Jordan Curve Theorem

A *curve* γ is the image of a continuous map $f_\gamma = f : [0,1] \to \mathbb{R}^2$. It is *closed* if $f_\gamma(0) = f_\gamma(1)$. Thus a closed curve can be thought of the continuous image of the standard unit circle $\mathbb{S}^1$. An *arc* is a subspace of $\mathbb{R}^2$ homeomorphic to $[0,1]$, i.e., an injective curve. A *Jordan curve* is a subspace of $\mathbb{R}^2$ homeomorphic to $\mathbb{S}^1$.

Let p be a point in X and ϵ be a positive real number. The *ϵ-neighborhood* $\mathsf{N}(p;\epsilon)$ of p in X is defined to be the collection of all points $q \in X$ such that $\mathrm{d}_X(p,q) < \epsilon$. A set $U \subseteq X$ is *open* if for any $p \in U$ there exists some $\epsilon > 0$ such that $\mathsf{N}(p;\epsilon) \subset U$. Empty set and X are open. Topology induced from the metric.

A point $p \in X$ is a *limit point* of a set $A \subset X$ if every ϵ-neighborhood of p contains a point $q \neq p$ of A. A set $B \subset X$ is *closed* if it contains all its limit points. If p is a limit point of both A and $X \setminus A$, then p is called a *frontier point* of A and $X \setminus A$. The *frontier* $\mathrm{fr}(A)$ of A is the collection of its frontier points. The intersection $\mathrm{cl}(A)$ of all closed sets that contain A is called the *closure* of A. The *interior* $\mathrm{int}(A)$ of A is the union of all open sets that are contained in A.

Some Useful Results

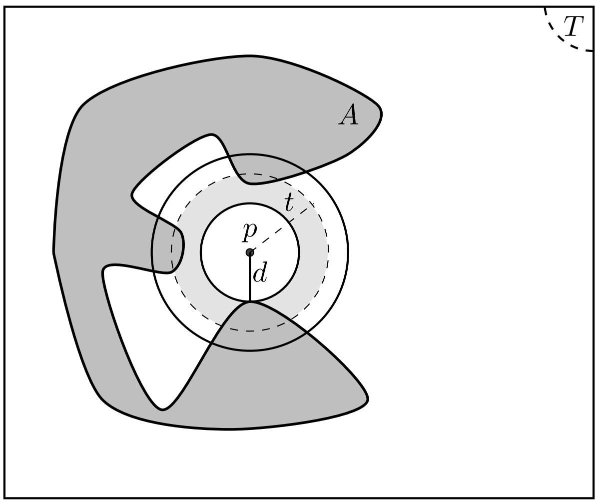
Let X be a topological space and A be a subspace of X. A *retraction* $r : X \to A$ is a continuous map such that the restriction of r to A is the identity map on A. If Y is a topological space and $f : A \to Y$ is a continuous map, then an *extension* of f is a continuous map $F : X \to Y$ such that the restriction of F to A is f.

Theorem 1 (No-Retraction Theorem). *There is no retraction from a disk D to its boundary S.*

Proof. Let $g : D \to S$ be a retraction and let $\alpha : S \to S$ be the antipodal map (the composition of a homeomorphism between S and the standard 1-sphere and the antipodal map in the standard case.). Then the map $\alpha \circ g : D \to D$ does not have any fix point, contradicting Brouwer's Fix Point Theorem. □

Theorem 2 (Tietze Extension Theorem). *Let T be a metric space and A be a closed subset of T. Then any continuous map $f : A \to [0,1]$ has a continuous extension to T.*

Proof. For a point $p \in T \setminus A$, consider the neighborhoods $N_1 = N(p,d)$ and $N_2 = N(p,2d)$ where $d = d(p)$ is the distance between p and A. For $d \le t \le 2d$ define $g_p(t)$ to be $\sup f(x)$ over all $x \in A \cap N(p,t)$.



The function $g_p(t)$ is bounded and nondecreasing on $[d(p), 2d(p)]$, and, in particular, it is integrable. Define

$$F(p) = \frac{1}{d(p)} \int_{d(p)}^{2d(p)} g_p(t)\,dt,$$

if $p \in T \setminus A$ and $F(p) = f(p)$ for $p \in A$. It is left to you as an exercise to show that F is continuous. □

Corollary 3. *Let T be a metric space and A be a closed subset of T. Then any continuous map $f : A \to [0,1] \times [0,1]$ has a continuous extension to T.*

Space Filling Curves

Theorem 4. *There are space filling curves, i.e. there are continuous surjections from $[0,1]$ to $[0,1] \times [0,1]$.*

Proof. Let $\mathcal{C} \subseteq [0,1]$ be the Cantor set. Define $f : \mathcal{C} \to [0,1] \times [0,1]$ by

$$f\left(\sum_{k\in\mathbb{N}} \frac{a_k}{3^k}\right) = \left(\sum_{k\in\mathbb{N}} \frac{a_{2k-1}}{2^{k+1}}, \sum_{k\in\mathbb{N}} \frac{a_{2k}}{2^{k+1}}\right).$$

We show that f is a continuous surjection. Then we use the Tietze extension theorem to extend it to the whole interval. □

Theorem 5 (Netto). *There is no continuous bijection from $[0,1]$ to $[0,1] \times [0,1]$.*

Proof. We sketch a proof here and leave the details to you as an exercise. Let f be a continuous bijection from $[0,1]$ to $[0,1] \times [0,1]$. Every closed subset C of $[0,1]$ is compact, and the continuous image $f(C)$ of a compact set is compact and, thus, $f(C)$ is a closed subset of $[0,1] \times [0,1]$. This shows that f^{-1} is continuous. Set $p = f(1/2)$. Then $f^{-1}([0,1] \times [0,1] - p)$ is not connected while $[0,1] \times [0,1] - p$ is connected which contradicts the fact that f^{-1} is continuos. □

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Lemma 6. *$\mathbb{R}^2 - J$ has exactly one unbounded component.*

Proof. Follows from the fact that J is compact. □

Theorem 7 (No Separation Theorem). *Let L be a Jordan arc in $\mathbb{R}^2$. Then $\mathbb{R}^2 - L$ is connected.*

Proof. Suppose not, then $\mathbb{R}^2 - L$ has a bounded component B. If A is any other component of $\mathbb{R}^2 - L$, then $\mathrm{cl}(B) \cap A = \emptyset$ since A is open and disjoint from B. In particular, $\mathrm{fr}(B) = \mathrm{cl}(B) \cap (\mathbb{R}^2 - B)$ is disjoint from A and, thus, $\mathrm{fr}(B) \subseteq L$. Let b be a point in B and let D be disk centered at b so that $L \subset \mathrm{int}(D)$. Since $L \cong [0,1]$, by Tietze Extension Theorem, the identity map $\mathrm{id}_L : L \to L$ has a continuous extension $r : D \to L$. Define $F : D \to D - \{b\}$ by

$$F(x) = \begin{cases} r(x) & \text{if } x \in \mathrm{cl}(B), \\ x & \text{if } x \in D - B. \end{cases}$$

Note that $(D - B) \cap \mathrm{cl}(B)$ is contained in L and, therefore, g is well-defined and continuous. Also, the boundary S of D is contained in $D - B$. So, $F(x) = x$ for all $x \in S$. Now, if $\mathfrak{p} : D - \{b\} \to S$ is the natural projection, then $\mathfrak{p} \circ F : D \to S$ is a retraction, contradicting the no-retraction theorem. □

Lemma 8. *Let J be a Jordan curve in $\mathbb{R}^2$. If $\mathbb{R}^2 - J$ is not connected, then every component of $\mathbb{R}^2 - J$ has J as its frontier.*

Proof. Let A be a component of $\mathbb{R}^2 - J$ and assume that frontier of A is a proper subset of J. Let L be the complement of a connected open component of $J - \mathrm{fr}(A)$. Then L is a Jordan arc and $\mathrm{fr}(A) \subseteq L$. Moreover, $\mathbb{R}^2 - L = A \cup (\mathbb{R}^2 - (A \cup L))$ where A and $\mathbb{R}^2 - (A \cup L)$ are open, disjoint and non-empty sets contradicting the No Separation Theorem. □

Lemma 9. *Let $f = (f_1, f_2)$ and $g(g_1, g_2)$ be two continuous maps from $[-1,1]$ to $[a,b] \times [c,d]$ such that*

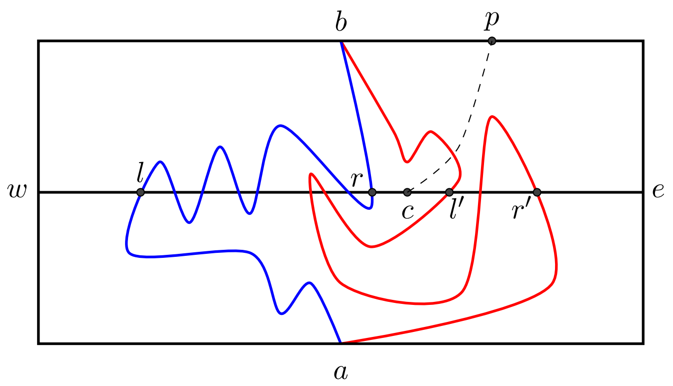
- $f_1(-1) = a$ *and* $f_1(1) = b$*;*
- $g_2(-1) = c$ *and* $g_2(1) = d$*.*

Then f *and* g *meet, that is,* $f(x) = g(y)$ *for some* $x, y \in [-1, 1]$*.*

Proof. □

Theorem 10 (Jordan Curve Theorem)**.** *Let* J *be a Jordan curve. Then* $\mathbb{R}^2 - J$ *has exactly two connected components.*

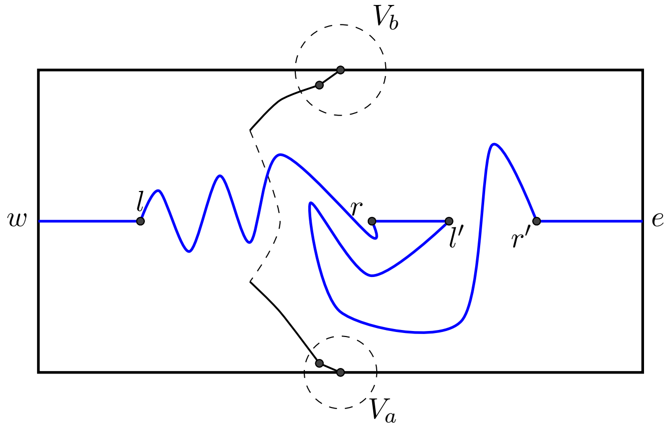
Proof. We start by introducing the notations. Let a and b be two points of J of the maximum distance. We can embed J, by rotating if necessary, in a rectangle R in such a way that $R \cap J = \{a, b\}$ and these intersection points have the same x-coordinate, say zero. Let w and e be the middle points of the vertical edges of R. Then J meets the segment $\overline{we}$. Let l be the left-most intersection point of J and $\overline{we}$. Let γ be the arc in J between a and b that passes contains l (the blue arc in the figure below) and let γ' be the other arc between a and b in J (the red one). Let r be the right-most intersection point of γ and $\overline{we}$. Note that r and l could be the same points. Let $\gamma_{[l,r]}$ denote the arc between l and r in γ. The arc γ' should meet $\overline{wl} + \gamma_{[l,r]} + \overline{re}$. However, γ' does not meet $\overline{wl} + \gamma_{[l,r]}$, since l is the left-most intersection of J and $\overline{we}$ and J cannot have a self-intersection. Hence, γ' meets $\overline{re}$. Let l' and r' be the left-most intersection and the right-most intersection points of γ' and $\overline{re}$, respectively. Finally, let c be the middle point of the segment $\overline{rl'}$.



At least one bounded component.
We claim that the component A of $\mathbb{R}^2 - J$ that contains c is bounded. Suppose not, then there is a path in A from c to a point outside R. Let p be the first point which this path intersects R and let α denote the part of the path from c to p. We assume that p has a positive x-coordinate, the other case (when the x-coordinate p is negative) is similar. In this case, there is a path $R_{[p,e]}$ from p to e that avoids both a and b. Since γ' meets $\overline{wl} + \gamma_{[l,r]} + \overline{rc} + \alpha + R_{[p,e]}$, it should meet α which is impossible as α is in A.
At most one bounded component.
Suppose B is a bounded component of $\mathbb{R}^2 - J$ different from A. Consider the path

$$\beta := \overline{wl} + \gamma_{[l,r]} + \overline{rl'} + \gamma'_{[l',r']} + \overline{r'e}$$

connecting w to e (the blue path in the figure below).



Since a and b are not in β, there are neighborhoods V_a and V_b of them that do not contain any point of β. On the other hand, there must be points a' and b' in B such that $a' \in V_a \cap B$ and $b' \in V_b \cap B$. Now, since B is path connected there is a path from a' to b' in B. This path must meet β which is impossible.

□