists of rational maps. We conclude with a proof of our main result, Theorem 2.9, which combines this background material with Galois-theoretic methods.

Section 3: We introduce the primary tool for understanding periodic points for morphisms of the projective line: dynatomic polynomials. We use arithmetic functions and elementary number theory to prove several lemmas about the coefficients of these polynomials for the family of maps $\phi_{k,b}(z) = k(z + b/z)$.

Section 4: Using the background in Section 3 and an application of the rational root theorem, we get the surprisingly strong Theorem 4.1 described above. Combining this Theorem with results from [6], we give all possible graphs associated to rational preperiodic points for maps of the form $\phi_b(z) = z + b/z$; there are at most 6 rational preperiodic points for such maps.

Section 5: Using the same techniques, we get a similar theorem and graphical description of rational preperiodic point structures for the family of maps $\psi_b(z) = -(z + b/z)$.

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2. UNIFORM BOUNDS FOR FAMILIES OF TWISTS

We begin with some background in arithmetic dynamics. Throughout, K will be a number field, and we will state explicitly when results hold for more general fields.

Definition 2.1. We define

$$\mathrm{Hom}_d^N(K) = \{\phi : \mathbb{P}^N(K) \rightarrow \mathbb{P}^N(K) : \phi \text{ is a morphism of degree } d^N\}.$$

That is, ϕ is defined in each coordinate by homogeneous polynomials of degree d with coefficients in K . (We follow the convention that Hom_d^N refers to $\mathrm{Hom}_d^N(\overline{K})$, and similarly for PGL_{N+1} .)

The primary tool in our proof is a classic theorem of Northcott [11].

Theorem 2.2 (Northcott). *Let $\phi \in \mathrm{Hom}_d^N(K)$. Then the set of preperiodic points $\mathrm{PrePer}(\phi, \mathbb{P}^N(\overline{K}))$ is a set of bounded height. In particular, $\mathrm{PrePer}(\phi, \mathbb{P}^N(K))$ is a finite set, and more generally, for any $D > 1$ the set*

$$\bigcup_{[L:K] \leq D} \mathrm{PrePer}(\phi, \mathbb{P}^N(L))$$

is finite.

Let $f \in \mathrm{PGL}_{N+1}$ act on the points of $\mathbb{P}^N(\overline{K})$ as a fractional linear transformation in the usual way. Then we define the morphism

$$\phi^f = f \circ \phi \circ f^{-1}.$$

Definition 2.3. Let $\phi, \psi \in \mathrm{Hom}_d^N(K)$. We say the morphisms are *conjugate* if there is some $f \in \mathrm{PGL}_{N+1}(\overline{K})$ such that $\phi^f = \psi$. They are *conjugate over K* if there is some $f \in \mathrm{PGL}_{N+1}(K)$ such that $\phi^f = \psi$.

If P is a point of period n for ϕ , then $f(P)$ has the same property for ϕ^f , and similarly for preperiodic points. It is also easily seen that $(\phi^n)^f = (\phi^f)^n$. So conjugate maps have essentially the same dynamical behavior. However, if we are concerned with the arithmetic of the (pre)periodic points, we must be a bit more careful. For a map $\phi \in \mathrm{Hom}_d^N(K)$,

$$\mathrm{Twist}(\phi/K) = \left\{ \begin{array}{l} K\text{-equivalence classes of maps } \psi \in \mathrm{Hom}_d^N(K) \\ \text{such that } \psi \text{ is } \overline{K}\text{-equivalent to } \phi \end{array} \right\}.$$

An element $\psi \in \mathrm{Twist}(\phi/K)$ is called a twist of ϕ .

Example 2.4. Let

$$\phi(z) = z - \frac{2}{z} \text{ and } \psi(z) = z - \frac{1}{z}.$$

Also let $f(z) = z\sqrt{2}$. One may check that $\phi^f(z) = \psi(z)$. So ψ is a (quadratic) twist of ϕ . Solving $\phi^2(z) = z$ gives the \mathbb{Q} -rational two-cycle $\{\pm 1\}$. But ψ does not have rational points of primitive period 2; solving $\psi^2(z) = z$ gives $\{\pm 1/\sqrt{2}\}$.

Definition 2.5. For any $\phi \in \mathrm{Hom}_d^N(\overline{K})$ define \mathcal{A}_ϕ to be the *automorphism group* of ϕ , i.e.,

$$\mathcal{A}_\phi = \{f \in \mathrm{PGL}_{N+1}(\overline{K}) \mid \phi^f = \phi\}.$$

From [5, 12], \mathcal{A}_ϕ is well-defined as a finite subgroup of $\mathrm{PGL}_{N+1}(\overline{K})$.

We introduce some notation to make the statement and proof of Lemma 2.7 more succinct. Let $\phi \in \text{Hom}_d^N(K)$, and $\psi \in \text{Twist}(\phi/K)$, meaning there is an $f \in \text{PGL}_{N+1}$ with $\phi^f = \psi$. Write $f = (a_{ij})$, a matrix. At least one of the a_{ij} is nonzero, say $a_{lm} \neq 0$. So we may also write $f = (a'_{ij})$ where $a'_{ij} = \frac{a_{ij}}{a_{lm}}$, since this represents the same element of PGL_{N+1} .

Definition 2.6. Let $K_f = K(a'_{ij})$ be the minimal field of definition for a given $f \in \text{PGL}_{N+1}$ such that $\phi^f = \psi$, and let L_f be the Galois closure of K_f .

Lemma 2.7. For any $\psi \in \text{Twist}(\phi/K)$ and any f satisfying $\phi^f = \psi$,

$$[K_f : K] \leq \#\mathcal{A}_\phi.$$

Proof. Choose $f \in \text{PGL}_{N+1}$ such that $\phi^f = \psi$, and let $\sigma \in \text{Gal}(L_f/K)$. Since ψ is defined over K ,

$$\begin{aligned} f\phi f^{-1} &= (f\phi f^{-1})^\sigma = f^\sigma \phi^\sigma (f^{-1})^\sigma \\ &= f^\sigma \phi (f^{-1})^\sigma. \\ \phi &= f^{-1} f^\sigma \phi (f^\sigma)^{-1} f. \end{aligned}$$

Hence $f^{-1}f^\sigma \in \mathcal{A}_\phi$. Define the map

$$\begin{aligned} \rho : \text{Gal}(L_f/K) &\rightarrow \mathcal{A}_\phi \\ \sigma &\mapsto f^{-1}f^\sigma \end{aligned}$$

Then

$$\begin{aligned} \rho(\sigma) = \rho(\tau) &\Leftrightarrow f^{-1}f^\sigma = f^{-1}f^\tau \\ &\Leftrightarrow f^\sigma = f^\tau \\ &\Leftrightarrow f = f^{\tau\sigma^{-1}}. \end{aligned}$$

So $\tau\sigma^{-1} \in \text{Gal}(L_f/K_f)$ since it fixes f . Clearly if $\tau_1\sigma^{-1} = \tau_2\sigma^{-1}$ as elements of $\text{Gal}(L_f/K_f)$, then they are equal as elements of $\text{Gal}(L_f/K)$, and $\tau_1 = \tau_2$. Hence ρ is $[L_f : K_f]$ -to-1, and we conclude that

$$[L_f : K_f][K_f : K] = [L_f : K] \leq [L_f : K_f](\#\mathcal{A}_\phi),$$

which gives the result since all of the extensions are finite. \square

Remark 2.8. If all elements of \mathcal{A}_ϕ are defined over K , then ρ is in fact a group anti-homomorphism; that is, it is a homomorphism with the direction of group multiplication reversed. This is because $\sigma\tau \mapsto f^{-1}f^{\sigma\tau}$ but now $(f^{-1}f^\tau)^\sigma = f^{-1}f^\tau$. So we have,

$$\begin{aligned} \rho(\sigma)\rho(\tau) &= f^{-1}f^\sigma f^{-1}f^\tau = f^{-1}f^\sigma (f^{-1}f^\tau)^\sigma \\ &= f^{-1}f^\sigma (f^{-1})^\sigma f^{\tau\sigma} = f^{-1}f^{\tau\sigma} \\ &= \rho(\tau\sigma). \end{aligned}$$

Inversion is also an anti-homomorphism, so the map ρ^{-1} , which maps each $\sigma \in \text{Gal}(L_f/K)$ to $\rho^{-1}(\sigma)$, is a homomorphism. Since the map is injective, we conclude that $\text{Gal}(L_f/K)$ is a subgroup of \mathcal{A}_ϕ .

The following is now a straightforward consequence of Northcott's Theorem.

Theorem 2.9. *Let K be a number field and let $\phi \in \text{Hom}_d^N(K)$. Then there is a uniform bound B_ϕ such that for all $\psi \in \text{Twist}(\phi/K)$,*

$$\#\text{PrePer}(\psi, \mathbb{P}^N(K)) \leq B_\phi.$$

Proof. Given $\psi \in \text{Twist}(\phi/K)$ and $f \in \text{PGL}_{N+1}$ such that $\phi^f = \psi$, every K -rational preperiodic point for ψ corresponds to a K_f -rational preperiodic point for ϕ . From Lemma 2.7, $[K_f : K] \leq \#\mathcal{A}_\phi := D$. By Northcott's Theorem,

$$\bigcup_{[L:K] \leq D} \text{PrePer}(\phi, \mathbb{P}^N(K_f))$$

is finite; call the cardinality of this set B_ϕ . Then for every $\psi \in \text{Twist}(\phi/K)$,

$$\#\text{PrePer}(\psi, \mathbb{P}^N(K)) \leq B_\phi. \quad \square$$

Remark 2.10. In fact Lemma 2.7 holds over an arbitrary field K ; hence Theorem 2.9 also holds when K is a function field over a finite field.

Example 2.11. Suppose that $\phi(z) : \mathbb{P}^1(K) \rightarrow \mathbb{P}^1(K)$ is a degree-2 morphism with a unique fixed point $P \in \mathbb{P}^1(\overline{K})$. Applying any Galois action to the equation $\phi(P) = P$ shows that any Galois conjugate of P is also fixed by ϕ . Hence, $P \in \mathbb{P}^1(K)$. Choose a change of coordinates $f \in \text{PGL}_2(K)$ moving P to infinity. Then we may write

$$\phi^f(z) = \frac{z^2 + az + b}{z + a} \text{ for some } a, b \in K \text{ with } b \neq 0.$$

The critical points of ϕ^f are $a \pm \sqrt{b}$, so conjugate by $g \in \text{PGL}_2(K)$ which fixes infinity and moves the critical points to $\pm\sqrt{b}$, and we see that ϕ is conjugate over K to a map of the form $\phi_b(z) = z + \frac{b}{z}$.

For any $b \in K^*$, ϕ_b is a quadratic twist of $\phi(z) = z + \frac{1}{z}$ with conjugating map $f_b = z\sqrt{b}$. So $f_b \in \text{PGL}_2(L)$ with $[L : K] \leq 2$. Let $Q \in \mathbb{P}^1(K)$ be a K -rational preperiodic point for ϕ_b . Then $f_b(Q)$ is an L -rational preperiodic point for ϕ .

By Theorem 2.9, $\#\text{PrePer}(\phi_b, \mathbb{P}^1(K)) \leq B$ for some absolute bound B , independent of the parameter b . In Section 4, we show that for this twisted family of quadratic rational maps when $K = \mathbb{Q}$, the bound is actually 6.

3. MAIN TOOL: DYNATOMIC POLYNOMIALS

For a morphism $\phi : \mathbb{P}^1(K) \rightarrow \mathbb{P}^1(K)$ (i.e. when the dimension $N = 1$), we may choose $F, G \in K[z]$ such that $\phi(z) = F(z)/G(z)$ with $\deg \phi = \max\{\deg F, \deg G\}$. Hence we use the terms ‘‘morphism’’ and ‘‘rational map’’ interchangeably when the map is on the projective line.

In this section, we will write rational maps using homogeneous coordinates:

$$\begin{aligned} \phi : \mathbb{P}^1 &\rightarrow \mathbb{P}^1 \\ [X : Y] &\mapsto [F(X, Y) : G(X, Y)], \end{aligned}$$

where F and G are homogeneous polynomials of the same degree with no common factor. Then for $n > 1$,

$$\phi^n[X : Y] = [F_n(X, Y) : G_n(X, Y)],$$

where F_n and G_n are given recursively by

$$F_n(X, Y) = F_{n-1}(F(X, Y), G(X, Y)) \text{ and } G_n(X, Y) = G_{n-1}(F(X, Y), G(X, Y)).$$

Definition 3.1. The n -period polynomial of ϕ is

$$\Phi_{\phi,n}(X, Y) = YF_n(X, Y) - XG_n(X, Y).$$

The n^{th} dynatomic polynomial of ϕ is the polynomial

$$\Phi_{\phi,n}^*(X, Y) = \prod_{d|n} \Phi_{\phi,d}(X, Y)^{\mu(\frac{n}{d})},$$

where μ is the Mobius function. When the function ϕ is clear, we will suppress it from the notation, writing simply Φ_n^* .

See [15, p.151] for a proof that $\Phi_n^*(X, Y)$ is indeed a polynomial. Clearly it is then a homogeneous polynomial in X and Y . Further, we let

$$\nu_2(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) 2^d = \text{degree of } \Phi_n^* \text{ for a quadratic rational map.}$$

By construction, all points of period $d \mid n$ are roots of the n -period polynomial Φ_n . One might hope that the roots of Φ_n^* are the points of primitive period n (eliminating as roots points with period $d < n$). This isn't quite the case, but it is true that every point with primitive period n is a root of Φ_n^* , and that fact is enough for our purposes. See [15, Chapter 4] for details about dynatomic polynomials and their properties.

Lemma 3.2. *The following products are positive powers of k for $n > 1$:*

- (1) $\prod_{d|n} \left(k^{2^d-d-1}\right)^{\mu(\frac{n}{d})}$.
- (2) $\prod_{d|n} \left(k^{2^d-1}\right)^{\mu(\frac{n}{d})}$.
- (3) $\prod_{d|n} \left(k^{2^{d-1}}\right)^{\mu(\frac{n}{d})}$.
- (4) $\prod_{d|n} \left(k^{b(d)}\right)^{\mu(\frac{n}{d})}$ where $b(d) = \left\lceil \frac{2(2^{d-1}-1)}{3} \right\rceil$.

Proof. Proofs will rely on facts about the properties of the arithmetic functions φ and μ , as well as the connection between them. We refer the interested reader to [8, Chapter 2] for these facts.

(1) Consider

$$\begin{aligned} \prod_{d|n} \left(k^{2^d-d-1}\right)^{\mu(\frac{n}{d})} &= k^{\sum_{d|n} \mu(\frac{n}{d})(2^d-d-1)} \\ &= k^{\sum_{d|n} \mu(\frac{n}{d})(2^d-d)}. \end{aligned}$$

The last step follows from the fact that when $n > 1$ the $\sum_{d|n} \mu\left(\frac{n}{d}\right) = 0$. Recall that $\sum_{d|n} \mu\left(\frac{n}{d}\right)d = \varphi(n)$ where $\varphi(n)$ is the Euler totient function.

Also,

$$\sum_{d|n} \mu\left(\frac{n}{d}\right) 2^d \geq 2^n - \sum_{d|n, d \neq n} 2^d \geq 2^n - 2^{n-1} = 2^{n-1}. \quad (3.1)$$

Hence

$$\begin{aligned}
\sum_{d|n} \mu\left(\frac{n}{d}\right) (2^d - d) &= \sum_{d|n} \mu\left(\frac{n}{d}\right) 2^d - \sum_{d|n} \mu\left(\frac{n}{d}\right) d \\
&= \sum_{d|n} \mu\left(\frac{n}{d}\right) 2^d - \varphi(n) \\
&\geq \sum_{d|n} \mu\left(\frac{n}{d}\right) 2^d - n \\
&\geq 2^{n-1} - n.
\end{aligned}$$

After taking the derivative of $2^{x-1} - x$, we see that the function is increasing as long as $x > 1$. Hence $\sum_{d|n} \mu\left(\frac{n}{d}\right) (2^d - d) > 0$.

(2) follows immediately from (1).

(3) follows immediately from equation (3.1), replacing d by $d - 1$ and using the fact that $n > 1$.

(4) Now consider

$$\sum_{d|n} \mu\left(\frac{n}{d}\right) b(d) \geq \sum_{d|n} \mu\left(\frac{n}{d}\right) \frac{2}{3} (2^{d-1} - 1).$$

Multiplying the result from (3) by $\frac{2}{3}$ and using the fact that $\sum_{d|n} \mu(n/d) = 0$ for $n > 1$, we see that (4) is also positive. \square

Let K be a number field and let

$$\begin{aligned}
\phi_{k,b} : \mathbb{P}^1 &\rightarrow \mathbb{P}^1 \\
[X : Y] &\mapsto [k(X^2 + bY^2) : XY]
\end{aligned}$$

for some $b, k \in K^*$. *A priori* Φ_n^* is a polynomial in X and Y . However, if we view k and b as parameters, we may ask if $\Phi_n^* \in \mathbb{Z}[k, b, X, Y]$. The following Lemma will help us address this question.

Lemma 3.3. *For every $n > 1$,*

- (1) *The coefficient of $X^{\nu_2(n)}$ in $\Phi_{\phi_{k,b};n}^*(X, Y)$ is a positive power of k times $C_n(k)$, where $C_n(k)$ refers to the n^{th} cyclotomic polynomial in the variable k .*
- (2) *The coefficient of $Y^{\nu_2(n)}$ in $\Phi_{\phi_{k,b};n}^*(X, Y)$ is a positive power of k times a positive power of b .*
- (3) *Each monomial of Φ_n^* is divisible by k .*

Proof. We have

$$\begin{aligned}
F_1(X, Y) &= k(X^2 + bY^2) & G_1(X, Y) &= XY \\
F_n(X, Y) &= k(F_{n-1}^2 + bG_{n-1}^2) & G_n(X, Y) &= F_{n-1}G_{n-1}.
\end{aligned} \tag{3.2}$$

A simple induction argument shows that for $n > 1$, we have

$$\deg_X(F_n) = \deg_X(G_n) + 1, \quad \text{and in fact } \deg_X(F_n) = 2^n \text{ and } \deg_X(G_n) = 2^n - 1.$$

(The same arguments hold for $\deg_Y F_n$ and $\deg_Y G_n$.) Now,

$$\text{coefficient of } X^{2^n} \text{ in } F_n = k \left(\text{coefficient of } X^{2^{n-1}} \text{ in } F_{n-1} \right)^2,$$

so inductively this coefficient is k^{2^n-1} . Similarly,

$$\begin{aligned} \text{coeff. of } X^{2^n-1} \text{ in } G_n &= \left(\text{coeff. of } X^{2^{n-1}} \text{ in } F_{n-1} \right) \left(\text{coeff. of } X^{2^{n-1}-1} \text{ in } G_{n-1} \right) \\ &= Y \prod_{i=0}^{n-1} k^{2^i-1} = Y k^{2^n-n-1}. \end{aligned}$$

Let c_d be the coefficient of X^{2^d} in $\Phi_d = YF_d - XG_d$ and c_d^* be the coefficient of $X^{\nu_2(d)}$ in Φ_d^* . So that

$$\begin{aligned} c_d &= Y \left(k^{2^d-1} - k^{2^d-d-1} \right) = Y k^{2^d-d-1} (k^d - 1) \\ c_n^* &= \prod_{d|n} c_d^{\mu(\frac{n}{d})} = \prod_{d|n} \left(Y k^{2^d-d-1} (k^d - 1) \right)^{\mu(\frac{n}{d})} \\ &= \prod_{d|n} Y^{\mu(\frac{n}{d})} \prod_{d|n} \left(k^{2^d-d-1} \right)^{\mu(\frac{n}{d})} \prod_{d|n} (k^d - 1)^{\mu(\frac{n}{d})}. \end{aligned}$$

(Here, we use the definition of Φ_n^* and the fact that we know it is a polynomial in X and Y .) When $n > 1$, the first term is 1, the second is a positive power of k by Lemma 3.2, and the third is $C_n(k)$ exactly as claimed.

Also,

$$\begin{aligned} \text{coefficient of } Y^{2^n+1} \text{ in } \Phi_n &= \text{coefficient of } Y^{2^n} \text{ in } F_n \\ &= k \left(\text{coefficient of } Y^{2^{n-1}} \text{ in } F_{n-1} \right)^2, \end{aligned}$$

so inductively this coefficient is $k^{2^n-1} b^{2^n-1}$. So then

$$\text{coefficient of } Y^{\nu_2(n)} \text{ in } \Phi_n^* = \prod_{d|n} \left(k^{2^d-1} b^{2^d-1} \right)^{\mu(\frac{n}{d})},$$

which is a positive power of k times a positive power of b by Lemma 3.2.

The proof for the final claim is similar, but the algebraic details are messier. We sketch the main points here and leave the details for the reader. Inductively one may show that

$$F_n(X, Y) = k^{a(n)} f_n(X, Y) \text{ and } G_n(X, Y) = k^{b(n)} g_n(X, Y),$$

where

$$a(n) = \frac{2^n - (-1)^n}{3}, \quad b(n) = \left\lceil \frac{2(2^{n-1} - 1)}{3} \right\rceil$$

and $f_n, g_n \in \mathbb{Z}[k, b, X, Y]$. So then

$$\Phi_n(X, Y) = k^{b(n)} \Psi_n(X, Y),$$

where $\Psi_n \in \mathbb{Z}[k, b, X, Y]$. Then exactly as above, it follows that a positive power of k divides each dynatomic polynomial Φ_n^* . \square

Since $\Phi_n^*(X, Y)$ is homogeneous, we may dehomogenize in the usual way. We will write $\Phi_n^*(z)$ for the dehomogenized dynatomic polynomial. Note that the lead coefficient of $\Phi_1^*(z) = k - 1 = C_1(k)$ and the constant term is bk .

Mobius inversion gives $\prod_{d|n} \Phi_{\phi_{k,b,d}}^*(z) = \Phi_{\phi_{k,b,n}} \in \mathbb{Z}[k, b, z]$. In other words, $\Phi_{\phi_{k,b,n}}(z)$ factors over $\mathbb{Q}(k, b)$, so by Gauss's Lemma it factors over $\mathbb{Z}[k, b]$. Lemma 3.3

and the remark above show that the polynomials $\Phi_{\phi_{k,b,d}}^*(z)$ in the product each have content a non-negative power of k , meaning that $\Phi_{\phi_{k,b,n}}^*(z) \in \mathbb{Z}[k, b, z]$.

Lemma 3.4. *Let $n > 1$. Then each monomial of the n^{th} dynatonic polynomial $\Phi_n^*(X, Y)$ has the form $c_i X^{2i} Y^{\nu_2(n)-2i} b^{\frac{\nu_2(n)-2i}{2}}$ where $c_i \in \mathbb{Z}[k]$.*

Proof. We first show that for all $n \geq 1$,

$$F_n(X, Y) = \sum_{i=0}^{2^{n-1}} c_i X^{2i} Y^{2^n-2i} b^{2^{n-1}-i}, \text{ and} \quad (3.3)$$

$$G_n(X, Y) = Y \sum_{j=1}^{2^{n-1}} d_j X^{2j-1} Y^{2^n-2j} b^{2^{n-1}-j}, \quad (3.4)$$

where $c_i, d_j \in \mathbb{Z}[k]$.

From equation (3.2), we see that F_1 and G_1 have the correct form. Assume F_{n-1} and G_{n-1} satisfy the claim above.

Consider

$$F_n = k \left(\sum_{i=0}^{2^{n-2}} c_i X^{2i} Y^{2^{n-1}-2i} b^{2^{n-2}-i} \right)^2 + kb \left(Y \sum_{j=1}^{2^{n-2}} d_j X^{2j-1} Y^{2^{n-1}-2j} b^{2^{n-2}-j} \right)^2.$$

If we look at the first term monomial by monomial we get

$$\begin{aligned} & \left(c_i X^{2i} Y^{2^{n-1}-2i} b^{2^{n-2}-i} \right) \left(c_j X^{2j} Y^{2^{n-1}-2j} b^{2^{n-2}-j} \right) \\ &= c_i c_j X^{2(i+j)} Y^{2^n-2(i+j)} b^{2^{n-1}-(i+j)}, \end{aligned}$$

so each monomial has the correct form. Now consider monomials from the second term:

$$\begin{aligned} & kb Y^2 \left(d_i X^{2i-1} Y^{2^{n-1}-2i} b^{2^{n-2}-i} \right) \left(d_j X^{2j-1} Y^{2^{n-1}-2j} b^{2^{n-2}-j} \right) \\ &= kd_i d_j X^{2(i+j-1)} Y^{2^n-2(i+j-1)} b^{2^{n-1}-(i+j-1)}, \end{aligned}$$

which has the correct form. This completes the proof for F_n , and the proof for G_n is similar.

It follows immediately from equations (3.3) and (3.4) that

$$\Phi_n(X, Y) = Y F_n - X G_n = Y \sum_{i=0}^{2^n} e_i X^{2i} Y^{2^n-2i} b^{2^{n-1}-i}.$$

We now compute the dynatonic polynomial:

$$\begin{aligned} \Phi_n^*(X, Y) &= \prod_{d|n} \left(Y \sum e_i X^{2i} Y^{2^d-2i} b^{2^{d-1}-i} \right)^{\mu\left(\frac{n}{d}\right)} \\ &= \prod_{d|n} Y^{\mu\left(\frac{n}{d}\right)} \prod_{d|n} \left(\sum e_i X^{2i} Y^{2^d-2i} b^{2^{d-1}-i} \right)^{\mu\left(\frac{n}{d}\right)} \\ &= \prod_{d|n} \left(\sum e_i X^{2i} Y^{2^d-2i} b^{2^{d-1}-i} \right)^{\mu\left(\frac{n}{d}\right)}, \end{aligned}$$

where the last step follows because $n > 1$.

Let

$$\alpha = \sum e_i X^{2i} Y^{D_\alpha - 2i} b^{\frac{D_\alpha}{2} - i} \text{ and } \beta = \sum f_j X^{2j} Y^{D_\beta - 2j} b^{\frac{D_\beta}{2} - j}.$$

Clearly the form is not affected if we add or subtract two such monomials. It is then easy to check that $\alpha\beta$ and $\frac{\alpha}{\beta}$ also have the correct form. \square

Lemma 3.5. *For all $n \geq 1$, there exists a homogeneous polynomial $\psi_n(w, b) \in \mathbb{Z}[w, b]$ such that $\psi_n(z^2, b) = \Phi_n^*(z, b)$.*

Proof. From Lemma 3.4, when $n > 1$ each monomial of $\Phi_n^*(z)$ has the form $c_i z^{2i} b^{\frac{\nu_2(n)-2i}{2}}$. A straightforward calculation shows that $\Phi_1^*(z)$ also has this form.

Now substitute $w = z^2$ to get $\Phi_n^*(w)$ with monomials of the form $c_i w^i b^{\frac{\nu_2(n)}{2} - i}$, which is homogeneous in w and b , with degree $\frac{\nu_2(n)}{2}$. \square

Definition 3.6. Let $F(X, Y)$ be a homogeneous polynomial. We define $\ell(F)$ to be the leading coefficient of the dehomogenized polynomial $F(z, 1)$ and $c(F)$ to be the constant term of $F(z, 1)$.

Lemma 3.7. *Let*

$$\begin{aligned} \phi_b(X, Y) &: \mathbb{P}^1 \rightarrow \mathbb{P}^1 \\ (X, Y) &\mapsto (X^2 + bY^2, XY). \end{aligned}$$

Then

$$\ell(\Phi_n^*(X, Y)) = \begin{cases} p & \text{if } n = p^e, e \geq 1 \\ b & \text{if } n = 1 \\ 1 & \text{otherwise,} \end{cases}$$

and $c(\Phi_n^*)$ is a non-negative power of b .

Proof. The result for $n = 1$ follows from the remark after Lemma 3.3.

For $n > 1$, we see from Lemma 3.3 part (1) that $\ell(\Phi_n^*)$ is a non-negative power of k times $C_n(k)$, where $C_n(k)$ is the n^{th} cyclotomic polynomial in the variable k . The result follows from evaluating this at $k = 1$. Similarly, the result for $c(\Phi_n^*)$ follows from part (2) of Lemma 3.3 and evaluating at $k = 1$. (See, for example, [4, p. 280 points 2. and 4.] to evaluate the cyclotomic polynomials at $k = 1$.) \square

4. THE EXPLICIT BOUND

In this section, we find an explicit uniform bound for the number of \mathbb{Q} preperiodic points for a one-parameter family of quadratic rational maps; namely, the quadratic maps with a unique fixed point. The existence of such a bound follows immediately from Theorem 2.9, so the content of this work is in finding the bound explicitly. By construction, a rational point (z_0, b_0) on the variety $V(\Phi_n^*(z, b))$ corresponds to a quadratic rational map $\phi(z) = z + \frac{b_0}{z}$, and a rational point z_0 of period n for ϕ . Note that $b_0 \neq 0$, since that value does not give a degree 2 rational map.

By Lemma 3.5, we may substitute $w = z^2$ in Φ_n^* , and the resulting polynomial $\psi_n(w, b) \in \mathbb{Z}[w, b]$ is homogeneous in w and b . So if $V(\Phi_n^*(z, b))$ has a rational point (z_0, b_0) with $b_0 \neq 0$, then $V(\psi_n(w, b))$ has a rational point (z_0^2, b_0) . Since $\psi_n(w, b)$ is homogeneous, we may equivalently ask if $\psi_n(w, 1)$ has a rational root.

Theorem 4.1. *The rational map $\phi(z) = z + \frac{b}{z}$ where $b \in \mathbb{Q}$ has no rational points with primitive period $n \geq 5$.*

Proof. From Lemma 3.7, we know that for $n > 1$ the lead coefficient of $\psi_n(w, 1) \in \mathbb{Z}[w]$ is either 1 (if n is not a prime power) or p (if $n = p^e$) and the constant term is 1. The only rational roots of such a polynomial are ± 1 when n is not a prime power, and ± 1 or $\pm \frac{1}{p}$ when n is a power of a prime.

In either case, we can have no more than four rational roots of ψ_n , which means no more than four rational points are on any given cycle, and this is independent of the parameter b . \square

We now combine Theorem 4.1 with earlier work on the two-parameter family $\phi(z) = kz + \frac{b}{z}$ to find exactly what primitive periods are possible for rational periodic points.

Corollary 4.2. *If $P \in \mathbb{P}^1(\mathbb{Q})$ is a periodic point for $\phi(z) = z + \frac{b}{z}$ where $b \in \mathbb{Q}$, then P is either the point at infinity or a point of primitive period 2.*

Proof. From Theorem 4.1, we know that the primitive period of P must be less than 5.

From [6, Theorem 4], a map of the form $\phi(z) = kz + \frac{b}{z}$ with $k, b \in \mathbb{Q}^*$ has a rational point of primitive period 4 if there is some $m \in \mathbb{Q} \setminus \{0, \pm 1\}$ such that $k = 2m/(m^2 - 1)$ and $b = -m/(m^4 - 1)$. In the proof of that Theorem, we see that the map has a rational point of primitive period 4 if and only if we can find a rational point on a certain algebraic curve. The parameterization of the curve given in the proof shows that in fact the condition $k = 2m/(m^2 - 1)$ is necessary. However, there is no such m with $1 = 2m/(m^2 - 1)$. We conclude that P has primitive period less than 4.

Similarly, [6, Theorem 3] says that if $\phi(z) = kz + \frac{b}{z}$ with $k, b \in \mathbb{Q}^*$, then $\phi(z)$ has no rational point of primitive period 3. Hence, P has primitive period one or two.

By construction, P is a fixed point if and only if P is the point at infinity. The only other possibility is a rational point of primitive period 2. \square

To finish finding the exact bound on the number of rational preperiodic points for a map of the form $\phi(z) = z + \frac{b}{z}$, we introduce a bit more notation.

Definition 4.3. Let m and n be positive integers. Given a rational map ϕ and a point P that is strictly preperiodic for ϕ (in other words, P is preperiodic but not periodic), we say that P has type m_n if P enters a cycle of primitive period m after n iterations. That is, $\phi^{n+m}(P) = \phi^n(P)$, where $m \geq 1$ and $n \geq 1$ are the smallest such integers.

Corollary 4.4. *Let $\phi(z) = z + \frac{b}{z} \in \mathbb{Q}(z)$ with $b \neq 0$. Then ϕ has either 2, 4, or 6 rational preperiodic points.*

Proof. First note that for every nonzero $b \in \mathbb{Q}$, the point at infinity is fixed, and $\phi(0)$ is infinity. Hence, every ϕ_b has a rational fixed point and a rational point of type 1_1 .

Applying [6, Proposition 6], we see that ϕ has a rational points of type 1_2 if and only if $b = -c^2$ for some $c \in \mathbb{Q}^*$. However, all of these maps are conjugate over \mathbb{Q} , so take $b = -1$ as a representative. From [6, Propositions 7 and 8], we conclude that ϕ has no rational points of type 1_n for $n > 2$.

Applying [6, Proposition 2], we see that ϕ has a rational point of primitive period 2 if and only if $b = -2c^2$ for some $c \in \mathbb{Q}^*$. Again, all such maps are

conjugate over \mathbb{Q} , so we take $b = -2$ as a representative. It is a simple matter to check that in this case, ϕ_b has two rational points of type 2_1 . (See Figure 1.) From [6, Proposition 8], we conclude that ϕ_b has no rational points of type 2_n for $n > 1$.

Finally, [6, Proposition 9] says that ϕ_b has both rational points of type 1_2 and rational points of primitive period 2 if and only if we can solve $1 = 1/(x^2 - 1)$ with $x \in \mathbb{Q} \setminus \{0, \pm 1\}$. Evidently, this is not possible. Hence for $b = -1$ (and all \mathbb{Q} -conjugate maps), ϕ_b has four rational preperiodic points. Similarly, for $b = -2$ (and all \mathbb{Q} -conjugate maps), ϕ_b has six rational preperiodic points. There are no b values for which ϕ_b has more than six rational preperiodic points. \square

We provide a graphical representation of all possible structures for rational preperiodic points for the family $\phi(z) = z + \frac{b}{z}$. In these graphs, the vertices represent points in $\mathbb{P}^1(\mathbb{Q})$, and an arrow from vertex P to vertex Q indicates $\phi(P) = Q$.

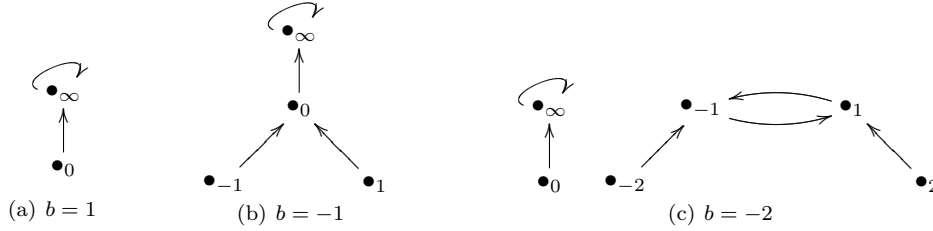


FIGURE 1. All possible rational preperiodic graphs for $\phi_b(z) = z + \frac{b}{z}$.

5. ANOTHER FAMILY OF TWISTS

We conclude with an abbreviated analysis of the possible preperiodic structures for another family of twists. Lemma 3.3 says the lead coefficient of the dynatomic polynomials are powers of k times a cyclotomic polynomial in k . With the help of [3, Proposition 1], we can evaluate cyclotomic polynomials at roots of unity. Hence, we consider $k = -1$. The proofs are similar to those in Section 4, so the details will be sketched here.

We now consider the maps $\psi_b(z) = -(z + \frac{b}{z})$. Again, each ψ_b is conjugate to ψ_1 via the map $f(z) = z\sqrt{b}$. The family of twists is distinct from the one already considered, since ψ_b has two finite fixed points at $\pm\sqrt{-b/2}$.

Note that Lemmas 3.3, 3.4, and 3.5 apply to the family ψ_b , as we are taking $k = -1$.

Lemma 5.1. *Let*

$$\begin{aligned} \psi_b(X, Y) &: \mathbb{P}^1 \rightarrow \mathbb{P}^1 \\ (X, Y) &\mapsto (-X^2 + bY^2, XY). \end{aligned}$$

Then

$$\ell(\Phi_n^*(X, Y)) = \begin{cases} \pm p & \text{if } n = 2p^e, e \geq 1, p \text{ prime} \\ -2 & \text{if } n = 1 \\ \pm 1 & \text{otherwise,} \end{cases}$$

and $c(\Phi_n^)$ is a non-negative power of b .*

Proof. The case $n = 1$ is found by computation. By Lemma 3.3, the lead coefficient of ψ_b is some power of k times a cyclotomic polynomial. We apply [3, Proposition 1] to evaluate the cyclotomic polynomials at $k = -1$, yielding the result. \square

Proposition 5.2. *Let $\psi_b(z) = -(z + \frac{b}{z}) \in \mathbb{Q}(z)$ with $b \neq 0$. Then ψ_b has either 2 or 4 rational preperiodic points.*

Proof. A proof identical to Theorem 4.1 using Lemma 5.1 shows there can be no rational points of primitive period $n \geq 5$, and we apply [6, Theorems 3 and 4] to see that there are no rational points of primitive period 3 or 4. From [6, Proposition 2], we conclude that no value of $b \in \mathbb{Q}^*$ gives a rational map with a rational point of primitive period 2. Hence, we need only consider fixed points and points of type 1_n for $n \geq 1$.

For every $b \in \mathbb{Q}^*$, there is a rational fixed point at ∞ and a rational point of type 1_1 at 0. By [6, Proposition 1], ψ_b has rational finite fixed points if and only if $b = -2c^2$. Since all such maps are conjugate over \mathbb{Q} , so we take $b = -2$ as a representative. There are no other type 1_1 rational points. Applying [6, Proposition 6] we have rational points of type 1_2 if and only if $b = -c^2$. Again, all such maps are conjugate over \mathbb{Q} , so we take $b = -1$ as a representative. By [6, Proposition 7 and 8] there are no rational points of type 1_n for $n > 2$. See Figure 2. \square

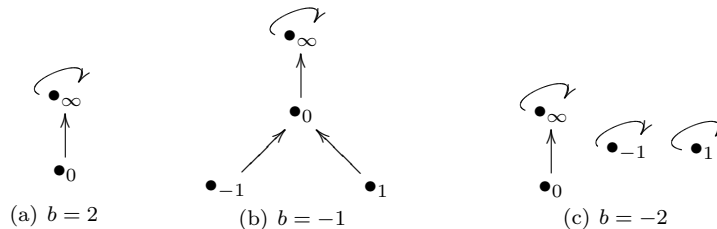


FIGURE 2. All possible rational preperiodic graphs for $\psi_b(z) = -(z + \frac{b}{z})$.

REFERENCES

- [1] Najmuddin Fakhruddin. Questions on self maps of algebraic varieties. *J. Ramanujan Math. Soc.*, 18(2):109–122, 2003.
- [2] E. V. Flynn, Bjorn , and Edward F. Schaefer. Cycles of quadratic polynomials and rational points on a genus-2 curve. *Duke Math. J.*, 90(3):435–463, 1997.
- [3] R. P. Kurshan and A. M. Odlyzko. Values of cyclotomic polynomials at roots of unity. *Math. Scand.*, 49(1):15–35, 1981.
- [4] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.
- [5] Alon Levy. The space of morphisms on projective space. *Acta Arith.*, 146(1):13–31, 2011.
- [6] Michelle Manes. \mathbb{Q} -rational cycles for degree-2 rational maps having an automorphism. *Proc. Lond. Math. Soc. (3)*, 96(3):669–696, 2008.
- [7] Loïc Merel. Bornes pour la torsion des courbes elliptiques sur les corps de nombres. *Invent. Math.*, 124(1-3):437–449, 1996.
- [8] Steven J. Miller and Ramin Takloo-Bighash. *An invitation to modern number theory*. Princeton University Press, Princeton, NJ, 2006.
- [9] Patrick Morton. Arithmetic properties of periodic points of quadratic maps. II. *Acta Arith.*, 87(2):89–102, 1998.
- [10] Patrick Morton and Joseph H. Silverman. Rational periodic points of rational functions. *Internat. Math. Res. Notices*, (2):97–110, 1994.

- [11] D. G. Northcott. Periodic points on an algebraic variety. *Ann. of Math. (2)*, 51:167–177, 1950.
- [12] Clayton Petsche, Lucien Szpiro, and Michael Tepper. Isotriviality is equivalent to potential good reduction for endomorphisms of \mathbb{P}^N over function fields. *J. Algebra*, 322(9):3345–3365, 2009.
- [13] Bjorn Poonen. The classification of rational preperiodic points of quadratic polynomials over q : a refined conjecture. *Math. Z.*, 228(1):11–29, 1998.
- [14] Joseph H. Silverman. Lower bounds for height functions. *Duke Math. J.*, 51(2):395–403, 1984.
- [15] Joseph H. Silverman. *The arithmetic of dynamical systems*, volume 241 of *Graduate Texts in Mathematics*. Springer-Verlag, 2007.
- [16] Michael Stoll. Rational 6-cycles under iteration of quadratic polynomials. *LMS J. Comput. Math.*, 11:367–380, 2008.

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