A short course on

Cohen-Macaulay Complexes

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The concept of Cohen-Macaulay complexes played an indispensable role in Stanly's prove of Upper Bound Conjecture. Since then, it has been one of the cornerstones of a rapidly developing mathematical discipline at the crossroads of combinatoics, commutative algebra and topology.

A Brief Historical Introduction

The concept of *shellability* is central in combinatorial geometry and topology. Shellability can be traced back at least to Schläfli's 1852 work *Theorie der Vielfachen Kontinuität*. In order to extend Euler polyhedron formula to heigher dimension, Schläfli needed to assume that the facets of each convex polytope can be ordered in a favourable way; *a shelling order*. The existence of shelling orders for convex polytopes had become questionable, especially since unshellable triangulations of the 3-dimensional ball and the sphere were constructed. However, in the late 1960s Bruggesser and Mani showed that all convex polytopes are shellable.

In 1970, McMullen made an elegant use of shellability to verify Motzkin's *Upper Bound Conjecture* for the number of faces in convex polytopes. It was conjectured that the same bound is valid for all triangulations of spheres. To every simplicial complex K, Stanley associated a certain algebra $\mathbb{C}[K] = \mathbb{C}[x_1, \dots, x_n]/I_K$, the so-called *Stanley-Reisner ring* of K. He knew that the number of faces in K are related to the Hilbert function of $\mathbb{C}[K]$. Also, he realized that if for a triangulated sphere K, $\mathbb{C}[K]$ is *Cohen-Macaulay*, then the face numbers of K satisfy the Upper Bound Conjecture.

Almost at the same time but for different motivations, Hochster got interested in Stanley-Reisner rings. He was looking for characterization of when $\mathbb{C}[K]$ is Cohen-Macaulay in terms of the topological properties of K. Reisner, a student of Hochster, solved this problem and, in particular, showed that $\mathbb{C}[K]$ is Cohen-Macaulay if K is a triangulated sphere.

Course Plan

There will be approximately eight 60-minute lectures. The first lectures are devoted to shellable complexes, their topological properties, the theorem by Bruggesser and Mani, and unshellable triangulated balls and spheres. Next, we study Cohen-Macaulay complexes in some depth. In particular, we will present topics like: the charaterization of face numbers of Cohen-Macaulay complexes, Hochster's formula, Reisner's theorem, and Stanley's proof of UBC. We also show that Cohen-Macaulayness for simplicial complexes is a topological property. Finally, we talk about other important families of complexes related to Cohen-Macaulay complexes, such as Buchsbaum and Gorenstein complexes.

Requirements

We will assume a knowledge of elementary facts in algebra and topology. Other than that, some familiarity with basic concepts from algebraic topology would be useful but not necessary.

Literature

We will not follow any specific book in this course. However, the following books are recommended for side reading.

Stanley (1996) *Combinatorics and Commutative Algebra.*

Miller, Sturmfels (2005) *Combinatorial Commutative Algebra.*

Ziegler (1995) *Lectures on Polytopes.*

Björner, Las Vergnas, Sturmfels, White, Ziegler (1999) *Oriented Matroids.* Sections 4 and 9.

Historical notes about the early days of Cohen-Macaulayness in combinatorics can be found in these articles.

Stanley, *How the Upper Bond Conjecture was proved*

Hochster, *Cohen-Macaulay Varieties, Geometric Complexes, and Combinatorics*

Björner, *Let* Δ *be a Cohen-Macaulay complex.*

Schedule

We will have weekly lectures from week 9 to week 16. Our first meeting will be on Friday March the 1st from 13:15 and it will take place at room 3418 KTH.

Examination

Those who are interested in taking credit for the course must write a report and give a presentation of a subject that will be assigned to them.