HOMOMORPHISMS TO $\mathbb{R}$ CONSTRUCTED FROM RANDOM WALKS

ANNA ERSCHLER AND ANDERS KARLSSON

ABSTRACT. We give a construction of homomorphisms from a group G into $\mathbb{R}$ using random walks on G. The construction is an alternative to [KL07] that works in more general situations. Applications include an estimate on the drift of random walks on groups of subexponential growth admitting no nontrivial homomorphism to $\mathbb{Z}$ and inequalities between the asymptotic drift $L(n)$ and the asymptotic entropy $H(n)$.

Some of the entropy estimates obtained have applications independent of the homomorphism construction, for example a Liouville-type theorem for slowly growing harmonic functions on groups of subexponential growth and on some groups of exponential growth.

1. INTRODUCTION

Let G be a finitely generated group and μ a probability measure on G. We say that μ is *non-degenerate* if its support generates G as a semi-group. We assume that μ has *finite first moment* (with respect to a word metric $l = l_S$ which we fix):

$$L(\mu) := \sum l(g)\mu(g) < \infty.$$

It is clear that the property to have a finite first moment does not depend on the choice of finite generating set S. Finite first moment implies the finiteness of the *entropy of μ*

$$H(\mu) := -\sum_G \mu(g)\log\mu(g) < \infty.$$

The *entropy of the random walk* is

$$h(\mu) = \lim_{n\to\infty} \frac{H(\mu^{*n})}{n}$$

This limit exists by subadditivity. If the entropy of μ is finite, then this limit is finite. The entropy criterion asserts that the Poisson boundary of (G, μ) is trivial if and only if $h(\mu) = 0$ ([KV83, Theorem 1.1] and [Der80]). Let $H(n) = H(\mu^{*n})$. Proposition 1.3 in [KV83] asserts that

$$H(n+1) - H(n) \searrow h(\mu)$$

The expectation of the displacement from the origin is

$$L(n) := \sum_{g\in G} l(g)\mu^{*n}(g).$$

The following limit exists in view of subadditivity:

$$l = l(\mu) = \lim_{n \to \infty} \frac{1}{n} L(n),$$

which is called the *linear drift* of a.s. random walk trajectory by the subadditive ergodic theorem.

In [KL07] Ledrappier and the second named author proved that if a finite first moment random walk has zero entropy and positive drift, then the group has a non-trivial homomorphism to $\mathbb{R}$. In this paper we establish criteria for the existence of homomorphisms to $\mathbb{R}$ even in cases when the drift of the random walk might be zero.

Theorem A. *Suppose that μ is non-degenerate with $\mu(e) > 0$, has finite second moment and that for some sequence n_k such that $L(n_k + 1) - L(n_k) > 0$ and*

$$\sqrt{H(n_k + 1) - H(n_k)}/(L(n_k + 1) - L(n_k)) \to 0$$

as $k \to \infty$. Then G admits a non-trivial homomorphism to $\mathbb{R}$.

This theorem has the following corollaries. Let G be a group of subexponential growth at most $\exp(n^b)$, $b < 1$ and without nontrivial homomorphisms to $\mathbb{R}$. Let μ be a non-degenerate, probability measure with $\mu(e) > 0$ and of finite second moment. Then there exists $a < 1$ (depending only on b) such that for some $C > 0$,

$$L(n) \leq C n^a.$$

This applies for example to the first Grigorchuk group - see Corollary 13. For symmetric finitely supported random walk this was already known [E03] and follows from Varopoulos' long range estimates [Var85]. Mathieu [Mat06] has shown that for a finitely supported measure on a group, such that all elements of this support are torsion elements, Varopoulos-Carne type estimates hold, and that, therefore, for such measures, entropy of random walks is zero if and only if the drift is zero. In particular, this shows that the drift of finitely supported probability measure on groups of subexponential growth generated by torsion elements is zero. A similar result, based on Mathieu's inequality, was announced in [RV05]. For non-symmetric random walk the result in [KL07] states that the drift of any finite first moment random walk is zero, that is, $L(n)$ is sublinear. Our results provide quantitative estimates of $L(n)$. In this context, let us recall that even if we restrict ourselves to the class of (symmetric) simple random walks on groups of subexponential growth, $L(n)$ can be arbitrarily close to linear [E06]. This is in the contrast with the fact that for any finitely supported measure μ on the first Grigorchuk group G (and, more generally, for any (G, μ) such that G is of growth at most $\exp(n^b)$, μ is a measure with finite second moment and the group generated by the support of μ does not admit non-trivial homomorphism to $\mathbb{Z}$), the drift of the random walk (G, μ) is at most $C n^{(b+1)/2}$, see Corollary 13.

Another corollary is:

Corollary. *For a symmetric, non-degenerated with support containing e, finite second moment random walk on a finitely generated group, there is a constant $C > 0$, such that for all $n > 1$,*

$$H(n) \geq C \frac{L(n)^2}{n}.$$

It is easy to see (see e.g. Lemma 7 in [E03]) that Gaussian long range estimates imply the conclusion of the corollary above. Such estimates hold for finitely supported symmetric measures on groups (Varopoulos [Var85]), more generally for any finitely supported centered measure on a group (here centered means that the projection of μ on the abelianization of G is centered, see Mathieu [Mat06] for a partial result in this direction and Dungey [Dun08] for the general statement). Though, at least in the symmetric case, this type of the estimates hold also for measures with a very quick decay [Pe08], observe that they certainly do not hold for the class of measures with finite second moment, as in the corollary above. It was previously known, however, that $l > 0$ implies $h > 0$ for symmetric finitely supported random walks (Varopoulos [Var85]) and, more generally, for centered finite first moment random walks ([KL07]). We stress that the more general inequality in the corollary above the condition of the second first moment cannot be removed - see the remark after Corollary 9.

Among the ingredients of the criteria for homomorphisms are entropy estimates in section 5. One of the direct corollaries of these estimates is for example the following Liouville-type result

Theorem B. *Consider a random walk on G and assume that the measure μ is non-degenerate and has finite support including the identity. Take an increasing function $f_{harm}(n) \geq 0$. Suppose that at least one of the following assumption holds:*

i) For all n, $H(n) \leq f_H(n)$, where $f_H(n)$ is a non-degreasing function. And $f_{harm}(n) \geq 0$ satisfies

$$f_{harm}(n)\sqrt{f_H(n+1) - f_H(n)} \to 0,$$

as $n \to \infty$.

ii) There exists an infinite sequence n_k such that

$$f_{harm}(n_k)\sqrt{H(n_k+1) - H(n_k)} \to 0,$$

as $k \to \infty$.

Then every harmonic functions on G of growth at most f_{harm} is constant.

If $H(n)$ grows linearly, the theorem implies the almost obvious fact that if a harmonic function tends to zero as the word length of its argument tends to infinity, then this function is equal to zero. The theorem is of interest when $H(n)$ is sublinear. In this case it is well-known that all bounded harmonic functions are constant, see Avez [Avez76] for this implication in the case of finitely supported measures and Kaimanovich, Vershik [KV83] and Derriennic [Der80] for the general form of the entropy criterion: for any non-degenerate finite entropy probability measure the two following conditions are equivalent. 1) All bounded harmonic functions with respect to this measure are constants and 2) $H(n)$ grows sublinear). The theorem shows that bounds on $H(n)$ can be used to show the absence of non-constant harmonic function of slow growth, and not only of non-constant bounded harmonic functions.

The theorem implies in particular, that if G is a group of subexponential growth $\leq \exp(n^a)$, for example the first Grigorchuk group, then there exists $b > 0$, depending only on a, such that all μ-harmonic functions of growth at most n^b are constant. Recall that for infinitely supported measures on groups of subexponential growth the situation could be quite different: some of such measure can admit non-constant bounded harmonic functions ([E04, E05]).

Another application of the theorem concerns simple (symmetric) random walks on iterated wreath products of $\mathbb{Z}$. For these random walks, $H(n)$ is bounded by n^a, $a < 1$ ([E03]), and hence there exists $b > 0$ such that all μ-harmonic functions of growth at most n^b are constant. Finally, the same conclusion holds for a certain simple random walk on the Basilica group [BV05] and for a more general class of groups that includes this group [Kai05].

2. Preliminaries on entropy and drift

Let G be a finitely generated group, with word length l_S and non-degenerate probability measure μ of finite first moment. When the generating set S is understood we often write $l(g)$ instead of $l_S(g)$. Recall the fundamental inequality [Gui80]:

$$h \leq lv,$$

where v is the exponential growth rate of the number of elements g with $l(g) \leq R$ in G. This inequality is implied by the following standard lemma:

Lemma 1. *Let μ be a probability measure of finite first moment on a group G. Then i) for any $\varepsilon > 0$, there is a constant $C > 0$, depending on cardinality of S and not depending on μ, such that for all n*

$$H(n) \leq (v + \varepsilon)L(n) + C.$$

ii) Moreover, if the growth function of the group G satisfies $v_G(n) \leq A\exp(n^b)$, $b < 1$, then $H(n) \leq Cn^b$ for some constant C. Here the constant C depends only on A and cardinality of S.

Proof. (Cf. [KL07].) i) Let S_* be an arbitrary finite generating set, and denote by l_* the corresponding word metric. Let a_n denote the number of elements g of word length $l_*(g) = n$, so $a_n \leq (2|S_*|)^n$. Let ν' be the probability measure on G defined by $\nu'(g) = 2^{-l_*(g)-1}/a_{l_*(g)}$. Then for any probability measure ν of finite first moment we have

$$H(\nu) - \sum_{g \in G} \nu(g)\log(2^{l_*(g)+1}a_{l_*(g)}) = -\sum_{g \in G} \nu(g)\log\frac{\nu(g)}{\nu'(g)} \leq 0,$$

where the last inequality comes from the elementary inequality $-\log t \leq 1/t - 1$. Therefore

$$H(\nu) \leq \sum_{g \in G} \nu(g)\log(2^{l_*(g)+1}) + \sum_{g \in G} \nu(g)\log(a_{l_*(g)})$$

$$\leq \sum_{g \in G} \nu(g)(l_*(g)+1)\log 2 + \sum_{g \in G} \nu(g)l_*(g)\log(2|S_*|)$$

$$\leq \log(4|S_*|)\sum_{g \in G} \nu(g)l_*(g) + \log 2.$$

Apply the above inequality to $\nu := \mu^{*n}$, with $S_* := S_k = \{g : l(g) \leq k\}$ and note that $l_*(g) \leq l(g)/k + 1$. It is then clear that given $\varepsilon > 0$, for all sufficiently large k and n there is a constant C making the inequality valid.

ii) From the proof of the first part of the lemma we know that for any generating set S_*

$$H(\mu^{*n}) = H(n) \leq \log(4|S_*|)\sum \mu^{*n}(g)l_*(g) + \log 2.$$

We apply it to $S_* = S_n = \{g : l(g) \leq n\}$. We have that

$$H(n) \leq \log(4A \exp(n^b)) \sum \mu^{*n}(g)(l(g)/n + 1) + \log 2$$

The triangle inequality applied to the word metric shows that the first moment of the convolution of two measures is not greater than the sum of their first moments, and thus $\sum_g \mu^{*n} l(g) \leq n \sum_g \mu^{*n} l(g) = L(\mu)n$. This implies that

$$H(n) \leq (L(\mu) + 1)n^b + \log(4A)(L(\mu) + 1) + \log 2 \leq Cn^b$$

for an appropriate constant C and all n. $\qquad\square$

We will furthermore make use of:

Lemma 2. *There are constants c and C such that for any probability measure ν on $\mathbb{Z}_+$ with finite first moment,*

$$H(\nu) \leq c \log(L(\nu) + 1) + C.$$

Moreover, for any $c = 1 + \epsilon$, $\epsilon > 0$ one can choose C such that the inequality holds.

Proof. Assume first that ν has finite support and let $p_i = \nu(i)$. Lemma 1.1 [Bow75] asserts that for any finite sequence a_i of numbers it holds that

$$H(\nu) = -\sum p_i \log p_i \leq -\sum p_i a_i + \log \sum e^{a_i},$$

where the sums are taken over the i such that $p_i \neq 0$. Fix an $\varepsilon > 0$. We apply this inequality to $a_i := -(1 + \varepsilon) \log(i + 1)$ and get

$$H(\nu) \leq (1 + \varepsilon) \sum p_i \log(i + 1) + \log \sum \frac{1}{(i + 1)^{1+\varepsilon}}.$$

Note that the following constant is independent of ν: $C := \log \sum_{i=0}^{\infty} \frac{1}{(i+1)^{1+\varepsilon}}$. Using the convexity of -log we get

$$H(\nu) \leq (1 + \varepsilon) \log \left(\sum p_i(i + 1) \right) + C = (1 + \varepsilon) \log \left(1 + \sum p_i i \right) + C.$$

Since ε and C are independent of ν, this inequality extends to general ν of finite first moment. $\qquad\square$

For estimates of $H(n)$ from below, one has Lemma 7 in [E03] which states that for any symmetric finitely supported measure μ it holds that for some $C > 0$:

$$H(n) \geq C \frac{1}{n} L(n)^2 - \log n,$$

and as $H(n) \geq C_1 \log(n)$ (which follows from the stronger general estimate on the transition probabilities $\mu^{*n}(g) \leq Cn^{-1/2+\varepsilon}$) this in turn implies that

$$H(n) \geq C_2 \frac{1}{n} L(n)^2.$$

This is a corollary of Varopoulos' long range estimate ([Var85]), see [Car85] for a simple proof and this precise formulation:

$$\mu^{*n}(g) \leq Ce^{-al(g)^2/n}$$

for some constants $C, a > 0$.

3. DEFINITION AND PROPERTIES OF T_n

Let G be a finitely generated group and μ a non-degenerate probability measure on G with finite first moment (with respect to a word metric l_S which we fix and sometimes denote by l). Define

$$T_n(g) = \sum_{h \in G} \left(l(gh) - l(h) \right) \mu^{*n}(h).$$

Note that $|T_n(g)| \leq l(g)$.

Lemma 3. *We have that*

$$\sum_{g \in G} T_n(g)\mu(g) = L(n+1) - L(n)$$

Proof. We have

$$\sum T_n(g)\mu(g) = \sum_{g \in G} \sum_{h \in G} \left(l(gh) - l(h) \right) \mu^{*n}(h)\mu(g)$$

$$= \sum_{h \in G} \left(\sum_{g \in G} l(gh)\mu(g) - l(h) \right) \mu^{*n}(h)$$

$$= \sum_{h \in G} l(h) \left(\mu^{*(n+1)}(h) - \mu^{*n}(h) \right)$$

$$= L(n+1) - L(n).$$

$\square$

Consider $\beta(n)$ such that for every $g \in G$ there exists a constant $C(g)$ such that

$$\sum_{h \in G} |g\mu^{*n}(h) - \mu^{*n}(h)| \leq C(g)\beta(n). \quad (*)$$

Remark. Observe, that if such $C(g)$ exists and if the measure is non-degenerate (or, more generally *adapted*, i.e. the support generates G as a group), we can choose $C(g) = Cl_S(g)$, where S is a finite generating set, and C is the maximum of $C(g)$ over $g \in S$. (β is then the maximum over S of the left hand side.)

Since the left hand side of (*) is bounded by 2, the existence of such $\beta(n)$ is not an issue. Note that if the entropy $h = 0$ and the random walk is aperiodic, one can choose a sequence $\beta(n) \to 0$ as $n \to \infty$ (see [KV83]). In general, if we do not assume that the measure is adapted, the claim of the Remark above does not need to be true.

Lemma 4. *For any $g_1, g_2 \in G$ and any S*

$$|T_n(g_1 g_2) - T_n(g_1) - T_n(g_2)| \leq l_S(g_2)C(g_1)\beta(n)$$

Proof. We have that

$$|T_n(g_1g_2) - T_n(g_1) - T_n(g_2)| = \left| \sum_{h \in G} (l_S(g_1g_2h) - l_S(g_1h) - l_S(g_2h) + l_S(h)) \, \mu^{*n}(h) \right|$$

$$= \left| \sum_{h \in G} (l_S(g_1g_2h) - l_S(g_1h) - (l_S(g_2h) - l_S(h))) \, \mu^{*n}(h) \right|$$

$$= \left| \sum_{h \in G} (l_S(g_2h) - l_S(h)) \, (g_1\mu^{*n}(h) - \mu^{*n}(h)) \right|$$

$$\leq l_S(g_2) \sum_{h \in G} |(g_1\mu^{*n}(h) - \mu^{*n}(h))| \leq l_S(g_2)C(g_1)\beta(n).$$

$\square$

Take a finite generating set S of G and denote by $\gamma_S(n)$ the maximum of the absolute value of $T_n(g)$, $g \in S$.

Lemma 5. *For any finite generating set S there exist positive constants C and C_1 such that for any g in the group, generated by the support of μ, it holds*

$$|T_n(g)| \leq \gamma_S(n)l_S(g) + C_1 l_S(g)^2 \beta(n)$$

Proof. Let $m = l_S(g)$. There exist g_1, g_2, $\ldots g_m \in S$ such that $g_1g_2 \ldots g_m = g$. Observe that by Lemma 4 for any j, $1 \leq j \leq m$

$$|T_n(g_1g_2 \ldots g_j) - T_n(g_1g_2 \ldots g_{j-1}) - T_n(g_j)| \leq l_S(g_j)C(g_1g_2 \ldots g_j)\beta(n)$$
$$\leq C_0 l_S(g_1g_2 \ldots g_j)\beta(n)$$
$$= C_0 j \beta(n).$$

Therefore,

$$|T_n(g_1g_2 \ldots g_n) - T_n(g_1) - T_n(g_2) - \cdots - T_n(g_m)| \leq C_0(1 + 2 + \cdots + m) \leq C_1 m^2 \beta(n)$$

Finally note that

$$|T_n(g_1) + T_n(g_2) + \cdots + T_n(g_m)| \leq m\gamma_S(n),$$

which completes the proof of the lemma. $\square$

Lemma 6. *If μ is a symmetric measure with finite second moment, then for some positive constant C and all g in the group, generated by the support of μ it holds*

$$|\sum_g T_n(g)\mu(g)| \leq C\beta(n)$$

and

$$|L(n+1) - L(n)| \leq C\beta(n).$$

Proof. From the definition of T_n we know that $T_n(e) = 0$. Therefore, by Lemma 4 and the remark preceeding it, we have that

$$|T_n(g) + T_n(g^{-1})| \leq C_1 l(g)l(g^{-1}))\beta(n) = C_1 l^2(g)\beta(n).$$

Summing over g and using the fact that the second moment of μ is finite, we get

$$\sum_g |T_n(g) + T_n(g^{-1})|\mu(g) \leq C_1 \sum_g l^2(g)\mu(g)\beta(n) \leq C\beta(n).$$

Therefore,

$$|\sum_g (T_n(g) + T_n(g^{-1}))\mu(g)| \leq C\beta(n)$$

If μ is symmetric, then

$$\sum_g T_n(g^{-1})\mu(g) = \sum_g T_n(g^{-1})\mu(g^{-1}) = \sum_g T_n(g)\mu(g),$$

and this implies the first claim of the lemma. The second one follows from the first one in view of Lemma 3. $\qquad\square$

Note that the conclusion of the lemma does not hold without the assumption on the second moment, see the discussion after Corollary 9.

4. CONSTRUCTION OF LIMITS OF T_n

4.1. **Simplest case.** We may take a subsequence $n_k \to \infty$ such that limit

$$T(g) = \lim_{k\to\infty} T_{n_k}(g)$$

exists for every g (by the diagonal process argument).

Lemmas 3 and 4 in the previous section imply that:

- if the entropy of the random walk is zero, then T is homomorphism. Indeed, it follows from Lemma 4 and the following property of the entropy: Zero entropy of an aperiodic random walk on a group is equivalent to that for every g

$$\sum_{h\in G} |g\mu^{*n}(h) - \mu^{*n}(h)| \to 0 \text{ as } n \to \infty.$$

  See [KV83, Theorem 4.2].
- if the drift l of the random walk is positive, then for an appropriately chosen subsequence of n's, T is not identically zero: Indeed, we can choose a sequence n_k in such a way that $L(n_k + 1) - L(n_k) \geq l - \varepsilon_k$ with $\varepsilon_k \to 0$ and such that $T(g) = \lim_{k\to\infty} T_{n_k}(g)$ converges. Lemma 3 then implies that

$$\sum_{g\in G} T(g)\mu(g) \geq l.$$

4.2. **General case.** Fix a finite generating set S in G. For every integer $i > 0$, let $\gamma_S(i)$ denote $\max |T_i(g)|$, where the maximum is taken over $g \in S$.

Assume that S and i are such that $\gamma_S(i) \neq 0$. Put $\alpha_S(i) = 1/\gamma_S(i)$ and take g with $l_S(g) = m$, $g = g_1 g_2 \ldots g_m$, for $g_j \in S$. By Lemma 5 we know that

$$\alpha_S(i)|T_i(g)| \leq \alpha_S(i)\gamma_S(i)m + C_1 m^2 \beta(i)\alpha_S(i) = m + C_1 m^2 \beta(i)\alpha_S(i)$$

In order to take the limit, we want that $\gamma_S(i) \neq 0$ and that $\alpha_S(i)\beta(i)$ remains bounded along some infinite subsequence. We may then take a subsequence $n_k \to \infty$ of our subsequence such that the limit

$$T_\alpha(g) = \lim_{k\to\infty} \alpha_S(n_k)T_{n_k}(g)$$

exists for every g. Observe, that for any i there exists $g \in S$ such that $\alpha_S(i)T_i(g) = 1$, and therefore by construction T_α is not identically zero.

Lemma 4 implies that under some conditions on μ and $\alpha(i)$ (so that $\alpha_S(i)\beta(i)$ tends to zero) the constructed map T_α is a homomorphism.

Therefore, to ensure that our construction provides a non-trivial homomorphism to $\mathbb{R}$ it is sufficient to know that there exists a generating set S and a subsequence such that
- $\gamma_S(n_k)$ is not zero, and
- $\beta(n_k)/\gamma_S(n_k)$ tends to zero.

Proposition 7. *Suppose that μ is non-degenerate, has finite second moment and that for some sequence n_k it holds that $L(n_k + 1) - L(n_k) > 0$ and*

$$\beta(n_k)/(L(n_k + 1) - L(n_k)) \to 0$$

as $n \to \infty$. Then G admits a non-trivial homomorphism to $\mathbb{R}$ (constructed as T_α with respect to some subsequence of n_k).

Proof. We know from Lemma 5 that

$$|T_n(g)| \le l_S(g)\gamma_S(n) + l_S(g)^2 C\beta(n),$$

and hence

$$\sum_{g \in G} |T_n(g)|\mu(g) \le C_1\gamma_S(n) + C_2\beta(n),$$

for suitable constants C_1 and C_2, since the first and the second moment of μ are finite (with respect to S). By Lemma 3, this implies that for all n

$$(L(n + 1) - L(n)) \le C_1\gamma_S(n) + C_2\beta(n)$$

In particular, for our subsequence n_k

$$L(n_k + 1) - L(n_k) \le C_1\gamma_S(n_k) + C_2\beta(n_k) \le C_1\gamma_S(n_k) + C_2\epsilon(L(n_k + 1) - L(n_k)),$$

where ϵ can be chosen arbitrarily small if k is large enough. Therefore,

$$(L(n_k + 1) - L(n_k))(1 - \epsilon C_2) \le C_1\gamma_S(n_k).$$

If k is large enough, we can choose ϵ such that $\epsilon C_2 < 1/2$. For such k

$$1/2(L(n_k + 1) - L(n_k)) \le C_1\gamma_S(n_k).$$

This implies that for sufficiently large k, $\gamma_S(n_k) \ne 0$. Therefore, $\alpha_S(n_k) = 1/\gamma_S(n_k)$ from the construction of T_α is well-defined. Observe that for sufficiently large k

$$\alpha_S(n_k)(L(n_k + 1) - L(n_k)) \le 2C_1$$

and therefore, by assumption, $\alpha(n_k)\beta(n_k)$ tends to 0 as $k \to \infty$. Since $\alpha(n_k)\beta(n_k)$ is bounded, T_α is well-defined along some subsequence of n_k. Since $\alpha(n_k)\beta(n_k)$ tends to zero, it follows from Lemma 4 that T_α is a homomorphism. By the construction of T_α it is not identically zero. $\square$

5. Entropy and differences of shifted convolutions

The results of this section, except for two of the three corollaries, are independent from previous sections.

Lemma 8. *Let S be a finite set in the support of μ. Assume that the identity e is in the support of μ*
i) Then there exists $C_0 > 0$ such that for any $g \in S$

$$H(n + 1) - H(n) \ge C_0 \sum_{h:\mu^{*n}(gh)+\mu^{*n}(h)>0} \frac{(\mu^{*n}(gh) - \mu^{*n}(h))^2}{\mu^{*n}(gh) + \mu^{*n}(h)}$$

ii) There exists $C_1 > 0$ such that for any $g \in S$

$$\sum_h |\mu^{*n}(gh) - \mu^{*n}(h)| \leq C_1 \sqrt{H(n+1) - H(n)}$$

Proof. i) Let $g_0, g_1, g_2, \ldots$ be the support of μ and we assume that $g_0 = e$ the identity, and $g_1 = g$ in the statement. Let $p_i = \mu(g_i)$ and $\nu_i = g_i \mu^{*n}$. Note that

$$\nu := \mu^{*(n+1)} = \sum p_i \nu_i$$

and that all ν_i have the same (weight) distribution, so in particular $H(\nu_i) = H(\nu_0)$ for all i. By the basic concavity property of entropy we have that

$$H(\nu) = H\left((p_0 + p_1)(\frac{p_0}{p_0 + p_1}\nu_0 + \frac{p_1}{p_0 + p_1}\nu_1) + p_2\nu_2 + p_3\nu_3 + \ldots\right)$$

$$\geq (p_0 + p_1)H\left(\frac{p_0}{p_0 + p_1}\nu_0 + \frac{p_1}{p_0 + p_1}\nu_1\right) + \sum_{i \geq 2} p_i H(\nu_i)$$

$$= (p_0 + p_1)H\left(\frac{p_0}{p_0 + p_1}\nu_0 + \frac{p_1}{p_0 + p_1}\nu_1\right) + (1 - p_0 - p_1)H(\nu_0).$$

Let

$$D := \sum_{h:\nu_1(h)+\nu_0(h)>0} \frac{(\nu_1(h) - \nu_0(h))^2}{\nu_1(h) + \nu_0(h)}$$

and $p = p_0/(p_0 + p_1)$. By symmetry we may assume that $p \leq 1/2$ and by concavity:

$$H\left(2p\left(\frac{1}{2}\nu_0 + \frac{1}{2}\nu_1\right) + (1 - 2p)\nu_1\right) \geq 2pH\left(\frac{1}{2}(\nu_0 + \nu_1)\right) + (1 - 2p)H(\nu_1).$$

Therefore, in order to show the desired inequality, it remains to show that

$$H\left(\frac{1}{2}(\nu_0 + \nu_1)\right) \geq \frac{1}{2}H(\nu_0) + \frac{1}{2}H(\nu_1) + CD$$

for some constant $C > 0$. This is proved by summation of the following inequality, for $a, b > 0$, there is a constant C such that

$$-\frac{1}{2}(a + b)\log((a + b)/2) + \frac{1}{2}(a \log a + b \log b) \geq C\frac{(a - b)^2}{a + b}.$$

To prove this inequality, we may assume that $a + b = 2$. Indeed, if not, then we multiply a and b with $x = 2/(a + b)$ and observe that

$$\frac{1}{2}\left(x(a + b)\log(x(a + b)/2) - (xa \log xa + xb \log xb)\right)$$

$$= \frac{1}{2}x\left((a + b)\log((a + b)/2) - (a \log a + b \log b)\right)$$

while the right hand side becomes $Cx\frac{(a-b)^2}{a+b}$.

So now we assume that $a + b = 2$. Let us assume also that $a \geq b$. Take ϵ such that $a = 1 + \epsilon$, $b = 1 - \epsilon$. We have to show that for all $\epsilon : 0 \leq \epsilon \leq 1$ and for some positive C

$$((1 + \epsilon)\log(1 + \epsilon) + (1 - \epsilon)\log(1 - \epsilon)) \geq 16C\epsilon^2$$

Observe that $((1 + \epsilon)\log(1 + \epsilon) + (1 - \epsilon)\log(1 - \epsilon)) > 0$ for any $\epsilon : 0 < \epsilon \leq 1$, and therefore it suffices to prove the inequality above for ϵ in the neighborhood

of zero. Let us write $\log(1+\epsilon)$ and $\log(1-\epsilon)$ as series in ϵ (at 0): $\log(1+\epsilon) = \epsilon - 1/2\epsilon^2 - 1/3\epsilon^3 \ldots$.

$$((1+\epsilon)\log(1+\epsilon) + (1-\epsilon)\log(1-\epsilon))$$
$$= -2(1/2\epsilon^2 + 1/4\epsilon^4 + 1/6\epsilon^6 + \ldots) + 2\epsilon(\epsilon - 1/3\epsilon^3 + \ldots)$$
$$\geq 2(\epsilon^2(1 - 1/2) + \epsilon^4(1/3 - 1/4) + \cdots \geq C_2\epsilon^2,$$

for any $\epsilon : 0 \leq \epsilon \leq 1$.

ii) Now we will show that i) implies ii). Put $E = H(n+1) - H(n)$. Observe that

$$\sum_h |\mu^{*n}(gh) - \mu^{*n}(h)| = \sum_{h:|\mu^{*n}(gh)-\mu^{*n}(h)|/(\mu^{*n}(gh)+\mu^{*n}(h))\geq\sqrt{E}} |\mu^{*n}(gh) - \mu^{*n}(h)|$$
$$+ \sum_{h:|\mu^{*n}(gh)-\mu^{*n}(h)|/(\mu^{*n}(gh)+\mu^{*n}(h))<\sqrt{E}} |\mu^{*n}(gh) - \mu^{*n}(h)|.$$

The second sum is at most

$$\sum_{h:|\mu^{*n}(gh)-\mu^{*n}(h)|/\mu^{*n}(gh)<2\sqrt{E}} |\mu^{*n}(gh) - \mu^{*n}(h)|$$
$$+ \sum_{h:|\mu^{*n}(gh)-\mu^{*n}(h)|/\mu^{*n}(h)<2\sqrt{E}} |\mu^{*n}(gh) - \mu^{*n}(h)|,$$

which is at most $4\sqrt{E}$. Now we estimate the first sum. This sum is at most the right hand side from i) divided by $\sqrt{E}$, and, therefore, it is smaller than $C_2\sqrt{E}$.   □

Corollary 9. *For a symmetric non-degenerate, finite second moment random walk with $\mu(e) > 0$ it holds*

i) for some $C > 0$ and all $n > N$

$$L(n+1) - L(n) \leq C\sqrt{H(n+1) - H(n)}$$

ii) for some $C > 0$ and all $n > N$

$$H(n) \geq C\frac{L(n)^2}{n}$$

Proof. i) follows from the second part of Lemma 8 and Lemma 6 in Section 3.

ii) follows from i): we know that for some $C > 0$

$$\sqrt{H(n+1) - H(n)} \geq C\left(L(n+1) - L(n)\right).$$

Therefore,

$$H(n+1) - H(n) \geq C^2(L(n+1) - L(n))^2$$

Hence by summation

$$H(n) \geq \sum_{i=0}^{n-1} C^2(L(i+1) - L(i))^2$$

To finish the proof of ii) observe that

$$\sum_{i=0}^{n-1}(L(i+1) - L(i))^2 \geq \frac{1}{n}(\sum_{i=0}^{n-1}(L(i+1) - L(i)))^2 = \frac{L(n)^2}{n}$$

□

Remark. The assumption that the second moment is finite is important. For any $\epsilon > 0$ there is a measure with finite $2 - \epsilon$ moment on $\mathbb{Z}$ for which the conclusion of the statement does not hold. Indeed, take $\alpha : 1, 2 - \epsilon < \alpha < 2$ and consider a symmetric finite first moment measure on $\mathbb{Z}$ in the domain of the attraction of the Stable Law with parameter α. Since this is a finite first moment measure on $\mathbb{Z}$, we have $H(n) \le C' \log(L(n) + 1) + C_1 \le C' \log(n) + C_1$ (follows from Lemma 2)

On the other hand, since the limit stable law has finite first moment, we have $L(n)/n^{1/\alpha} \ge C$.

This implies that the assumption about the second moment is also necessary in Lemma 6, since the corollary follows from this lemma and Lemma 8, and the latter holds for any measure.

Moreover, the same example shows that the assumption about second moment is important in Proposition 7. Indeed, observe that for a symmetric measure on any abelian G it holds $T_n(g) = T_n(g^{-1})$ for any g and any n. This implies that, whatever normalizing we consider, if in this case the constructed limit T_α is a homomorphism, then it is identically zero.

Now we are going to show that the second part of Lemma 8 implies Theorem B.

Proof of Theorem B. We start with showing that the assumption i) implies the assumption ii), namely that that if i) holds, then there exists a subsequence n_k such that

$$f_{harm}(n_k)\sqrt{H(n_k + 1) - H(n_k)} \to 0.$$

First observe, that since $f_{\mathrm{harm}}(n)\sqrt{f_H(n+1) - f_H(n)}$ tends to 0, there exists a sequence tending to infinity of positive numbers M_n such that

$$M_{n+1}(f_{\mathrm{harm}}(n)\sqrt{f_H(n+1) - f_H(n)})$$

tends to 0. Put $K_n = M_n^2/2$. Observe that we can choose a sufficiently slowly growing sequence M_n above satisfying in addition the following property (**):

$$\frac{K_{n+1}}{K_n} \le \frac{1}{(2 - f_H(n+1)/f_H(n))},$$

since by the assumption of the theorem for all n the expression on the right hand side is greater than 1.

K_n tends to infinity, and therefore, for infinitely many n it holds $H(n+1) - H(n) \le K_{n+1}f_H(n+1) - K_n f_H(n)$. (Indeed, otherwise for all sufficiently large n we have $H(n+1) - H(n) \ge K_{n+1}f_H(n+1) - K_n f_H(n)$. This implies that $H(n) \ge K_n f_H(n) - Const$, and therefore that $1 \ge H(n)/f_H(n) = K_N - Const/H(n) \ge K_n - Const/H(0)$. This shows that K_n is bounded and we get a contradiction).

Now observe that the property (**) implies that $K_{n+1}f_H(n+1) - K_n f_H(n) \le 2K_{n+1}(f_H(n+1) - f_H(n))$. Combining this with the previous observation we see that there exists an infinite subsequence n_k along which we have

$$H(n_k+1) - H(n_k) \le 2K_{n_k+1}(f_H(n_k+1) - f_H(n_k)) = M_{n_k+1}^2(f_H(n_k+1) - f_H(n_k)).$$

It holds for all k

$$f_{\mathrm{harm}}\sqrt{H(n_k + 1) - H(n_k)} \le \left(f_{\mathrm{harm}}\sqrt{f_H(n_k+1) - f_H(n_k)}\right)M_{n_k+1}$$

and hence $f_{\mathrm{harm}}\sqrt{H(n_k+1) - H(n_k)}$ tends to 0 as $k \to \infty$.

Therefore it suffices to prove the theorem under the assumption ii). Consider a harmonic function $\phi : G \to R$. Take $g_1, g_2 \in G$. Observe that for all k

$$\phi(g_1) - \phi(g_2) = \sum_h \phi(g_1 h) \mu^{*n_k}(g_1 h) - \sum_h \phi(g_2 h) \mu^{*n_k}(g_2 h)$$

This is at most $f_{harm}(n_k)$ multiplied by

$$\sum_h \left(\mu^{*n_k}(h) - \mu^{*n_k}(g_2 g_1^{-1} h) \right) \leq C_1 \sqrt{H(n_k + 1) - H(n_k)},$$

where the inequality above is from the claim of the second part of Lemma 8. Thus we see that as $k \to \infty$,

$$\sum_h \phi(g_1 h) \mu^{*(n_k)}(g_1 h) - \sum_h \phi(g_2 h) \mu^{*(n_k)}(g_2 h) \to 0,$$

and this shows that for any $g_1, g_2 \in G$ it holds $\phi(g_1) = \phi(g_2)$. $\qquad\square$

Lemma 10. *Suppose that the entropy of a random walk is such that for all n we have $H(n) \leq f_H(n)$, where the function $f_H(n)$ satisfies $f_H(0) = 0$. Then for all n we have $H(n + 1) - H(n) \leq f_H(n)/n$.*

In particular, if for all n we have $H(n) \leq Cn^\alpha$, $0 < \alpha < 1$. Then for some $C > 0$ and all n

$$H(n + 1) - H(n) \leq \frac{C}{n^{(1-\alpha)}}$$

Proof. Follows from the fact that $H(n + 1) - H(n)$ is non-increasing. $\qquad\square$

Corollary 11. *Take a group G such that its growth function is bounded above by $\exp(Cn^a)$, for some $a < 1$. There exists $b > 0$ such that every harmonic function on G with respect to a finitely supported non-degenerate measure, of growth at most n^b is constant. Moreover, if a harmonic function ϕ with respect to some finitely supported measure satisfies the following condition: there exist an infinite sequence n_k such that for all k and all $g : l_S(g) \leq n_k$ the value $\phi(g) \leq n_k^b$, then the function ϕ is constant.*

Example: G is the first Grigorchuk group, [Gri82].

Proof. The assumption implies that for all n it holds $H(n) \leq Cn^a$. Then by Lemma 10 for all n it holds $H(n + 1) - H(n) \leq \frac{C}{n^{(1-\alpha)}}$. Take b such that $2b < 1 - a$, put $f_{harm}(n) = n^b$ and apply Theorem B. $\qquad\square$

Recall that for finitely generated groups of polynomial growth, harmonic functions with respect to symmetric finitely supported measures and of sublinear growth are constant (see [HS93], Theorem 6.1).

5.1. **Corollaries of the statements about homomorphisms taking into account the estimates of entropy.** .

Proof of Theorem A. Follows from Proposition 7 and the second part of Lemma 8. $\qquad\square$

Corollary 12. *Suppose that μ is non-degenerate with $\mu(e) > 0$, has finite second moment and that G admits no nontrivial homomorphism into $\mathbb{R}$. Then there is a constant c such that for all n,*

$$H(n) \geq c \frac{L(n)^2}{n}.$$

Proof. In view of Theorem A and the assumption on G, there must be a $c > 0$ such that for every n such that $L(n+1) - L(n) > 0$,

$$\sqrt{H(n+1) - H(n)}/(L(n+1) - L(n)) \geq c.$$

For all n we hence have

$$\sqrt{H(n+1) - H(n)} \geq c(L(n+1) - L(n)).$$

Therefore, as in the proof of Corollary 9,

$$H(n) \geq cL(n)^2/n.$$

$\square$

Corollary 13. *Consider a non-degenerate, with e in the support, finite second moment random walk on a group G of intermediate growth at most $\exp(n^b)$, $b < 1$, which do not admit non-zero homomorphisms to $\mathbb{R}$. There exists a constant C such that for all sufficiently large n it holds $L(n) \leq Cn^{(b+1)/2}$.*

Proof. This is an immediate consequence of the previous corollary and Lemma 1. $\square$

References

[Avez72] A. Avez, *Entropie des groupes de type fini,* C.R.Acad.Sci.Paris, Sér. A, 275,(1972),1363-1366.

[Avez76] A. Avez, *Harmonic functions on groups,* Differential Geometry and Relativity, Dordrecht-Holland, 1976, 27-32.

[Bow75] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms,* Lecture Notes in Mathematics, Vol. 470. Springer-Verlag, Berlin-New York, 1975.

[BV05] L. Bartholdi, B. Virág, *Amenability via random walks,* Duke Math. J. 130, No. 1, 39-56 (2005).

[Car85] Th.C. Carne, *A transmutation formula for Markov chains,* Bull. Sci. Math., II. Sér. 109, 399-405 (1985).

[Der80] Y. Derriennic, *Quelques applications du théoreme ergodique sous-additif,* Astérisque 74, (1980), 183-201.

[Dun08] N. Dungey, *Properties of random walks on discrete groups: time regularity and off-diagonal estimates,* Bull. Sci. Math. 132 (2008), no. 5, 359–381.

[E03] A. Erschler, On drift and entropy growth for random walks on groups, Ann. Prob. 31 (2003) 1193-1204

[E06] A. Erschler, *Piecewise automatic groups,* Duke Math. J. 134, no. 3, 591-613 (2006).

[E04] A. Erschler, *Boundary behavior for groups of subexponential growth,* Ann. Math, 160, no 3, 2004, 1183-1210

[E05] A. Erschler, *Critical constants for recurrence of random walks on homogeneous G-spaces,* Ann. Inst. Fourier, 2005, 55, fas.2, 493-509

[Gri82] R.I. Grigorchuk, *Degrees of growth of finitely generated groups, and the theory of invariant means,* Math USSR Izv, 25:2, 1985, 259-300.

[HS93] W. Hebisch, L. Saloff-Coste, *Gaussian Estimates for Markov Chains and Random Walks on Groups* The Annals of Probability, Vol. 21, No. 2 (Apr., 1993), pp. 673-709

[Gui80] Y. Guivarch, *Sur la loi des grands nombres et le rayon spectral d'une marche aléatoire,* Astérisque 74, 47-98, Soc.Math.France, 1980.

[Kai05] V.A. Kaimanovich, *"Münchhausen trick" and amenability of self-similar groups,* Int. J. Algebra Comput. 15, No. 5-6, 907-937 (2005).

[KV83] V.A. Kaimanovich, A.M. Vershik, Random walks on discrete groups: boundary and entropy, Ann. Prob. 11 (1983) 457-490

[KL07] A. Karlsson, F. Ledrappier, Linear drift and Poisson boundary for random walks, Pure Appl. Math. Q. 3 (2007) 1027-1036

[Mat06] P. Mathieu, *Carne-Varopoulos bounds for centered random walks,* Ann. Probab. 34, No. 3, 987-1011 (2006).

[Pe08] R. Peyre, *A probabilistic approach to Carne's bound,* Potential Anal. 29 (2008), no. 1, 17–36.

[RV05] D. Revelle, B. Virag, Research announcement at a Banff workshop 2005

[Var85] N.Th. Varopoulos, *Long range estimates for Markov chains,* Bull. Sci. Math., II. Sér. 109, 225-252 (1985).