SOME GROUPS HAVING ONLY ELEMENTARY ACTIONS ON METRIC SPACES WITH HYPERBOLIC BOUNDARIES

A. KARLSSON AND G. NOSKOV

ABSTRACT. We study isometric actions of certain groups on metric spaces with hyperbolic-type bordifications. The class of groups considered includes $\mathrm{SL}_n(\mathbb{Z})$, Artin braid groups and mapping class groups of surfaces (except the lower rank ones). We prove that in various ways such actions must be elementary. Most of our results hold for non-locally compact spaces and extend what is known for actions on proper CAT(-1) and Gromov hyperbolic spaces. We also show that $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$ cannot act on a visibility space X without fixing a point in $\overline{X}$. Corollaries concern Floyd's group completion, linear actions on strictly convex cones, and metrics on the moduli spaces of compact Riemann surfaces. Some remarks on bounded generation are also included.

1. INTRODUCTION

The philosophy of this paper is somewhat similar to Steinberg's article [Ste85], in which he deduces significant special cases of important theorems for some matrix groups "by using certain obvious relations among the simplest elements of these groups in rather simple-minded ways". In our setting, we also do not treat general lattices in some class of topological groups as is common in rigidity theory, on the other hand we are able to incorporate important groups arising in other contexts. In contrast to [Ste85] however, our results are of mainly one type: degeneracy of actions on metric spaces with hyperbolic boundaries.

Although several of the mechanisms behind our results are well-recognized, a few novel points in our proofs allow for a more unified and generalized treatment. The metric spaces considered here are not necessarily locally compact, CAT(0), geodesic or δ-hyperbolic, and our arguments may in particular prepare the ground for a more thorough study of actions on infinite dimensional hyperbolic spaces. In some situations (e.g. $SL_{n\geq 3}(\mathbb{Z})$-actions on classical hyperbolic spaces) we reprove some very special cases of the more sophisticated theory of superrigidity ([Mar77]). At other instances (e.g. Theorem 3) one might say that we contribute to this theory. Corollaries 1, 2, 3, and 4 below are some consequences of our considerations. Even though actions on trees are included in our study, the reader should consult Culler-Vogtmann [CV96] for more precision in this case.

Research partly supported by the DFG research group "Spektrale Analysis, asymptotische Verteilungen und stochastische Dynamik".

We propose a notion of metric spaces with hyperbolic bordification, which include CAT(-1)-spaces, visibility manifolds, Gromov hyperbolic spaces, and Hilbert's metric spaces of strictly convex domains with their standard boundaries, as well as, graphs with the end-compactification and general geodesic metric spaces completed following Floyd-Gromov. See section 2 for the definition and section 7 for more details on the examples.

Let us now proceed to define the classes of groups involved in this paper. Let S be a subset of a group Γ and P a property of groups. Define a graph, called the *P-graph of* S, which has S as the vertex set and two vertices s_1 and s_2 are joined by an edge if and only if the group generated by s_1 and s_2 has the property P. For P, we will consider the properties of: Commutation (C), Nilpotency (N), not containing a non-abelian free semigroup (NoFS_2), and finally, not containing a non-abelian free subgroup (NoF_2).

Let $\mathcal{A}$ (respectively, $\mathcal{B}$) be the class of groups which are generated by a set S whose NoFS_2-graph is connected (respectively, whose NoF_2-graph is connected). Clearly, $\mathcal{A} \subset \mathcal{B}$. When we speak of the generators of a group in these classes, we will always mean the elements in the special generating set S.

Examples include many amenable groups, products of infinite groups, $SL(n,\mathbb{Z})$ and $SL(n,\mathbb{R})$ for $n \geq 3$, braid groups B_n for $n \geq 5$, mapping class groups $\mathrm{Mod}(\Sigma_g)$ for $g \geq 2$, and some automorphism groups of free groups $SAut(F_n)$, for $n \geq 5$. See the last section for more details.

A sequence of isometries g_n is called *unbounded* if $d(g_n x, x)$ is unbounded as $n \to \infty$ for some (or any) x, and a single isometry g is called *unbounded* if $g_n := g^n$ is unbounded. An action is called *metrically proper* if whenever g_n leaves every finite subset of Γ, it holds that $d(x, g_n x) \to \infty$ for some (or any) x. An action is called *elementary* (following e.g. [Gro87]) if the limit set of any orbit consists of at most two points. (Note that some authors define an action to be elementary if the whole group fixes a point on the boundary or in the space.)

Let $\overline{X}$ be a hyperbolic type bordification, or more generally, a contractive bordification (see section 2) of a complete metric space X and such that the isometries of X extend to homeomorphisms of $\overline{X}$. We obtain:

Theorem 1. *Let Γ be a group in $\mathcal{A}$ and assume that Γ acts on X by isometries such that the elements in S are unbounded. Then Γ fixes a point in ∂X. If there are two fixed points, then the action is in addition elementary.*

Theorem 2. *Let Γ be a group in $\mathcal{B}$ and assume that Γ acts on X by isometries such that the elements in S are unbounded. Suppose moreover that every two elements in S generate a metrically proper action. Then the action of Γ is elementary and Γ fixes a point in ∂X.*

For example, X can be a Gromov hyperbolic space, CAT(-1)-space, or a locally visible CAT(0)-space with their usual bordifications, see [BH99] for details. Note that the hypothesis in Theorem 2 is clearly satisfied if every generator has infinite order and the action by Γ is metrically proper. These theorems are sharp in the sense that any hyperbolic Coxeter group (which all belong to $\mathcal{A}$) acts non-trivially on its Cayley graph but the generators certainly act boundedly.

Combining Theorem 1 with the fact that $SL(n,\mathbb{Z})$, $n \geq 3$, is boundedly generated by elementary matrices (see sections 6 and 8) we answer the question formulated in the introduction of Fujiwara's paper [Fuj99]:

Theorem 3. *Any isometric action of $SL(n,\mathbb{Z})$, $n \geq 3$, on a uniformly convex complete metric space X with a contractive bordification (e.g. a visibility manifold) must have a global fixed point in $\overline{X}$.*

For a statement without the assumption of uniform convexity we refer to section 6.

As already indicated above, our results have a large overlap with previously known results; we now try to briefly make some references to these. The theory of Fuchsian and Kleinian groups is concerned with discrete groups of isometries of real hyperbolic spaces and in these situations the statements in section 4 are classical. The phenomenon of certain groups not admitting non-degenerate actions on certain spaces occurs frequently in rigidity theory. One milestone here is Margulis' work [Mar77] with its impressive generality and depth. This work has been extended in various directions by many people, we mention only Burger-Mozes [BM96], Gao [Gao97], Fujiwara [Fuj99], and Monod-Shalom [MS02] as these works consider the setting of CAT(-1)-spaces (or visibility manifolds in the case of [Fuj99]) and are therefore most directly related to our paper. For trees there is the theory of Bass-Serre [Ser77], and further contributions by Tits [Tit77], Margulis [Mar81], Alperin [Alp82], and Watatani [Wat82]. Among more recent developments for trees, we have Bogopolski [Bog87], Pays-Valette [PV91], Noskov [Nos93], Lubotzky-Mozes-Zimmer [LMZ94], Culler-Vogtmann [CV96], and Shalom [Sha00]. There are of course many other related and notable works that could be mentioned. Except in the tree-case and for [Gao97] non-locally compact situations are usually not treated.

Many groups do admit elementary proper actions on some CAT(-1)-space in view of a simple warped product construction, see [Gro93, p. 157]. On the other hand, it follows from [KM99, Corollary 6.2] that an action of a nonamenable group on a CAT(0)-space with orbit function growing at most exponentially (e.g. a properly discontinuous action on a locally compact Cartan-Hadamard manifold with curvature bounded from below) always has an infinite limit set, hence cannot be elementary. In view of Theorem 2 we therefore get (recall that the fundamental group of a nonpositively curved manifold is always torsion-free):

Corollary 1. *A nonamenable group in the class $\mathcal{B}$ cannot be the fundamental group of a manifold which admits a complete metric for which the universal cover is a visibility manifold with at most exponential volume growth.*

More specifically we also have:

Corollary 2. *Let $\mathcal{T}$ be the Teichmüller space of a surface of genus g and with n punctures. Assume that $3g-3+n \geq 3$. Then there is no invariant complete CAT(0)-metric for which $\mathcal{T}$ is a visibility space and such that the number of orbit points of $Mod_{g,n}$ in balls grows at most exponentially with the radius.*

Theorem 2 (at least for X a locally compact δ- hyperbolic space) and Corollary 2 (at least for pinched negative curvature) for the mapping class groups ($g \geq 2$) were already known to McMullen [McM00, p. 327] and Brock-Farb [BF01]. In the latter reference several other interesting and related theorems about invariant metrics on Teichmüller spaces are proved. Our corollaries generalize Theorem 1.3 in [BF01]. Note the contrast of these results with the fact that the Weil-Petersson metric on the moduli spaces is a non-complete(!) metric of negative (but not pinched!) curvature and the result of Masur-Minsky [MM99] that the complex of curves, on which $\mathrm{Mod}(\Sigma_{g,n})$, $g \geq 1$, acts unboundedly (but not properly!), are Gromov hyperbolic.

Another application one can get, employing Hilbert's metric on convex sets, is:

Corollary 3. *Assume that a group Γ in class $\mathcal{B}$ with infinite order generators acts properly discontinuously by projective automorphisms on a strictly convex bounded domain C in $\mathbb{R}^N$. Then for any $v \in C$, the limit set $\overline{\Gamma v} \cap \partial C$ consists of at most two points.*

As a finitely generated group Γ acts metrically properly by isometries on itself and a Floyd boundary $\partial\Gamma$, which is a hyperbolic boundary, equals the limit set $L(\Gamma)$, we get from Theorem 2:

Corollary 4. *Any Floyd boundary of a finitely generated group in the class $\mathcal{B}$ with infinite order generators consists of at most two points.*

This provides more examples for the discussion in the last paragraph of [Flo84] and has a certain consequence for harmonic functions on such groups, see [Kar02b]. We refer to the article of Gromov-Pansu [GP91] for a discussion on the Floyd boundary, see also [Gro93] and [Kar02a].

The authors would like to thank Herbert Abels and the Universität Bielefeld for making this work possible. The first named author also gratefully acknowledges the support of FIM and thanks Marc Burger for some useful discussions related to this paper.

2. Hyperbolic Bordifications

Let (X,d) be a complete metric space. A *bordification of X* is a Hausdorff topological space $\overline{X}$ with X embedded as an open dense subset. The *boundary* is $\partial X := \overline{X} \setminus X$.

Fix a point $x_0 \in X$ and following Gromov we define for any two points $z, w \in X$:

$$(z|w) := \frac{1}{2}(d(z,x_0) + d(w,x_0) - d(z,w)).$$

For a set $W \subset \overline{X}$, we let $(x|W) := \sup\{(x|w) : w \in W \cap X\}$.

The space $\overline{X}$ is called a *hyperbolic bordification* of X if:

HB 1: For any $\xi \in \partial X$, there is a countable family of neighborhoods $\mathcal{W}$ of ξ in $\overline{X}$, such that the collection of open sets

$$\{x : (x|W) > R\} \cup W$$

where $W \in \mathcal{W}$ and $R > 0$, constitutes a fundamental system of neighborhoods of ξ in $\overline{X}$.

HB 2: Any sequence $\{x_n\}$ in X for which $(x_n|x_m) \to \infty$ as $n, m \to \infty$, converges to a point in ∂X.

Note further that **HB 1** implies **HB 2** when $\overline{X}$ is sequentially compact. (In the case X is a geodesic space, we could alternatively use the expression $d(x_0, [z,w])$ instead of $(z|w)$, although it seems that in general this would not be quite equivalent.)

The one-point bordification is a trivial hyperbolic bordification of any complete metric space. For non-trivial examples, see section 7.

An *$Isom(X)$-bordification* is a bordification of X, where the action of $Isom(X)$ on X extends continuously to an action by homeomorphisms of $\overline{X}$ and ∂X. A *contractive bordification* is an $Isom(X)$-bordification which satisfies:

CB 1: (*Contractivity*) Whenever $g_n \in Isom(X,d)$ such that $g_n x_0 \to \zeta \in \partial X$ and $g_n^{-1} x_0 \to \eta \in \partial X$, then $g_n z \to \zeta$ for every $z \in \overline{X} \setminus \{\eta\}$ and this convergence is uniform outside every neighborhood of η.

CB 2: For any unbounded isometry g, there are numbers n_k such that $g^{n_k} x_0$ and $g^{-n_k} x_0$ converge to some point(s) in ∂X as $k \to \infty$.

CB 3: For any unbounded sequence g_n such that $g_n \xi \to \xi$ for some $\xi \in \partial X$, there is a subsequence g_{n_k} such that at least one of $g_{n_k} x_0$ or $g_{n_k}^{-1} x_0$ converges to ξ as $k \to \infty$.

The first condition is taken from [Woe93] and implies the last two conditions for a sequential compact $\overline{X}$. Let us emphasize that the axioms **CB 2** and **CB 3**, which may not have been much considered previously, play a crucial role in this paper. They are justified by the following theorem.

Theorem 4. *Let $\overline{X}$ be a hyperbolic $Isom(X)$-bordification of a complete metric space X. Then $\overline{X}$ is a contractive bordification.*

Proof. The proof of **CB 1** follows a nice argument of Woess [Woe93]:

Let g_n be a sequence of isometries with $g_n x_0 \to \zeta$ and $g_n^{-1} x_0 \to \eta$. Let U and V be neighborhoods in $\overline{X}$ of ζ and η respectively. By **HB 1** we can find $R > 0$, and neighborhoods W and W' of ζ and η respectively such that

$$A := \{x : (x|W) > R\} \subset U$$
$$B := \{x : (x|W') > R\} \subset V.$$

For any z outside B we have

$$\begin{aligned}(g_n x_0|g_n z) &= \frac{1}{2}(d(x_0, g_n x_0) + d(x_0, g_n z) - d(z, x_0)) \\ &= d(x_0, g_n x_0) - \frac{1}{2}(d(x_0, g_n x_0) - d(g_n^{-1} x_0, z) + d(z, x_0)) \\ &= d(x_0, g_n x_0) - (g_n^{-1} x_0|z) \geq d(x_0, g_n x_0) - R\end{aligned}$$

for all n such that $g_n x_0 \in W$ and $g_n^{-1} x_0 \in W'$. Therefore we have that there is $N > 0$ such that

$$g_n z \in A \subset U$$

for all $n \geq N$ and every $z \in B^c \supset V^c$, as required.

Assuming **HB 2,** the proof of **CB 2** is essentially given in [Kar01a, p. 1454], or see [Kar02a, Proposition 4]. The proof of **CB 3** basically follows the proof of Lemma 2 in [Kar02a] (this is the point where we use the countability assumption in **HB 1**). □

Axiomatizations of hyperbolic-type compactifications occur for example in Beardon [Bea97] and Kaimanovich [Kai00]. Axioms for contractive compactifications, or so-called convergence group actions, have also been considered by several people, including Gehring-Martin [GM87], Gromov [Gro87], Woess [Woe93], and Bowditch [Bow99]. From a less hyperbolic point of view, various notions of contractive boundaries were introduced by Furstenberg around 1970.

3. Individual isometries

The following proposition provides a classification of the elements in $Isom(X, d)$ into *elliptic isometries* (bounded orbits), *parabolic isometries* (unbounded orbits and one fixed point) and *hyperbolic isometries* (unbounded orbits and two fixed points):

Proposition 1. *Let X be a complete metric space and $\overline{X}$ an $Isom(X)$-bordification satisfying **CB 1** and **CB 2**. Let g be an unbounded isometry. Then there exist two not necessarily distinct points $\xi^{\pm} \in \partial X$ and n_k such that $g^{n_k} x \to \xi^+$ and $g^{-n_k} x \to \xi^-$ for all x. Furthermore the set of fixed points in $\overline{X}$ of g consists exactly of the point(s) $\xi^{\pm}$.*

Proof. The convergence statement is asserted by **CB 2**. From continuity and **CB 1** we have

$$g(\xi^{\pm}) = g(\lim_{k\to\infty} g^{\pm n_k} x) = \lim_{k\to\infty} g^{\pm n_k}(gx) = \xi^{\pm}.$$

Moreover, **CB 1** also implies that g cannot fix any other point. □

The following discussion will not be needed later and we shall therefore be rather brief.

First note that in general, it is not true that for an unbounded isometry g the whole sequence $g^n x$ converges when $n \to \infty$ as is illustrated by Edelstein's example ([Kar01a, p. 1453]) together with a warped product giving an infinite dimensional hyperbolic space. However, in the case g has two fixed points in ∂X, then, at least under some extra assumptions, one can say more:

For example, one can embed the complete metric space X isometrically in $C(X)/\mathbb{R}$ via

$$\Phi : x \mapsto [d(x,\cdot)]$$

(as noted by Kuratowski and others) and let $\overline{X} = \overline{\Phi(X)}$. For CAT(0)-spaces this gives the usual bordification, see [BH99, II.8.13]. Assume now that this $\overline{X}$ satisfies **HB 1** and **HB 2** (which is the case for CAT(-1)-spaces). Say that g fixes $\xi \in \partial X$, then choosing the representative b for which $[b] = \xi$ and $b(x_0) = 0$, we have that

$$b(gy) = b(y) + T_g$$

for any $y \in X$ and some constant T_g. Hence $b(g^n x_0) = nT_g$.

We now claim that if g fixes also another point, then $|T_g| > 0$. Indeed, for $g^{n_i} \to [b]$ we get from $T_g = 0$ that for any given m and $\varepsilon > 0$ that

$$(g^{n_i} x_0 | g^m x_0) \geq d(g^m x_0, x_0) - \varepsilon$$

for all large i. This makes it impossible for any subsequence to converge to anything else than ξ in view of **HB 1**.

In general it holds that

$$A_g := \lim_{n\to\infty} \frac{1}{n} d(g^n x_0, x_0) \geq |T_g|$$

and for CAT(0)-spaces one has in addition that $A_g = \inf_x d(gx, x)$. Assume that $A_g > 0$, then one can show that $g^n x_0 \to \xi$ and $b(g^n x_0) = -A_g n$, *cf.* [Kar01a]. To sum up we have:

Theorem 5. *Assume that $\overline{X} := \overline{\Phi(X)}$ is a hyperbolic bordification and let g be an isometry of X. Then one of the following occurs:*

1. *g has bounded orbits;*
2. *$A_g = 0$, but orbits are unbounded and there is a unique fixed point in $\overline{X}$ and which is the unique accumulation point in ∂X;*
3. *$A_g > 0$, the forward and the backward orbits converge respectively to two distinct points in ∂X and these constitute the fixed point set of g.*

4. Groups generated by two unbounded isometries

Assume throughout that $\overline{X}$ is an $Isom(X)$-bordification of X and assume that the conditions **CB 1** and **CB 2** hold.

Proposition 2. *Assume g and h are two unbounded isometries and that the group generated by these two elements does not contain a non-abelian free semigroup. Then $Fix(g) = Fix(h)$.*

Proof. If $|\partial X| \leq 2$, then any unbounded isometry fixes every point of ∂X. Assume now that ∂X contains at least three points. From **CB 2** we have $g^{\pm n_k} x_0 \to \xi^{\pm}$ and $h^{\pm n_l} x_0 \to \eta^{\pm}$, for some, not necessarily all disjoint, boundary points. Suppose now that $\mathrm{Fix}(g)$ is not equal to $\mathrm{Fix}(h)$, so say $\xi^+ \neq \eta^-$. In the case $\eta^- = \eta^+ = \xi^-$, we replace h by hgh^{-1}. (The point is that this element will have forward limit point $h\xi^+$ which is neither equal to η^- nor equal to ξ^+; the backward limit point has to be η^- as before.)

Therefore we can assume that ξ^+ is different from η^+ and η^-, and that ξ^- is different from η^+. As $\overline{X}$ is Hausdorff we may find disjoint neighborhoods U, V and W around ξ^+, η^+ and $\{\eta^-, \xi^-\}$. From **CB 1** there are m, $n > 0$ such that

$$g^n(W^c) \subset U$$
$$h^m(W^c) \subset V.$$

These two elements generate a free non-abelian semigroup, see for example Proposition VII.2 in [dlH00]. Hence $\mathrm{Fix}(g)$ must be equal to $\mathrm{Fix}(h)$. □

Proposition 3. *Let g and h be two unbounded isometries. If their fixed point sets are disjoint, then the group generated by g and h has a non-abelian free subgroup.*

Proof. First note that we may assume that the cardinality of ∂X is at least three (hence infinite). Since $\overline{X}$ is Hausdorff we may find disjoint open sets X_1 containing $\mathrm{Fix}(g)$ and X_2 containing $\mathrm{Fix}(h)$. The statement now follows from **CB 1** and the standard table-tennis argument (Klein's criterion) as in [dlH00], Proposition II.24. □

Proposition 4. *Assume in addition that **CB 3** holds. Suppose g and h are unbounded isometries of X and which generate a group which acts metrically properly on X. If the fixed point sets of g and h are not equal, then the fixed point sets are in fact disjoint.*

Proof. This proof is a modification of a nice argument of Gehring-Martin [GM87]. Both fixed point sets are nonempty by Proposition 1. In the case both g and h have one fixed point then the statement is trivial. Therefore we now assume that $\mathrm{Fix}(h) = \{\xi^+, \xi^-\}$, $(h^{\pm n_j} x_0 \to \xi^{\pm})$ and $\xi^- \in \mathrm{Fix}(g)$. We need to show that ξ^+ is fixed also by g and we may therefore assume that ∂X contains at least three (hence infinite number of) points. Choose neighborhoods U^-, U^+ in $\overline{X}$ of ξ^- and ξ^+ respectively so that

$$hU_- \cap U_+ = \emptyset, \tag{1}$$

and

$$E := \overline{X} \setminus (U^+ \cup U^-) \neq \emptyset,$$

which is possible because h is a homeomorphism, $\xi^{\pm}$ are fixed points of h and $\overline{X}$ is Hausdorff. Since h^{-n_j} contracts toward ξ^- and g is a homeomorphism fixing ξ^-, we have that

$$gh^{-n_j}(E) \subset U_- \setminus \{\xi^-\}$$

for every large j. Because of (1) we can find a $k = k(j)$ such that

$$h^{k(j)} g h^{-n_j} E \cap E \neq \emptyset. \tag{2}$$

Now let $g_j = h^{k(j)} g h^{-n_j}$ and note that

$$g_j \xi^- = \xi^- \text{ and } \lim_{j \to \infty} g_j \xi^+ = \xi^+, \tag{3}$$

since $k(j) \to \infty$ as $j \to \infty$ (g is a homeomorphism) and $gh^{-n_j}\xi^+ = g\xi^+$.

We assert that g_j is bounded. Suppose not, then by **CB 3** (applied twice) and (3), there is a subsequence n_i such that $g_{n_i}^{\pm 1} x_0 \to \eta^{\pm} \in \partial X$, where $\{\eta^{\pm}\} = \{\xi^{\pm}\}$. But from **CB 1** we should then have that $g_{n_i} E \subset U^+$ or U^- for all large i, which contradicts (2). Hence g_j is bounded and by the properness assumption, $g_j = g_i$ for some distinct i and j. Therefore $h^k = gh^l g^{-1}$ for some non-zero integers k and l.

We claim that it now follows that $g\xi^+ = \xi^+$. Indeed, applying the obtained equality to $g\xi^+$ we have

$$h^k(g\xi^+) = gh^l g^{-1} g\xi^+ = g\xi^+.$$

As h^k is unbounded it can have at most two fixed points, namely the same as h, that is, ξ^+ and ξ^-. So we have that $g\xi^+$ is either ξ^+ or ξ^-, but the latter is impossible because $g\xi^- = \xi^-$ and g is bijective. $\square$

5. Main results: unbounded cases

Let X be a complete metric space and $\overline{X}$ an $Isom(X)$-bordification satisfying **CB 1** and **CB 2**. For a group Γ acting by isometry on X, we denote by $L(\Gamma)$ the limit set of Γ, that is, the set of accumulation points in ∂X of an orbit Γx. The set $L(\Gamma)$ is independent of which orbit we consider by **CB 1** and by continuity it is Γ-invariant. We call an action of Γ on X *elementary with respect to* $\overline{X}$ if the limit set $L(\Gamma)$ consists of 0, 1 or 2 points. If the orbit is bounded, then the action is said to be *bounded.* An action of Γ is called *quasi-parabolic with respect to* $\overline{X}$ if Γ fixes exactly one point of the boundary ∂X.

Theorem 6. *Let Γ be a group in $\mathcal{A}$. Assume that Γ acts on X such that the elements of S are unbounded. Then, either the action is quasi-parabolic, or, Γ fixes two fixed points of ∂X and the action is elementary.*

Proof. Inductively using Propositions 1, 2 and that the NoFS$_2$-graph is connected, we have that the non-empty fixed point sets of each element in S coincides. Hence Fix(Γ), which equals this common set by **CB 1**, has cardinality 1 or 2. Now assume the fixed point set contains two points η and

ζ, and that $g_n x_0 \to \xi \in \partial X$. Then **CB 3** guarantees that there is a subsequence of g_n^{-1} converging to η in view of the fact that $g_n \eta = \eta$. Now **CB 1** implies that $\xi = \zeta$. $\square$

Theorem 7. *Let Γ be a group in $\mathcal{B}$. Assume that $\overline{X}$ also satisfies **CB 3** and that Γ acts on X such that the elements of S are unbounded. Suppose that the group generated by any pair of elements of S acts metrically properly. Then the Γ-action is elementary and fixes one or two points in ∂X.*

Proof. Inductively using Propositions 1, 3, 4 and that the F_2-graph is connected, we have that the non-empty fixed point sets of each element in S coincides. Hence $\mathrm{Fix}(\Gamma)$, which equals this common set by **CB 1**, has cardinality 1 or 2. Finally, in view of Propositions 1 and 4, we have that the limit set must equal the fixed point set arguing using **CB 3** similarily to the previous proof. $\square$

If G is a finite extension of a group in $\mathcal{A}$ or $\mathcal{B}$, then the above conclusions about elementariness of the action also holds for G because limit sets, if finite, can only consists of at most two points for contractive boundaries.

Note that if two elements s and r are conjugate then both are either bounded or unbounded as can be seen from the following simple calculation:

$$d(x, r^n x) = d(x, (gsg^{-1})^n x) = d(x, gs^n g^{-1} x) = d(g^{-1}x, s^n g^{-1} x).$$

6. Main results: bounded cases

As usual we denote by X a complete metric space with a contractive bordification $\overline{X}$.

Proposition 5. *Assume that $\overline{X}$ is sequentially compact and let Γ be a group of isometries of X. Suppose that every element of Γ is bounded. Then either the G-orbit is bounded or $L(\Gamma) = Fix(\Gamma) \subset \partial X$ is one point.*

Proof. We may suppose that the orbit is unbounded and that G does not fix a point in ∂X. By compactness we can find a sequence g_n such that $g_n x_0 \to \xi^+ \in \partial X$ and $g_n^{-1} x_0 \to \xi^- \in \partial X$. Moreover, if $\xi^+ = \xi^-$ then as G does not fix this point there is an element $p \in G$ such that $pg_n x_0 \to p\xi^+ \neq \xi^+$ and $(pg_n)^{-1} x_0 = g_n^{-1} p x_0 \to \xi^-$. We may hence assume that $\xi^+ \neq \xi^-$. By **CB 1,** for every large N, g_N is unbounded, cf. [Kar02a, Lemma 3]. This is a contradiction. $\square$

We do not know if it can happen that the action is unbounded but the limit set is empty in the case X is not locally compact.

A metric space (X, d) is called *uniformly convex* if it is geodesic and there is a strictly decreasing continuous function g on $[0, 1]$ with $g(0) = 1$, such that for any $x, y, w \in X$ and midpoint z of x and y,

$$\frac{d(z, w)}{R} \leq g\left(\frac{d(x, y)}{2R}\right),$$

where $R := \max\{d(x,w), d(y,w)\}$. Cartan-Hadamard manifolds and more generally CAT(0)-spaces, as well as L^p-spaces for $1 < p < \infty$ are standard examples of uniformly convex spaces. A *circumcenter* of a bounded set D is a point $x \in X$ such that $\overline{B_r(x)} \supset D$ and the radius r is minimal, that is

$$r = \inf\{R : \exists y, \overline{B_R(y)} \supset D\}.$$

The following lemma is well known, but we include the proof for completeness and in lack of a reference.

Lemma 1. *Let D be any bounded subset of a uniformly convex complete metric space (Y,d). Then there exists a unique circumcenter of D.*

Proof. We may assume that D contains at least two points so that

$$r := \inf\{R : \exists y, D \subset \overline{B_R(y)}\} > 0.$$

To prove the existence, let $\{x_n\}$ be a sequence of points such that $D \subset \overline{B_{r_n}(x_n)}$ and r_n converges to r. We need to show that $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$, $\delta = \max\{r_n, r_m\}/r$, and $z = z_{n,m}$ be the midpoint of x_n and x_m. Then there is a point $w = w_{n,m}$ such that

$$d(z,w) \geq \frac{r}{\delta}.$$

By uniform convexity we have

$$d(z,w) \leq g\left(\frac{d(x_n,x_m)}{2\max\{r_n,r_m\}}\right)\max\{r_n,r_m\} < g\left(\frac{d(x_n,x_m)}{2r}\right)\delta r.$$

Hence

$$g\left(\frac{d(x_n,x_m)}{2r}\right) > \frac{1}{\delta^2}.$$

There is a large number M such that δ is so close to 1 so that

$$d(x_n,x_m) < 2rg^{-1}(\delta^{-2}) < \varepsilon$$

for all $n,m \geq M$. By completeness x_n converges to a point which is a circumcenter. Note that because $\{x_n\}$ was arbitrary it is clear that this circumcenter is unique. □

In view of the above two statements we obtain the following corollary which generalizes a special case of a fixed point result in [Ser77]:

Corollary 5. *Assume in addition that X is uniformly convex and proper. If each element of Γ has a fixed point in X, then Γ has a fixed point in $\overline{X}$.*

Proof. By Proposition 5 we may assume that Γx_0 is a bounded set. Since the map assigning to a bounded set its unique circumcenter (Lemma 1) commutes with isometries and because $\Gamma(\Gamma x_0) = \Gamma x_0$ we have that Γ must fix the circumcenter of this orbit. □

Let us now record a statement from the theory of locally compact hyperbolic spaces which carries over:

Proposition 6. *The stabilizer of three distinct points in ∂X is bounded.*

Proof. Let Γ be the group which pointwise fixes the three points in question ξ_1, ξ_2, and ξ_3. It has finite index in the stabilizer. Suppose that Γ is unbounded, so there is a sequence $g_n \to \infty$ and $g_n\xi_i = \xi_i$ for every n and i. Then **CB 3** implies that there is a subsequence n_k such that $g_{n_k}x_0 \to \xi_1$ (or $g_{n_k}^{-1}x_0 \to \xi_1$). Applying **CB 3** to this subsequence we get a finer subsequence n_l for which $g_{n_l}^{\varepsilon}x_0 \to \xi_1$ and $g_{n_l}^{-\varepsilon}x_0 \to \xi_2$, for $\varepsilon = 1$ or -1. As this sequence also fixes ξ_3 we have a contradiction to **CB 1**. □

In the theory of convergence groups (see [GM87] and [Bow99]) one deduces that Γ acts properly on triples of points in the compact space ∂X.

A group is *boundedly generated by a finite set of generators S* if there is a constant ν such that every element $g \in \Gamma$ can be written as a product $g = r_1^{k_1}r_2^{k_2}...r_m^{k_m}$ where r_i are some elements in S, k_i integers and $m \leq \nu$.

Note that the following lemma is related to an idea in Shalom's proof of [Sha01, Thm. 2.6]:

Lemma 2. *Let Γ be a group boundedly generated by a finite set S. If Γ acts on a metric space such that every element in S is bounded, then every Γ-orbit is bounded.*

Proof. Let r_i be a sequence of generators, $1 \leq i \leq \nu$. Denote by B_1 a bounded subset, e.g. $\{x_0\}$. As r_1 is bounded, we have that also

$$B_2 := \bigcup_{n \in Z} r_1^n B_1$$

is bounded. By finite induction ($1 \leq i \leq \nu$) we can conclude that

$$\bigcup_{(k_1,\ldots,k_m)\in\mathbb{Z}^m} r_m^{k_m}\cdots r_2^{k_2}r_1^{k_1}B_1$$

is bounded. As the number of sequences r_i, $1 \leq i \leq \nu$ of elements in S is finite, we can continue the induction to obtain that ΓB is bounded for any bounded set B. □

Now putting together this last lemma with Theorem 6, the remark in the end of section 4 and Lemma 1, we obtain Theorem 3 and the following more general result:

Theorem 8. *Assume that $\Gamma =< S >$ is a group in $\mathcal{A}$ such that in addition Γ is boundedly generated by S consisting of a finite number of elements all of which are conjugate. Whenever Γ acts by isometry on a complete metric space X with a contractive bordification, then either the orbit is bounded or there is a point in ∂X fixed by all of Γ.*

7. Examples of bordifications

Here follows a list of hyperbolic $Isom(X)$-bordifications:

- the one-point bordification of any complete metric space. Trivial;

- the bordification construction of Floyd (or the conformal boundary in the terminolgy of [Gro87]) of geodesic complete metric spaces. The proof is essentially Lemma 1 in [Kar02a], see also [Kar01b];
- the hyperbolic boundary of Gromov hyperbolic spaces (e.g. CAT(-1)-spaces). This is almost immediate from the definition, but it is also in fact a special case of the previous example. See [Gao97] for material on nonlocally compact spaces;
- the bordification with the space of ends introduced by Freudenthal. This is simple;
- the usual compactification of proper visibility spaces (first introduced by Eberlein-O'Neill, see [BH99]). (A variant is the concept of a locally visible CAT(0)-space, whose definition immediately verifies **HB 1**). It seems that **HB2** is not so clear in the non-proper case;
- the ordinary closure of a strictly convex bounded domain in $\mathbb{R}^n$ with Hilbert's metric. This is due to Beardon [Bea97] and [KN00, section 5]. Note that here the extension property of isometries is not automatic.

8. Examples of groups

Recall the notion of P-graph defined in the introduction. Some of the following series of examples are adaptions from [CV96]:

8.1. **Trivial examples.** Nilpotent groups and finitely generated groups of subexponential growth do not contain any non-abelian free semigroup. Finitely generated amenable groups, in particular solvable groups, do not contain any non-abelian free groups. The product of two groups is generated by a set whose C-graph is connected. Coxeter groups also belong to $\mathcal{A}$ since any pair of the (standard) generators generate a finite or infinite dihedral group.

8.2. **SL(n,A),** $n \geq 3$**.** Let A be a commutative $\mathbb{Z}$-algebra with 1, for example $\mathbb{Z}$, a ring of S-integers, or a field. An *elementary matrix* is an element $x_{ij}(a) := 1 + ae_{ij}$ of $SL(n,A)$ where $a \in A$ and e_{ij} is the matrix with 1 at the place (i,j), $i \neq j$, and zero everywhere else. It is straightforward to verify the following relations for any $a, b \in A$:

$$[x_{ij}(a), x_{kl}(b)] = x_{il}(ab)$$

if $i \neq l$, $j = k$, and

$$[x_{ij}(a), x_{kl}(b)] = 1$$

if $i \neq l$, $j \neq k$.

Let $E(n,A)$ denote the subgroup of $SL(n,A)$ generated by all the elementary matrices. The N-graph of the set of elementary matrices is connected provided $n \geq 3$. To see this, let $x_{ij}(a)$ and $x_{kl}(b)$ be two elements, so $i \neq j$ and $k \neq l$. If $i \neq l$, and $j \neq k$, then they commute. If $j = k$ and $i \neq l$ (or $j \neq k$ and $i = l$), then it also follow from the above commutation rules that the two elements in question generate a 2-step nilpotent group. Finally, if

$i = l$ and $j = k$, then there is an m different from i and j (since $n \geq 3$). Now $[x_{ij}(a), x_{im}(a)] = 1$ and $x_{kl}(b) = x_{ji}(b)$ together with $x_{im}(a)$ generate a nilpotent group as in the second case. This shows that $E(n, A) \in \mathcal{A}$ for $n \geq 3$.

Assume from now on that $n \geq 3$. If A is a field or the integers of a number field, then $E(n, A) = SL(n, A)$, see [BMS67]. More generally, if A satisfies Bass' stable range condition S_d (see [Bas68]) and $SK_1(A)$ is finite, then $E(n, A)$ has finite index in $SL(n, A)$ for $n \geq d + 2$, and hence the results in section 5 apply. (Recall that $SL(2, \mathbb{Z})$ acts non-trivially on the hyperbolic plane.)

Assume that A is generated by $a_1, ..., a_m$. One can then generate $E(n, A)$ with all $x_{ij}(a_k)$ and $x_{ij}(1)$ (assuming $n \geq 3$). The group $E(n, A)$ contains the group of monomial matrices which act transitively by conjugation on the set of generators $\{x_{ij}(a_k)\}$ for a fixed k. If the a_is generate A, then any two $x_{ij}(a_k)$ and $x_{ij}(a_l)$ are also conjugate.

In [CK83] it is proved that when A is the ring of integers of a number field, then $SL_n(A)$ is boundedly generated by elementary matrices. For $A = \mathbb{Z}$ see also [AM92], and for some interesting discussions about bounded generation we also refer to [Sha01]. In these cases A is isomorphic to $\mathbb{Z}^n$ additively, which implies that every elementary matrix $x_{ij}(a)$ can be written as a product of $x_{ij}(a_k)$, $1 \leq k \leq m$ and $x_{ij}(1)$. Therefore such $SL(n, A)$ are boundedly generated by a finite set of elementary matrices.

In conclusion we have that, if A is a ring of integers of an algebraic number field and $\mathbb{Z}A^{\times} = A$ (the units generate A), then Theorems 3 and 8 apply to $\Gamma = SL(n, A)$ for $n \geq 3$.

8.3. **Chevalley groups over rings.** Let A be a finitely generated commutative $\mathbb{Z}$-algebra with 1 and Φ a reduced irreducible root system of rank greater than 3. The *Steinberg group* $St(\Phi, A)$ associated to Φ with coefficients in A is generated by symbols $x_\alpha(a)$, where $\alpha \in \Phi$ and $a \in A$ subject to the relations

(1) $x_\alpha(a + b) = x_\alpha(a)x_\alpha(b)$
(2) $[x_\alpha(a), x_\beta(b)] = x_{\alpha+\beta}(N_{\alpha,\beta}ab)$, $\alpha + \beta \neq 0$,

where $N_{\alpha,\beta} = -N_{\beta,\alpha}$ if α, β, $\alpha + \beta \in \Phi$, and $N_{\beta,\alpha} = 0$ if $\alpha, \beta \in \Phi$, $\alpha + \beta \notin \Phi$, [Ste68].

Proposition 7. *The standard generating system* $S = \{x_\alpha(a) : \alpha \in \Phi\}$, $a \in A \setminus \{0\}$ *has a connected N-graph.*

Proof. If Φ^+ is the set of positive roots relative to some ordering of Φ then the corresponding subgroup is nilpotent, in particular the N-graph for the root subgroups $x_\alpha(A)$, $\alpha \in \Phi^+$ is connected. Now say that two positive systems are *adjacent* if they have a non-empty intersection. This gives a graph structure on the set of positive systems. If we prove that this graph is connected this would give the desired connectedness of the N-graph. The Weyl group acts transitively on the set of positive roots [Hum90], thus we need only connect Φ^+ to $w\Phi^+$ where w is a reflection in a simple root α of Φ^+. But w fixes all simple roots but α and since Φ has rank at least 2, we conclude that Φ^+ and $w\Phi^+$ are adjacent. □

Given a reduced irreducible root system Φ_n of rank n and a commutative ring A with 1, one can define a Chevalley group $G(\Phi_n, A)$, a group $E(\Phi_n, A)$ generated by the unipotents in $G(\Phi_n, A)$, and a Steinberg group $St(\Phi_n, A)$. There is a natural homomorphism $\pi : St(\Phi_n, A) \to G(\Phi_n, A)$ with image $E(\Phi_n, A)$, and one defines $K_1(\Phi_n, A) = \mathrm{coker}\pi$ and $K_2(\Phi_n, A) = \ker \pi$; these constructions are all functorial in A. It follows from the above that the N-graph of $E(\Phi_n, A)$ is connected. If $K_1(\Phi_n, A)$ is finite one could apply Theorems 6 and 7 to the Chevalley group $G(\Phi_n, A)$. In particular this is the case when A is the ring of integers of a global field. Results on bounded generation are obtained in [Tav91]. If the Weyl group acts transitively on the roots then $x_\alpha(a)$ and $x_\beta(a)$ are conjugate as in the previous subsection. This shows that there are some further situations where Theorems 3 and 8 apply.

Note however that in the case X is a complete separable CAT(-1)-space, the papers [BM96] and [Gao97] provide superrigidity results for general lattices in higher rank simple groups.

8.4. **Braid groups B_n, $n \geq 5$.** The Artin presentation of the classical braid groups $B_n =< s_1, s_2, ..., s_{n-1} >$ is

$$s_i s_j = s_j s_i \text{ if } |i-j| \geq 2$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

for $1 \leq i \leq n-2$. Hence the C-graph of $\{s_1, s_2, ..., s_n\}$ is connected for $n \geq 5$. (More generally, any Artin group, for which the associated standard Coxeter generating system has a connected C-graph, belongs to $\mathcal{A}$.)

These generators are all conjugate. Note that B_n is not boundedly generated. Indeed, recall that the corresponding pure braid group P_n, which has finite index in B_n, surjects onto P_3 which is isomorphic to $F_2 \times \mathbb{Z}$, and bounded generation passes to finite index subgroup and to homomorphic images.

When a group has an element in the center which acts unboundedly, then the arguments in section 4 imply that the whole group fixes a boundary point. It is well-known that this remark applies to the braid groups since the element

$$\Omega^2 = ((s_1 s_2 ... s_{n-1})(s_1 s_2 ... s_{n-2}) s_1 ... (s_1 s_2) s_1)^2$$

is a central element of infinite order.

8.5. **Mapping Class groups Mod$(\Sigma_{g,n})$, $g \geq 2$.** Let Mod$(\Sigma_{g,n})$ be the mapping class group of an oriented surface $\Sigma_{g,n}$ of genus g and n punctures.

It is known that if the genus is at least one, these groups are generated by a finite number of Dehn twists around non-separating simple closed curves and any two of these curves are either disjoint or intersect transversally in one point. If disjoint, then the corresponding two Dehn twists generate $\mathbb{Z}^2$. If $3g-3+n \geq 3$ then the C-graph is connected for this generating set and so $\mathrm{Mod}(\Sigma_{g,n}) \in \mathcal{A}$.

Since any non-separating simple closed curve can be moved to any other by a homeomorphism, the elements of S are all conjugate. For more information, see [BF01] or [CV96].

Farb-Lubotzky-Minsky proved in [FLM01] that $\mathrm{Mod}(\Sigma_{g,0})$ has finite index subgroups whose pro-p completion is not p-analytic, and as a corollary they establish that $\mathrm{Mod}(\Sigma_{g,0})$, for $g \geq 1$ is not boundedly generated by any finite set.

To see that $\mathrm{Mod}(\Sigma_{g,n})$ (for $g \geq 1$, or $g = 0$ and $n \geq 4$) is not boundedly generated by any finite set of Dehn twists, we propose a quite different approach. Namely, it is not a very deep fact that $\mathrm{Mod}(\Sigma_{g,n})$ acts unboundedly by isometry (see [MM99, p.105, p.124]) on the complex of curves. Moreover, it is immediate that a Dehn twist around a curve α fixes α in the complex. Our claim now follows from Lemma 2.

In fact, taking advantage of the deeper aspects of [MM99], Bestvina and Fujiwara recently managed to show that the space of non-trivial quasihomomorphisms of $\mathrm{Mod}(\Sigma_{g,n})$ as above, is infinite dimensional. This property is incompatible with bounded generation (this remark we learnt from M. Burger).

8.6. **Automorphism groups of free groups.** We follow [CV96] here. Let F_n be the free group of rank $n \geq 3$, with generators $x_1, ..., x_n$. We first consider the index two subgroup $SAut(F_n)$ of $Aut(F_n)$ consisting of special automorphisms (an automorphism is *special* if the determinant of the induced automorphism of $\mathbb{Z}^n$ is equal to $+1$). For $i \neq j$, let $\rho_{i,j}$ (respectively $\lambda_{i,j}$) be the automorphism which sends x_i to $x_i x_j$ (respectively $x_j x_i$) and fixes x_k for $k \neq i$. The elements $\rho_{i,j}$ and $\lambda_{i,j}$ generate $SAut(F_n)$. If i, j, and k are distinct, then

$$[\rho_{i,j}, \rho_{j,k}] = \rho_{i,k}$$
$$[\lambda_{i,j}, \lambda_{j,k}] = \lambda_{i,k}.$$

It follows that the automorphisms $r_i := \rho_{i,i+1}$ and $l_i := \lambda_{i,i+1}$ for $1 \leq i \leq n-1$, together with $r_n := \rho_{n,1}$ and $l_n := \lambda_{n,1}$ generate $SAut(F_n)$. Set $S = \{r_1, l_1, ..., r_n, l_n\}$. The group $SAut(F_n)$ contains the alternating group on n letters and the automorphisms which send exactly two generators to their inverses. Conjugating by appropriate such automorphisms one sees that the elements of S are all conjugate. It is known, due to Sury, that $Aut(F_n)$ and $SAut(F_n)$ are not boundedly generated.

Now let s_i denote r_i or l_i. It is easily checked that s_i commutes with r_j and l_j if $2 \leq |i-j| \leq n-2$. Thus the C-graph of S is connected if $n \geq 5$ and most of our results apply to $SAut(F_n)$ which belongs to $\mathcal{A}$ and therefore also to $Aut(F_n)$.

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FIM, ETH-Zentrum, 8092 Zürich, Switzerland

Faculty of Mathematics, University of Bielefeld, 33501 Bielefeld, Germany; Institute of Mathematics, Omsk, 644099 Pevtsova 13, Russia