Transfer Operators for Hecke Triangle Groups

F. Strömberg,
joint work with T. Mühlenbruch and D. Mayer

DNA Seminar, Uppsala University, 2006-09-04

What is Known for the Modular Group

Geodesic Flow on $S(\Gamma\backslash\mathcal{H})$		Closed geodesics on $\Gamma\backslash\mathcal{H}$
$\downarrow$		$\downarrow$
Cross section & Return Map		Continued Fractions
$\downarrow$		$\downarrow$
Interval Map	$\leftarrow$	Reduction Theory
$\downarrow$		
Transfer Operator	$\longrightarrow$	Functional Equation
$\downarrow$ (Dynamics or Reduction)		$\downarrow$ (Bruggeman-Lewis-Zagier)
Selberg Zeta Function		Cohomology
$\downarrow$STF(implicit)		$\downarrow$ (explicit)
	Maass waveforms	

What is Known for the Modular Group

Geodesic Flow on $S(\Gamma\backslash\mathcal{H})$		Closed geodesics on $\Gamma\backslash\mathcal{H}$
$\downarrow$		$\downarrow$
Cross section & Return Map		Continued Fractions
$\downarrow$		$\downarrow$
Interval Map	$\leftarrow$	Reduction Theory
$\downarrow$		
Transfer Operator	$\longrightarrow$	Functional Equation
$\downarrow$ (Dynamics or Reduction)		$\downarrow$ (Bruggeman-Lewis-Zagier)
Selberg Zeta Function		Cohomology
$\downarrow$STF(implicit)		$\downarrow$ (explicit)
Maass waveforms		

What is Known for the Modular Group

Geodesic Flow on $S\left(\Gamma\backslash\mathcal{H}\right)$		Closed geodesics on $\Gamma\backslash\mathcal{H}$
$\downarrow$		$\downarrow$
Cross section & Return Map		Continued Fractions
$\downarrow$		$\downarrow$
Interval Map	$\leftarrow$	Reduction Theory
$\downarrow$		
Transfer Operator	$\longrightarrow$	Functional Equation
$\downarrow$ (Dynamics or Reduction)		$\downarrow$ (Bruggeman-Lewis-Zagier)
Selberg Zeta Function		Cohomology
$\downarrow$STF(implicit)		$\downarrow$ (explicit)
	Maass waveforms	

Current Status

Our aim is to

- Generalize everything from $PSL(2,\mathbb{Z})$ ($q = 3$) to Hecke Triangle Groups G_q, $q \neq 3$.

Current Status

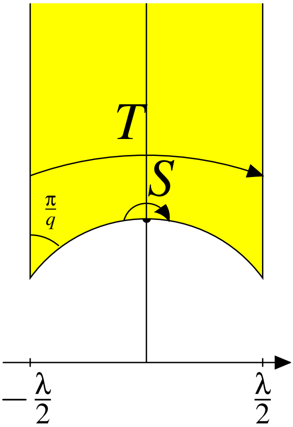
- Good candidate for Continued Fractions and Reduction.
- A Transfer Operator related to the Selberg Zeta Function.
- Partial results for Functional Equations and cohomology.
- We have yet to:
  - Find an explicit crossection and first return map.
  - Prove uniqueness of identified orbits for $q \neq 3$.
  - Connect Maass forms explicit to this representation of the cohomology for $q \neq 3$.

Hecke Triangle Groups

Let $q \geq 3$ and
$\lambda = \lambda_q = 2\cos\left(\frac{\pi}{q}\right)$

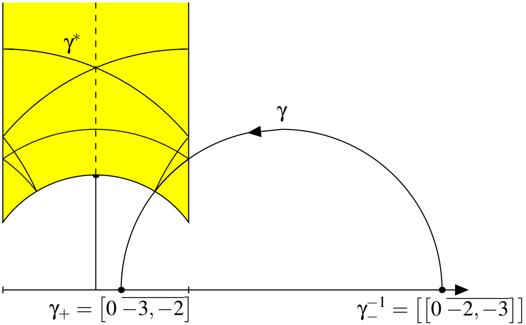
$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$T = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}.$$

$$G_q = \langle S, T \rangle,$$
$$S^2 = (ST)^q = Id$$



Geodesics on $G_q\backslash\mathcal{H}$

- γ = lift of γ^* on $G_q\backslash\mathcal{H}$
- γ^* closed $\Leftrightarrow$ $\gamma_\pm$ pair of hyp. fixpts. of G_q.
- γ_+ = attracting
- γ_- = repelling
- End points coded by Continued Fractions



Continued Fractions

The classical "+", Gauss, or simple CF

For $x \in \mathbb{R}$:

$$x = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \cdots}}, \; a_j \in \mathbb{Z}, a_j \geq 0, j > 0.$$

- Used by Artin (code modular billiard flow), Mayer (transfer operator), etc.
- A slight problem: $z \to \frac{1}{z}$ has determinant -1 in $PGL(2, \mathbb{R})$.
- A major problem: The generalization to $q > 3$ gives a bad (non trace-class) Transfer operator.

General λ (or G_q)-Continued Fractions.

Definitions

We identify a sequence of integers, $\{a_j\}_{j\geq 0}$, with

$$x = T^{a_0} S T^{a_1} S T^{a_2} \cdots (0) = a_0 \lambda - \cfrac{1}{a_1 \lambda - \cfrac{1}{\ddots}} \in \mathbb{R},$$

and say that it is a

- *non-regular* (formal) CF, $[[a_0; a_1, a_2, \ldots]]_\lambda \in \mathcal{A}$ in general.
- *regular* CF, $[a_0; a_1, a_2, \ldots]_\lambda \in \mathcal{A}_{\mathrm{Reg}}$, if it is generated by some function or satisfies some "regularity conditions" (e.g. avoids certain "forbidden blocks").

Repetitions in a CF are denoted by powers (finite) or bars (infinite).

Nearest λ-multiple Fractions

- From now on, consider the Nearest λ-CF (Hurwitz for $q = 3$ and Nakada for $q > 3$).
- Write $\lambda = \lambda_q$ and let $(x) = \left\lfloor \frac{x}{\lambda} + \frac{1}{2} \right\rfloor$ be the nearest λ-multiple of x.

Example

Here $\lambda_3 = 1$, $\lambda_4 = \sqrt{2}$, $\lambda_5 = \frac{1+\sqrt{5}}{2}$, $\lambda_6 = \sqrt{3}$, etc.

- $1 = [1]_3 = \left[1; \overline{2}\right]_4 = [1; 1]_5 = \left[1; \overline{1, 2}\right]_6 = [1; 1, 1]_7 = \cdots$

Note that finite/eventually periodic CFs correspond to parabolic/hyperbolic points (i.e. rational/quadratic irrational for $q = 3$, much harder to characterize for $q > 3$, i.e. cusps=$\mathbb{Q}(\lambda)$ only for $q = 3$ and 5).

The Generating Map

Definition (Generating Map)

Let $I_q = \left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right]$. The map $f_q : I_q \to I_q$ defined by

$$f_q(x) = \frac{-1}{x} - \left(\frac{-1}{x}\right)\lambda,\ x \in I_q$$

is a *generating map* for the nearest λ-multiple CF.
If $x = [0; a_1 a_2 \ldots]_\lambda$ then $\{a_n\}$ is obtained by setting $x_1 = x$ and then recursively for $n \geq 1$:

$$a_n = \left(\frac{-1}{x_n}\right),$$

$$x_{n+1} = f_q(x_n) = \frac{-1}{x_n} - a_n\lambda.$$

Action by f_q

The map f_q acts by a left-shift on regular λ-CF's inside I_q, i.e. if $x = [0; a_1, a_2, \ldots] \in \mathcal{A}_{\mathrm{Reg}}$, then

$$\begin{aligned} f_q(x) &= T^{-a_1} S x = \frac{-1}{x} - a_1 \lambda = [0; a_2, a_3, \ldots], \\ f_q^k(x) &= [0; a_{k+1}, a_{k+2}, \ldots], \; k \geq 2. \end{aligned}$$

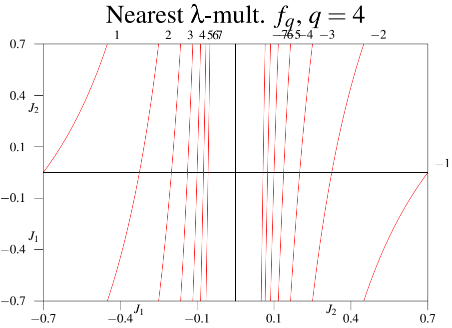
We extend the map f_q by setting $f_q(x) = x - (x)\lambda$ for $x \notin I_q$.

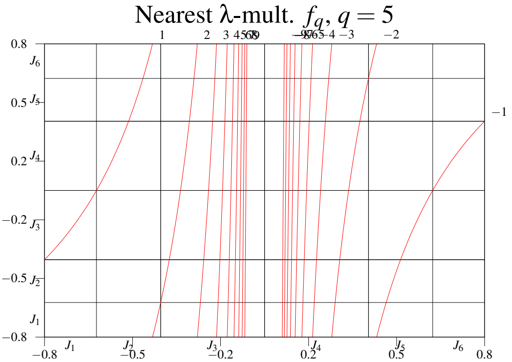
Definition

For $x \in \mathbb{R}$ we denote the f_q-orbit of x inside I_q by

$$\mathcal{O}(x) = \left\{ f_q^k(x) \right\}, \begin{cases} k \geq 0, & x \in I_q, \\ k \geq 1, & x \notin I_q. \end{cases}$$

Examples of Generating Maps





Black boxes = partition of unique set of inverses, orbit of $\pm\frac{\lambda}{2}$.

Markov partition = $\{\mathcal{I}_n\}_{n\in\mathbb{Z}}$, $\mathcal{I}_n = \left\{x \in I_q \mid f_q(x) = \frac{-1}{x} - n\lambda\right\}$.

Properties of Nearest λ-multiple CF

- If $A \in G_q$ is hyperbolic it is equivalent to $ST^{a_1}ST^{a_2}\cdots ST^{a_l}$ with fixedpoints $x, \overline{x}$ s.t.
  - $x = [0; \overline{a_1, a_2, \ldots, a_l}] \in \mathcal{A}_{\mathrm{Reg}}$ and
  - $\frac{1}{\overline{x}} = [[0; \overline{a_l, a_{l-1}, \ldots, a_1}]] \in \mathcal{A}$.

Theorem (Conjecture for $q > 3$, Theorem for $q = 3$)

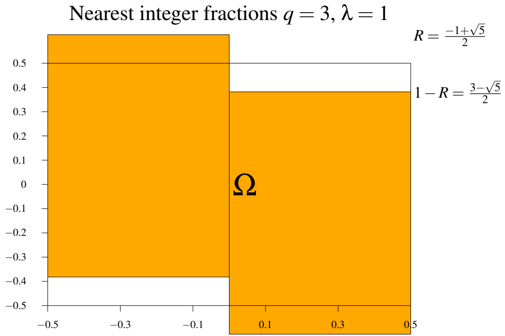
If $x = [0; \overline{a_1, a_2, \ldots, a_l}] \sim_{G_q} y = \left[0; \overline{b_1, b_2, \ldots, b_k}\right] \in \mathcal{A}_{Reg}$ either $\mathcal{O}(x) = \mathcal{O}(y)$, or $\mathcal{O}(x) = \mathcal{O}(R)$ and $\mathcal{O}(y) = \mathcal{O}(-R)$ (or vice versa) where $\frac{\lambda}{2} < R \leq 1$ is given by

$$R = \begin{cases} \frac{-1+\sqrt{5}}{2} = \left[1; \overline{3}\right], & q = 3, \\ \left[1; \overline{1^h, 2, 1^{h-1}, 2}\right], & q = 2h+3 \geq 5, \ odd, \\ 1 = \left[1; \overline{1^{h-1}, 2}\right], & q = 2h+2, \ even. \end{cases}$$

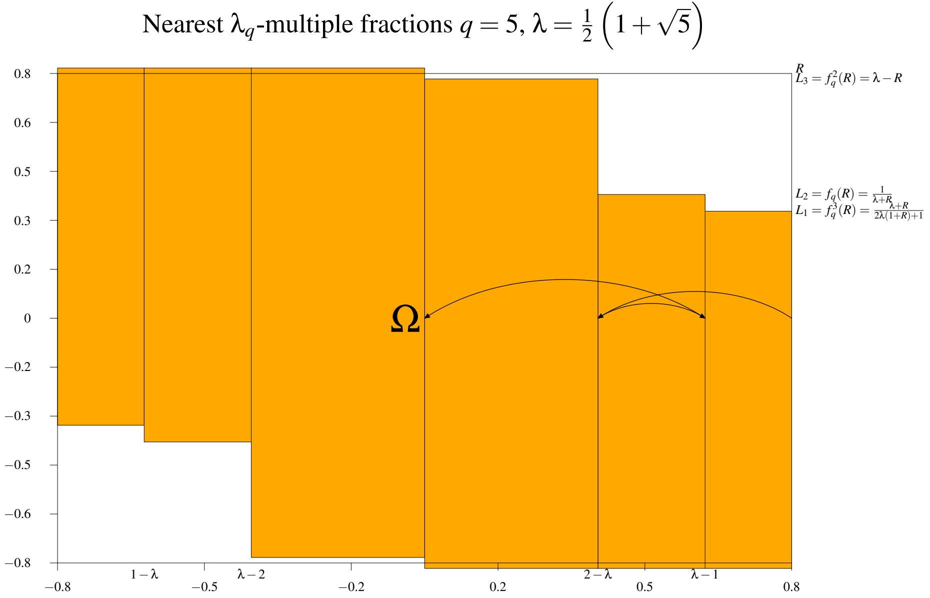
Reduction Theory

There exist a domain $\Omega = \cup J_i \times K_i$ with the following properties:

- Any closed geodesic in $\mathcal{H}$ is G_q- equivalent to a $\gamma \sim (\gamma_+, \gamma_-)$ with $\left(\gamma_+, \gamma_-^{-1}\right) \in \Omega$.
- The endpoints of the J_i coincide with $\mathcal{O}\left(\pm\frac{\lambda}{2}\right)$.
- The endpoints of the K_i (heights) coincide with $\mathcal{O}(\pm R)$.
- $|\gamma_-| \geq \frac{1}{R} \geq 1$.



The domain Ω.



Nearest λ_q-multiple fractions $q = 5$, $\lambda = \frac{1}{2}\left(1+\sqrt{5}\right)$
R
$L_3 = f_q^2(R) = \lambda - R$
$L_2 = f_q(R) = \frac{1}{\lambda+R}$
$L_1 = f_q^3(R) = \frac{\lambda+R}{2\lambda(1+R)+1}$
Ω
0.8
0.6
0.5
0.3
0.2
0
−0.2
−0.3
−0.5
−0.6
−0.8
$1-\lambda$
$\lambda-2$
$2-\lambda$
$\lambda-1$
−0.8
−0.5
−0.2
0.2
0.5
0.8

Reduction of Geodesics

Definition

The set of Reduced Geodesics

$$\begin{aligned}\mathcal{X}_{\text{Red}} &= \left\{\gamma \text{ closed geodesic of } G_q \,\middle|\, \gamma \sim \left(\gamma_+, \frac{1}{\gamma_-}\right) \in \Omega\right\} \\ &\simeq \left\{[0; \overline{a_1, a_2, \ldots, a_l}] \text{ regular} \,\middle|\, l \geq 1\right\}\end{aligned}$$

- f_q maps any closed geodesic γ (eventually) into $\mathcal{X}_{\text{Red}}$, and
- $f_q(\mathcal{X}_{\text{Red}}) \subseteq \mathcal{X}_{\text{Red}}$.
- $\mathcal{X}_{\text{Red}}$ falls into f_q-equivalence classes, $\mathcal{O}(\gamma)$, and up to $\mathcal{O}(\gamma_{\pm r})$ ($r = \lambda - R$) these are also G_q-equivalence classes.
- Let $\mathcal{Y}_{\text{Red}} = f_q \backslash \mathcal{X}_{\text{Red}}$ and $\mathcal{Y}^*_{\text{Red}} = G_q \backslash \mathcal{X}_{\text{Red}}$.

Transfer Operator for f_q

The Transfer (Generalized Perron-Frobenius) Operator

for the interval map $f_q : I_q \to I_q$ is defined as

$$\mathcal{L}_\beta f(x) = \sum_{y \in f_q^{-1}(x)} \left| \frac{df_q^{-1}(x)}{dx} \right|^\beta f(y) .$$

Example: The Perron-Frobenius operator $\mathcal{L}_1$

If μ_q is the invariant density for f_q then

$$\begin{aligned} \mathcal{L}_1 \mu_q &= \mu_q, \text{ and} \\ \mathcal{L}_1^l f &\to \mu_q \text{ exponentially as } l \to \infty \end{aligned}$$

for almost all f (in some Banach space with sup norm).

The Principal Series Representation

The principal series representation π_s

For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PGL(2,\mathbb{R})$, $s \in \mathbb{C}$ and f defined in a nbhd of $\mathbb{R}$

$$\pi_s(A) f(z) = \left((cz+d)^2\right)^{-s} f\left(\frac{az+b}{cz+d}\right).$$

Let D be a disk and consider the B-Space (with sup norm)

$$\mathcal{B}(D) = \left\{ f \mid f \text{ analytic in } D \text{ and cont. in } \overline{D} \right\}.$$

If A is hyperbolic with norm $\mathcal{N}(A)$ and $A\left(\overline{D}\right) \subset D$ then $\pi_s(A)$ is nuclear (Grothendieck) and has trace (easy exercise)

$$\mathrm{Tr}\left[\pi_s(A)\right] = \frac{\mathcal{N}(A)^{-s}}{1-\mathcal{N}(A)^{-1}},$$

The Transfer Operator for f_q

For $x = [0; a_1, a_2 \ldots] \in \mathcal{A}_{\mathrm{Reg}}$ and $\Re\beta > \frac{1}{2}$

$$\mathcal{L}_\beta f(x) = \sum_{n.x \in \mathcal{A}_{\mathrm{Reg}}} \pi_\beta (ST^n) f(x).$$

We sum over $n \in \mathbb{Z}$ s.t $n.x := [0; n, a_1, a_2 \ldots] \in \mathcal{A}_{\mathrm{Reg}}$ and

$$\mathcal{L}_\beta^l f(x) = \sum_{(n_1 \ldots n_l).x \in \mathcal{A}_{\mathrm{Reg}}} \pi_\beta (ST^{n_1} ST^{n_2} \cdots ST^{n_l}) f(x).$$

Reduction Theory $\Rightarrow A_{\vec{n}} = ST^{n_1} \cdots ST^{n_l}$ is hyperbolic and $A_{\vec{n}} \sim [0; \overline{n_1, \ldots, n_l}] \sim \gamma \in \mathcal{X}_{\mathrm{Red}}$!

- Let D be a disk based on $I = \left[-\frac{1}{R}, \frac{1}{R}\right] \supset I_q$. Then D is *invariant* for all $A_{\vec{n}}$ appearing in the $\mathcal{L}_\beta^l$!
- We will work on products of the Banach space, $\mathcal{B}(D)$.

Trace and Fredholm Determinant of $\mathcal{L}_\beta$

For $\Re\beta > \frac{1}{2}$, the operator $\mathcal{L}_\beta^l$ is nuclear of order zero, and

- has trace

$$\mathrm{Tr}\mathcal{L}_\beta^l = \sum_{[0;\overline{n_1 n_2 \ldots n_l}]\in\mathcal{A}_{\mathrm{Reg}}} \mathrm{Tr}\pi_\beta\left(ST^{n_1}ST^{n_2}\ldots ST^{n_l}\right)$$

$$= \sum_{\{\gamma\}\in\mathcal{Y}_{\mathrm{Red}},l(\gamma)=l} \frac{\mathcal{N}(\gamma)^{-\beta}}{1-\mathcal{N}(\gamma)^{-1}}$$

- and Fredholm determinant given by

$$-\mathrm{Log}\det\left(1-\mathcal{L}_\beta\right) = \sum_{l=1}^{\infty}\frac{1}{l}\mathrm{Tr}\mathcal{L}_\beta^l = \sum_{l=1}^{\infty}\frac{1}{l}\sum_{\{\gamma\}\in\mathcal{Y}_{\mathrm{Red}},l(\gamma)=l}\frac{\mathcal{N}(\gamma)^{-\beta}}{1-\mathcal{N}(\gamma)^{-1}}$$

Here $l\left(\gamma\right)=l$ if $\gamma\sim[0;\overline{a_1 a_2\ldots a_l}]\in\mathcal{A}_{\mathrm{Reg}}$ and $\mathcal{N}\left(\gamma\right)=\mathcal{N}\left(ST^{a_1}\ldots ST^{a_l}\right)$.

The Selberg Zeta function for G_q

For $\Re s > 1$ trivial manipulations yield:

$$\begin{aligned}
-\mathrm{Log} Z(s) &= -\mathrm{Log}\left[\prod_{\{\gamma_0\}\in \text{Prim.Hyp.}} \prod_{k=0}^{\infty}\left(1-\mathcal{N}(\gamma_0)^{-s-k}\right)\right] \\
&\vdots \\
&= \sum_{l=1}^{\infty}\frac{1}{l}\sum_{\{\gamma\}\in\mathcal{Y}_{\text{Red}}{}^*, l(\gamma)=l}\frac{\mathcal{N}(\gamma)^{-s}}{1-\mathcal{N}(\gamma)^{-1}}.
\end{aligned}$$

Connecting $\mathrm{Log}Z(s)$ and $\det(1-\mathcal{L}_\beta)$

- $\mathcal{L}_s \sim$ sum over $\mathcal{Y}_{\mathrm{Red}}$ (f_q-equivalence classes)
- $Z(s) \sim$ sum of $\mathcal{Y}^*_{\mathrm{Red}}$ (G_q-equivalence classes), hence

$$\begin{aligned} -\mathrm{Log}Z(s) &= \sum_{l=1}^{\infty} \frac{1}{l}\left(\mathrm{Tr}\mathcal{L}_s^l - \mathrm{Tr}\mathcal{K}_s^l\right) \\ &= -\mathrm{Log}\det(1-\mathcal{L}_s) + \mathrm{Log}\det(1-\mathcal{K}_s), \end{aligned}$$

where $\mathcal{K}_s$ correspond to $\mathcal{Y}_{\mathrm{Red}} \setminus \mathcal{Y}^*_{\mathrm{Red}}$.

The Auxillary Operator $\mathcal{K}_\beta$

If γ_r is the hyperbolic fixing $r = -R + \lambda$ we conjecture (know for $q = 3$) that

$$\mathcal{K}_\beta = \pi_\beta\left(\gamma_{-r}\right).$$

Theorem

$\det\left(1 - \mathcal{K}_\beta\right)$ *has no poles and only simple zeros at*

- $\beta_{n,k} = -n + \frac{2\pi ik}{\ln(\gamma_r'(-r))}, n \geq 0, k \in \mathbb{Z}.$

The zeros of $\det\left(1 - \mathcal{K}_\beta\right)$ are thus known explicitly!

The "Main Theorem" of the Transfer operator method

Theorem

For $\beta \in \mathbb{C}$

$$\det\left(1 - \mathcal{L}_\beta\right) = Z(\beta) \det\left(1 - \mathcal{K}_\beta\right),$$

where (conjecturally for $q > 3$) $\mathcal{K}_\beta = \pi_\beta\left(\gamma_{-r}\right)$.

Proof.

1. For $\Re\beta > 1$ this follows from the above simple calculations.
2. Writing $\mathcal{L}_\beta$ in terms of Hurwitz Zeta functions gives a meromorphic continuation.
3. This gives us the desired identity in the rest of $\mathbb{C}$.

[Incidentally this also proves the meromorphic continuation of $Z(\beta)$.]

$\square$

Implications

Corollary

- $\mathcal{L}_\beta$ *has eigenvalue* 1 *if and only if* $Z(\beta) = 0$ *for* $\beta \neq \beta_{n,k}$.
- $\mathcal{L}_\beta$ *has unbounded eigenvalues as* $\beta \to \beta_0 \in \mathbb{C}$ *only if* $Z(\beta)$ *has a pole at* $\beta = \beta_0$.
- *At* $\beta = 0, -1, -2, \ldots$ $\mathcal{L}_\beta$ *has eigenvalue* 1 *of the same order as the zero of* $Z(\beta) + 1$.

What about eigenvalue 1?

- What can we deduce about the poles and eigenvalue$=1$ of $\mathcal{L}_\beta$?
  - $\beta = 1$ corresponding to the invariant measure gives a simple eigenvalue 1.
  - We can see the poles at all $\beta \in \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \ldots$ to the correct order (=1) (theoretically for $q = 3$).
  - At the negative integers it is possible to compute the correct order of eigenvalue 1 either manually (the first few) or with computer aid (a few more) by finding correct dimension of spaces of eigenfunctions (polynomials) ($q = 3, 4, 5, 6, 7$).
- Unfortunately, except for the contribution from $\mathcal{K}_\beta$ we can not say much about $\beta \notin \mathbb{R}$, e.g. especially $\Re\beta = \frac{1}{2}$ (Maass forms) and for $q = 3$ $\Re\beta = \frac{1}{4}$ (Riemann zeros).

Comparison Of Comp. Methods (Maass forms)

Automorphy (standard)

- Real Analytic Theory.
- Fourier S. = Fast conv.
- K-Bessel
- Heuristic tests depend continuously on $R \in \mathbb{R}$.
- Solve $A_R \vec{X} = \vec{X}$.
- Rigorous methods by Booker-Strömbergsson-Venkatesh.

Transfer Operator

- Holomorphic Theory.
- Power S. = Slow conv.
- Riemann ζ.
- λ_β depends analytically on $\beta \in \mathbb{C}$.
- Solve $A_\beta \vec{X} = \lambda_\beta \vec{X}$ (and try to find $\lambda_\beta = 1$).
- Argument principle (in absence of poles of φ) and norm estimates.

Numerical Verifications

The following were verified:

- For $q \in \{3,4,5,6,7\}$ all Laplace-eigenvalues up to $\rho = 14$ ($\lambda = \frac{1}{4} + i\rho$) given by the Automorphy method to the predicted accuracy.
- For $q = 3$, 80 digits of the first R as given by Booker-Strömbergsson-Venkatesh.
- The Zeros ρ of the scattering matrix (here scalar) gives $1 - \rho$ as zeros of $Z(s)$ and:
  - For $q = 3$: $2\beta = \frac{1}{2} + i\rho$ with $\frac{1}{2} + i\rho$ the first few zeros of Riemann Zeta Function (from Odlyzkos tables). (The first two to 105 digits).
  - For $q = 5$ The first 7 values of ρ (zeros of $\varphi(s)$) in Helen Avelins work to her stated accuraccy (14-15 digits).

Extensions: Consider Selberg Zeta Functions

- Finite-index subgroups of $\Gamma \subset G_q$,

$$Z_\Gamma(s) = \prod_{\gamma \in \text{Prim.Hyp.}} \prod_{k \geq 0} \left(1 - \chi_\Gamma(\gamma) \mathcal{N}(\gamma)^{-s-k}\right),$$

  where χ_Γ is induced by the trivial rep. of $\Gamma \subset G_q$, and
- $PSL(2, \mathbb{Z})$ with non-trivial Multiplier v,

$$Z_{v,k}(s) = \prod_{\gamma \in \text{Prim.Hyp.}} \prod_{k \geq 0} \left(1 - v(\gamma) \mathcal{N}(\gamma)^{-s-k}\right).$$

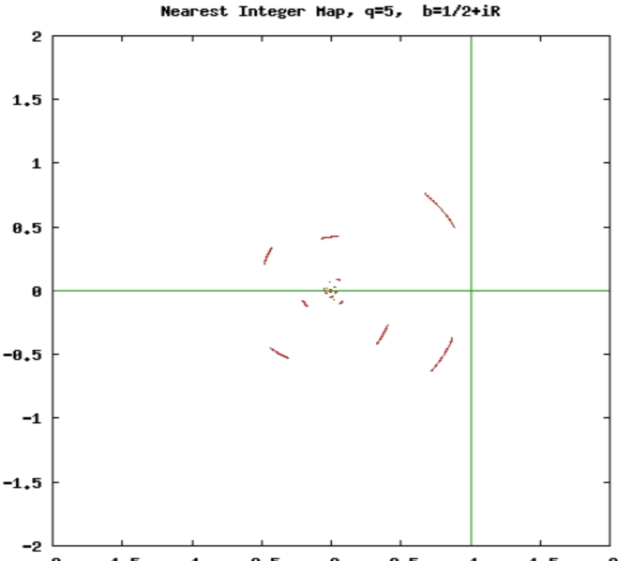
- The corr. transfer operators are obtained by replacing $\pi_s(\gamma)$ with

$$\pi_s^\Gamma(\gamma) = \chi_\Gamma(\gamma)\pi_s(\gamma), \text{ and } \pi_s^{v,k}(\gamma) = v(\gamma)\pi_s(\gamma), \text{ resp.}$$

Extensions: Worked Out

- Numerically, characters and multipliers lead to Lerch Zeta Functions,
$$L(\lambda, s, z) = \sum_{n=0}^{\infty} \frac{e(n\lambda)}{(z+n)^s}$$
- For rational weights λ is rational and L is a sum of Hurwitz Zeta functions.
- Explicit expressions have been worked out for
  - $\Gamma_0(p) \subset PSL(2,\mathbb{Z})$, p prime, and
  - $PSL(2,\mathbb{Z})$ with v_η and rational weight k.
- Some of these cases have also been tested numerically, e.g $\Gamma_0(p)$ for $p \leq 31$ and $PSL(2,\mathbb{Z})$ with weight $k = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{10}$.

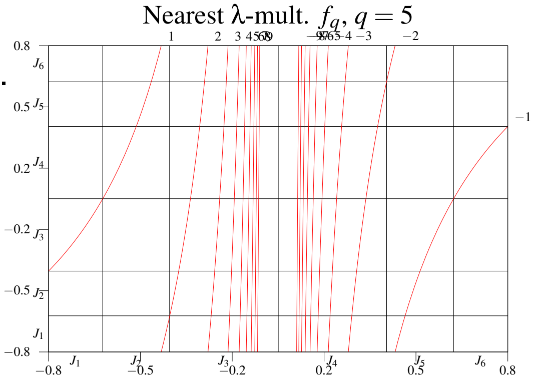
An Animation of Eigenvalues λ_β for $q = 5$, $\beta = \frac{1}{2} + i\rho$.



Explicit Form of $\mathcal{L}_\beta$ is needed for

- Numerical computations.
- Meromorphic continuation.

The explicit form of $\mathcal{L}_\beta f(x)$ depends on x.
We representing f as f_k in J_k and let $\mathcal{L}_\beta$ act on a product of Banach-spaces



$$(\mathcal{L}_\beta f(x))_i = \sum_{j=1}^{H} \sum_{n \in \mathcal{I}_{ij}} \pi_\beta (ST^n) f_j(x)$$

Example of Explicit Representation

Example: Let $q = 3$ then the dimension is 2 and $R = \frac{-1+\sqrt{5}}{2} = [1; \overline{3}]$

Recall the definition of $\pi_\beta\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) f(z) = (cz+d)^{-2\beta} f\left(\frac{az+b}{cz+f}\right)$.

$$\mathcal{L}_\beta \vec{f} = \begin{pmatrix} \sum_{n=3}^{\infty} \pi_\beta\left(ST^n\right) & \sum_{n=-2}^{-\infty} \pi_\beta\left(ST^n\right) \\ \sum_{n=2}^{\infty} \pi_\beta\left(ST^n\right) & \sum_{n=-3}^{-\infty} \pi_\beta\left(ST^n\right) \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

- The building blocks are always of the type $\sum \pi_\beta\left(ST^n\right)$!
- Using a simple example we will demonstrate the meromorphic continuation to $\Re\beta > -\frac{N}{2}$.

Continuation of Example

Example: Continued

Let $N \geq 1$ consider one of the “primitive” operators, e.g. $\tilde{\mathcal{L}}_\beta = (\mathcal{L}_\beta)_{11}$:

$$\begin{aligned}
\tilde{\mathcal{L}}_\beta f &= \sum_{n=3}^{\infty} \pi_\beta (ST^n) f = \sum_{n=3}^{\infty} \left(\frac{1}{z+n}\right)^{2\beta} f\left(\frac{-1}{z+n}\right) \\
&= \sum_{n=3}^{\infty} \left(\frac{1}{z+n}\right)^{2\beta} \left[\sum_{k=0}^{N} a_k \left(\frac{-1}{z+n}\right)^k + O\left(\left(\frac{1}{z+n}\right)^{N+1}\right)\right] \\
&= O(1) \sum_{n=3}^{\infty} \left(\frac{1}{z+n}\right)^{2\beta+N+1} + \sum_{k=0}^{N} a_k (-1)^k \sum_{n=3}^{\infty} \left(\frac{1}{z+n}\right)^{2\beta+k} \\
&= \tilde{\mathcal{L}}_\beta^{(N)} f + \tilde{\mathcal{A}}_\beta{}^{(N)} f.
\end{aligned}$$

Continuation of Example

Example: Continued

Note that the Hurwitz Zeta function is

$$\zeta(s,z) = \sum_{n=1}^{\infty} \frac{1}{(z+n)^s}, \ \Re s > 1$$

and it has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ and residue 1.
By analytic continuation we also have a power series expansion

$$\zeta(s,z+1) = \sum_{n=0}^{\infty} \frac{(s)_n (-1)^n \zeta(s+n)}{n!} z^n.$$

Continuation of Example

Example: Continued

$$\tilde{\mathcal{L}}_\beta^{(N)} f(z) = O(1)\zeta\left(2\beta + N + 1, z + 3\right)$$

- is analytic in $\Re\beta > -\frac{N}{2}$,
- annihilates polynomials of deg. $\leq N$.

$$\tilde{\mathcal{A}_\beta}^{(N)} f = \sum_{k=0}^{N} \frac{f^{(k)}(0)}{k!} (-1)^k \zeta\left(2\beta + k, z + 3\right)$$

- is of finite rank, and
- has simple poles at $\beta_k = \frac{1-k}{2}$, $k = 0, 1, 2, \ldots, N$.

One can show that all poles of $\mathcal{L}_\beta$ arise like this.

Continuation of Example

Example: Numerical implementation for $q = 3$

Project $\mathcal{L}_\beta$ onto the subspace of polynomials of degree $\leq N$, the degree N polynomial approximation of $\mathcal{A}_\beta^{(N)}$ using the PS of $\zeta(2\beta + k, z + 3)$:

$$\tilde{\mathcal{A}_\beta}^{(N)} f(z) = \sum_{n=0}^{N} z^n \sum_{k=0}^{N} a_k \left[\frac{(-1)^{n+k} (2\beta + k)_n \zeta(2\beta + k + n, 3)}{n!} \right]$$

and we can represent this operator by a $N \times N$-matrix. Analytic functions have convergent Power series and hence $\mathcal{B}$ can be approximated by spaces of polynomials, P_N and

$$\mathcal{A}_\beta^{(N)} = \mathcal{L}_{\beta|P_N} \to \mathcal{L}_\beta.$$