# Evaluating cable forces in cable supported bridges using methods of structural vibrations

*- An essay in theory of science and research methodology*

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## Abstract

This essay deals with the assessment of cable forces in existing cable supported bridges. The proposed method comprises measuring structural vibrations from which the cable force can be estimated. The estimation of the cable force is based on established theories that are modified to contain the problem studied. The aim of the study from an engineering aspect is to estimate the cable force within a sufficient tolerance. In the extent, the results are used in assessing global structural behaviour of the bridge, commonly performed by calibration of numerical models. From a scientific point of view, the main object is to propose a refined method for cable force determination, to be used in as general cases as possible and with as high tolerances as possible. The main content of this essay focus on the philosophical aspects in how to interpret measured data and how to use it to develop the proposed method of analysis. An important part of the study lies within determining the dependent parameters, their influence on the results and how to attenuate error sources. Furthermore, the use and application of numerical models in the process of testing the theory is discussed.

Keywords: Cable force, cable supported bridge, structural vibration, natural frequency, finite element methods, finite differences

## 1 Introduction

A significant part of civil engineering in general and structural engineering in particular is the maintenance of existing structures. In bridge engineering assessments of existing bridges are regularly performed by commission of the bridge owner. In order to perform a valid bridge assessment, the structural system must be known. In the case of cable supported bridges the structural system consists of cables subjected to tensile forces. The force depend both on physical properties of the cable and the geometry of the cable system. The geometry dependencies are often nonlinear. Due to the

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complexity of the system, theoretical models may result in large deviations compared to real bridge situation, if not all properties are given with sufficient accuracy. A method to gain more information of the structure is to perform in-situ measurements. In the present case, vibration measurements have been performed on a cable suspension bridge in order to evaluate the forces in the system. The concept is to use the relation between natural frequency and tensile force in structural members.

### 1.1 Theory dependence

In-plane vibrations of a cable subjected to a tensile force $S$, having a flexural rigidity $EI$ and a mass per unit length $m$ can be formulated as [9]:

$$EI\frac{\partial^4 y}{\partial x^4} - S\frac{\partial^2 y}{\partial x^2} = -m\frac{\partial^2 y}{\partial t^2} \tag{1.1}$$

Eq. 1.1 is dependent on the boundary conditions and for the bi-pinned case the tensile force can be expressed as

$$S = \left(\frac{2f_i l}{i}\right)^2 m - EI\left(\frac{i\pi}{l}\right)^2 = \frac{m}{\varkappa_{\mathrm{pp},i}^2}\left(\frac{2f_i l}{i}\right)^2 \tag{1.2}$$

where the last step of Eq. 1.2 is yet to be scrutinized. The solution is expressed as a function of the natural frequency $f_i$ of orders $i$. The main obstacle using Eq 1.1 is that it can only be solved exactly for the bi-pinned end condition. An approximate solution for the bi-clamped case has been presented by [9] and successfully used in bridge assessment by [5]. In the last step of Eq. 1.2, the term $\varkappa_{\mathrm{pp},i}$ is a function of $i$, depending on the tensile force, the length $l$, the flexural rigidity and the boundary conditions. For an ideal string, *i.e.* a cable with no flexural rigidity, $\varkappa$ is unity and the solution is therefore independent on the boundary conditions.

The main object of the research project is to validate the proposed method involving the $\varkappa$-function and expand it to the general case for arbitrary boundary conditions. Since only in-situ measurements have been performed, where none of the dependent parameters are readily known, a crucial part of the project is to estimate the error sources.

## 2 The research project

### 2.1 Background

The main object with the study, from a scientific point of view, is to accurately estimate the cable force. Using currently existing theories has shown great success in several similar cases, [4, 5, 6]. In the present case however, there are good reasons to believe that these methods results in large errors. What is here meant as currently existing theories are those covered by either the purely bi-pinned or the bi-clamped case, were even the later are rather limited.

In the case covered by this study, measurements were first carried out by [3], who also conducted analysis to evaluate the cable forces. The results from these analysis pointed out that some structural members behaved in a non-expected way, in contrast to both engineering experience and computer models. One explanation at hand was that the forces had been redistributed during earlier repairs of the bridge. New measurements were carried out, presented in [2], in order to evaluate the forces in several parts of the structural system, including the parts earlier shown to give unexpected results. The measured response in terms of natural frequencies showed good agreement with earlier studies. However, more detailed measurements were performed in order to extract further information. In addition, more refined methods were performed, based on the concepts of Eq. 1.2, further to be discussed. The results finally indicated that the earlier measurements were correct, but that the methods of analysis were not sufficiently valid to some extent, which the refined methods propose to solve.

# 3 Discussion

## 3.1 Dependent variables

Using the concepts of Eq. 1.2, the cable force can be calculated based on measured natural frequencies, $f_i$. The accuracy of the estimated cable force depend primary on the accuracy of the dependent parameters, which are the flexural rigidity $EI$, the mass per unit length $m$, the length of the cable $l$ and the boundary conditions.

The flexural rigidity depends highly on the structure of the cable and the types studied in the present case are primary of the type "locked coil cables". These cables consists of a core of typically 20 to 100 parallel wires, bundled together with an outer cover consisting of two or three layers of twisted wires. The flexural rigidity of the cable depends on the friction and interaction between the different wires. Since it is dependent on the friction, it may also depend on the tensile force. During measurements of a cable, the force is more or less constant, yielding the flexural rigidity to be rather constant. The influence of flexural rigidity increases with the mode order, *i.e.* the number of the overtone $i$. Both the tensile force and the flexural rigidity are constant for all $i$, provided reasonably small deformation. A better estimation of the flexural rigidity can therefore be performed using higher order modes. For higher order modes on the other hand, effects of transverse shear deformation increases. These effects are not included in Eq. 1.1 and are consequently not accounted for in the present analysis. Solving a differential equation including such effects has been performed by [7], resulting in a far more complicated solution than Eq. 1.2, even for the simplest boundary conditions. However, the influence of transverse shear deformation may be accounted for using numerical approaches. This has been done, and the results have shown that such effects are not influencing the results in the range of the present analysis, and are only relevant in cases of very thick structural members or for very high overtones $i$.

The mass of the cable is constant under all relevant conditions and the force is according to Eq. 1.2 linear proportional to the mass. The accuracy of the tensile force can therefore never be better than the accuracy of the mass. The mass can often be determined quite accurate, since the exact number of wires forming the cable is known. In the present case, some spare cables were measured, showing good agreement with the calculated mass performed according to [10].

The length of the cable is probably the easiest factor to measure with high accuracy. In some structural members however, it may be difficult to define from which exact points the length are to be defined. The length influences the result both explicitly, as shown directly in Eq. 1.2, but also more implicitly as a component in the $\varkappa$-function.

Since Eq. 1.1 only can be solved analytically for the bi-pinned boundary conditions, approximate solutions are required for all other cases. An approximate solution of Eq. 1.1 for the bi-clamped case has been shown by [9], along with a valid interval. Near the limit of this interval, the error in the approximation increases rapidly. Outside this range, or for any arbitrary boundary condition, general numerical methods are preferable. Numerical methods on the other hand, require the tensile force to be known. This can be circumventing by successive iteration, using the measured natural frequency as input data.

The measured natural frequency is also afflicted with errors, depending both on the measurement equipment and the duration of the measurement. The measured response consists of acceleration, which is transformed from time domain to frequency domain using Fast Fourier Transformation (FFT). Depending on the frequency of interest, the sampling frequency can be adjusted to give sufficiently accurate results. The accuracy of the results is also limited to the sampling time, *i.e.* the duration of the measurement. While the sampling rate sets the limit for the highest frequency to study, the sampling time sets the limit for the lowest. The former is due to that a sufficient number of measurement points must be covered in each period of frequency, while the later is due to that a sufficient number of periods must be covered within the total measurement time. The upper limit of the frequency range is theoretically limited by the Nyqvist frequency, defined as half the sample frequency. For signal processing reasons, filters must be applied to account for effects of so called aliasing, decreasing the valid frequency range. The accuracy in the instruments used is much higher than all other depending factors, which is provided by the manufacturer and also has been verified under laboratory conditions.

Since the study is based on in-situ measurements, performed on an actual bridge rather than in a well controlled laboratory environment, exterior factors may influence the results in a non-negligible manner, even if all dependent variables mentioned above are under full control. Since not all dependent variables are fully predefined, this will complicate the error estimation. Despite of this, it is proposed that several of the dependent variables can be determined more accurately, using Eq. 1.2 and different measurement techniques. Generally, most of the dependent parameters can be expressed on an interval and sensitivity analysis can be used in determining the range of the estimated cable force.

## 3.2 Modifications of current theory

The main limitation with the current theory is that it only provides adequate solutions for simple boundary conditions, namely the bi-pinned and bi-clamped ones. In many cases, such as for reasonably slender members, the difference between these two extremes is rather small, often within the tolerances of the dependent parameters. For significantly thicker cables, *i.e.* lower $l/EI$ ratios, the influence of the boundary conditions increases. The proposed method of analysis also aims at refining the determination of the dependent variables, basically by using higher order frequencies, which further increases the significance of the boundary conditions. Since the cable force is not known in advance and never can be readily determined, much of the work on extending existing theories and their validity is based on numerical analysis, preferably using methods of finite differences or finite elements. Code implementation using finite differences has shown to be suitable in calculating the $\varkappa$-functions presented in Eq. 1.2. The degree of restraint can then be expressed either as a ratio from bi-pinned to bi-clamped or as a definite value in the unit Nm/rad. From Eq. 1.2 it is realised that $\varkappa$ is a function of the mode orders $i$, and also includes the flexural rigidity $EI$ and the length $l$. For the bi-clamped case approximate solutions propose that it also depends on the tensile force $S$. The mere existence of the $\varkappa$-function is not to be seen as theoretically deduced, but rather as a practical fabrication striving for simplicity. In the end, it is used as a numerical solution of the differential equation in Eq. 1.1, where it includes all information of the boundary conditions, *i.e.* the degree of restraint.

The degree of restraint is often afflicted with large uncertainties. In some cases it is possible to express it as a function of the structural system, in terms of $EI$ and $l$ derived using structural mechanics, also found in handbook formulas. In other cases instrumental methods can be used, involving modal analysis. In the present study both these approaches have been used. If the degree of restraint can be expressed in terms of $EI$ and $l$ it can be implemented directly in the code. It is then only related to the structural system and does not include effects of *e.g.* friction. The method of modal analysis on the other hand, is based on measured responses of the actual structure, including friction and other more or less unobservable effects. The main concept of modal analysis, not thoroughly discussed in this paper, is to measure the response, commonly the acceleration, in several points of a structure. By integrating the acceleration signal, the deflection can be determined and with a sufficient amount of measurement positions the shape of deflection. Using this “real deflection shape” of the structure, numerical models can be calibrated in order to agree with the measured response. In practice, it has for the present case shown to be difficult to determine this real deflection shape with sufficient accuracy. Furthermore, using it to calibrate numerical models has also shown to be rather inaccurate, since a large difference in boundary condition often results in only a small change in mode shape. The modal analysis conducted indicates on the other hand that the shape corresponds to the one derived using handbook formulas. In some cases it is relevant to express the degree of restraint in relative terms, as a dimensionless scale from bi-pinned to bi-clamped. Numerical studies on the $\varkappa$–function has shown a rather linear correlation between those extremes, and a linear function has been proposed as

$$\varkappa_i = \varkappa_{\mathrm{pp},i} + \alpha\left(\varkappa_{\mathrm{cc},i} - \varkappa_{\mathrm{pp},i}\right) \tag{3.1}$$

where $\varkappa_i$ corresponds to the dimensionless degree of restraint $\alpha$, in the interval $0 \leq \alpha \leq 1$ where $\alpha = 0$ corresponds to the bi-pinned end conditions and $\alpha = 1$ to bi-clamped. A structure clamped at one end and pinned at the other has shown to correspond to $\alpha = 0.5$. Again, since no claim is made to the physical existence of the $\varkappa$-functions, the $\alpha$-factor is also to be seen as a practical tool of analysis rather than a true entity of the cable.

## 3.3 Attenuating error sources

One of the main concepts in attenuating error sources is to use as many overtones of the frequency as possibly. Using the modified theory proposed then yields an equation system as large as the number of overtones $i$. Since the cable force is assumed to be constant for all $i$, an error estimation of the dependent variables can be performed, provided that Eq. 1.2 is valid. One problem is still to be solved, namely the error sources in the measured signal. Sensitivity analysis has shown that a small deviation in measured frequency results in a rather large error in estimated tensile force. These errors may vary in a nonlinear way, regarding the overtone $i$. The reason is that for some modes the gauge is found to be near an inflection point, *i.e.* a point of zero motion. A signal from such a gauge will result in much larger error than a gauge at maximum deflection of the cable. To account for this, several gauges has been mounted on the structure, at carefully chosen positions. In order to further improve the accuracy of the results, a least square fit of the measured response has been proposed. This method may to some extent appear improper, since it more or less results in altering the measured response, regarding the part of the signal not to fit as ambient errors. It is probably more scientific to present the results within an interval based on estimated error sources rather than as a definite value. The trend-fitted result can however be shown to lie within the former interval. To justify the trend-fitting, one can rearrange Eq. 1.2 finding that the ratio $f_i/\varkappa_i$ should be linear proportional to overtone $i$. An iterative process involving the $\varkappa$–function is carried out, until the equation system yields a constant tensile force $S$. Numerical simulations introducing error sources in the frequency domain for a known tensile force, has shown to produce rather accurate results, using the above procedure. It should nevertheless be stressed that the scientific treatment of such trend-fitting may not be generally applied.

## 3.4 Application and validity of the proposed methods

The proposed methods were initially rather tailor-made to suite the actual problem at hand, namely to estimate the tensile force of a given structure. The main application were cases when the boundary conditions in some sense could be defined and had large influence of the results, but not readily evaluated using current theories. In the present study, measurements on different cable structures were carried out, and relating them to each other has shown good agreement with numerical models of the entire bridge.

The proposed methods of analysis have also been applied to other real case objects, reported in [8]. In that case, the cable force was monitored directly using a built-in measurement system of the bridge. The results showed good agreement with the force directly measured.

# 4 Conclusions

The problem stated in this paper has been to determine the tensile force in cable structures with sufficient accuracy, with its main applications on large bridges. The main problem in general is that the values of the dependent parameters often is difficult to accurate determine. Beyond that, as the influence of the flexural rigidity increases, the effects of boundary conditions increase. Using conventional tools of analysis has shown to give insufficient accuracy in such cases. The proposed modification of these theories to properly account for the boundary conditions has shown successful in producing accurate results in the cases studied, [1, 2, 8]. Slightly similar approaches has been proposed and applied by [4, 5, 6] mainly focusing on the bi-clamped case. They estimate the error in their analysis to approximately ±1 %, which is in the same range as the results using the methods proposed in the present study.

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