Combinatorial Topology-Problem set II.

Deadline: 22nd March 2020

1. (5 points) Let Σ be a triangulated 2-manifold without boundary. Let v and u be two adjacent vertices in Σ. Let Σ' be the simplicial complex obtained from Σ by contracting the edge uv. Show that Σ and Σ' are homeomorphic if and only if

$$\operatorname{link}(uv; \Sigma) = \operatorname{link}(v; \Sigma) \cap \operatorname{link}(u; \Sigma).$$

2. (2 points) Let M be a compact closed surface of characteristic χ. Let Σ be a triangulation of M with n vertices. Show that

$$f(\Sigma) = (n, 3n - 3\chi, 2n - 2\chi).$$

3. (i) (2 points) Let M be compact closed surface. Show that if M has a triangulation with n vertices, then one can triangulate M with m vertices for all $m \geq n$.
   (ii) (3 points) Give a triangulation of the torus $\mathbb{T}^2$ with 7 vertices and show that there is no triangulation of $\mathbb{T}^2$ with 6 vertices.
   (iii) (2 points) Characterize all possible f-vectors of triangulations of $\mathbb{T}^2$.

4. (3 points) Let $\mathcal{A} = \{L_1, \ldots, L_k\}$ be a line arrangement in $\mathbb{R}^2$ and S be the set of singular points. For $p \in S$ let m_p be the number of lines in $\mathcal{A}$ passing through p. Show that the number of regions in $\mathbb{R}^2 \setminus \mathcal{A}$ is

$$1 + k + \sum_{p \in S} (m_p - 1).$$

5. Let $\mathcal{A} = \{H_1, \ldots, H_k\}$ be an arrangement of hyperplanes in $\mathbb{R}^n$.
   (i) (2 points) Show that every region of $\mathbb{R}^n \setminus \mathcal{A}$ is homeomorphic to an open n-ball.
   (ii) (2 points) What is the maximum possible number of regions in $\mathbb{R}^n \setminus \mathcal{A}$?