Combinatorial Topology-Problem set I.

Deadline: 14th February 2020

1. (2 points) Show that in the game of Hex if all the tiles are colored, then not both players can win.

2. Let P be a Jordan polygon with consecutive vertices $v_1, v_2, \ldots, v_n$ and let B be the bounded component of $\mathbb{R}^2 - P$. Consider the closed triangle Δ formed by three consecutive vertices[1] v_{i-1}, v_i, v_{i+1} of P. We say that Δ is an *ear* of P if the interior of the segment $\overline{v_{i-1}v_{i+1}}$ (i.e., $\overline{v_{i-1}v_{i+1}} - \{v_{i-1}, v_{i+1}\}$) is a subset of B.

   (i) (5 points) Let P be a Jordan polygon with $n > 3$ vertices. Show that P has at least one ear.

   (ii) (1 point) Use part (i) to show that P can be triangulated without adding any extra vertices.

   (iii) (2 points) For any $n > 3$ give an example of a Jordan polygon with exactly two ears.

   Let Δ be an ear of P which is formed by v_{n-1}, v_n, v_1. Let P' be the Jordan polygon on $v_1, \ldots, v_{n-1}$ obtained from removing the vertex v_n and the edges that contain it and then connecting v_{n-1} to v_1. Let B' be the bounded component of the complement of P'.

   (iv) (4 points) Show that there is a homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(P) = P'$ and $f(B) = B'$.

   (v) (2 points) Use (iv) to show that B is homeomorphic to the open unit ball $\{x \in \mathbb{R}^2 | \ \|x\| < 1\}$.

3. Let $\mathbb{S}^d$ denote the standard d-sphere.

   (i) (2 points) Let $f : \mathbb{S}^1 \to \mathbb{S}^2$ be a continuous injection. Use Jordan curve theorem to show that $\mathbb{S}^2 - f(\mathbb{S}^1)$ has exactly two connected components.

   (ii) (1 point) Give an example to show that a continuous injection from $\mathbb{S}^1$ to torus is not necessarily disconnect it.

   (iii) (2 points) Conclude that there is no continuous bijection from 2-sphere to torus, and, in particular, the 2-sphere and torus are not homeomorphic.

[1] the indices are mod n.