Solution to problem 4

4. Is $h(x) = |x| + \max\{e^x, 10 + 37x + x^6\}$ a convex function on $\mathbb{R}$?

Solution:

The real line is a convex set.

By the triangle inequality

$$|tx + (1-t)y| \le t|x| + (1-t)|y|$$

for all $x, y \in \mathbb{R}$ and $t \in (0,1)$, so $|x|$ is convex. Since the sum of two convex functions is convex (Problem 1) and

$$\begin{aligned} (\exp(x))'' &= \exp(x) > 0, && \forall x \in \mathbb{R}, \\ \left(10 + 37x + x^6\right)'' &= 30x^4 \ge 0, && \forall x \in \mathbb{R}, \end{aligned}$$

we need only to show that the maximum of two convex functions is convex. Let $f, g : \mathbb{R} \to \mathbb{R}$ be convex functions. Then for $x, y \in \mathbb{R}$ and $t \in (0,1)$

$$\begin{aligned} e(tx + (1-t)y) &:= \max\{f(tx + (1-t)y), g(tx + (1-t)y)\} \\ &\le \max\{tf(x) + (1-t)f(y), tg(x) + (1-t)g(y)\} \\ &\le t\max\{f(x), g(x)\} + (1-t)\max\{f(y), g(y)\} \\ &= te(x) + (1-t)e(y). \end{aligned} \tag{1}$$

Hence h is convex.

Solution to problem 6

Let for some $n \in \mathbb{N}$

$$f(x) = \sum_{j=1}^{n} (x_j^4 - x_j^3 + x_j^2 - x_j)$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.
a) Show that f is a convex function.
b) You want to minimize f. Make one iteration of Newton's algorithm and start in $x^{(0)} = (1, \ldots, 1)$.

Solution:

The Hessian of f is

$$F(x) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2}(x) & & 0 \\ & \ddots & \\ 0 & & \dfrac{\partial^2 f}{\partial x_n^2}(x) \end{bmatrix}$$

where

$$\frac{\partial^2 f}{\partial x_i^2}(x) = 12x_i^2 - 6x_i + 2 = 12\left((x_i - 0.25)^2 + 1.25\right).$$

Since $\frac{\partial^2 f}{\partial x_i^2}(x) > 0$ for all $x \in \mathbb{R}^n$, $F(x)$ is positive definite and hence f is (strictly) convex on $\mathbb{R}^n$.

To make a step in the Newton method, we need to calculate then gradient,

$$\nabla f(x) = (4x_1^3 - 3x_1^2 + 2x_1 - 1, \ldots, 4x_n^3 - 3x_n^2 + 2x_n - 1).$$

In $x^{(0)}$, we have

$$\nabla f(x^{(0)}) = (2, \ldots, 2) \quad F(x^{(0)}) = 8I_{n\times n}.$$

From the solution of a), we know that F is positive definite. The Newton direction is therefore given by

$$d^{(0)} = -F(x^{(0)})^{-1}\nabla f(x^{(0)})^T = -\frac{1}{8}(2, \ldots, 2)^T$$

and we try to update with $t_0 = 1$:

$$x^{(1)} = x^{(0)} + t_0 d^{(0)} = \frac{3}{4}(1, \ldots, 1)^T.$$

The function value decreases,

$$f(x^{(0)}) = \sum_{i=1}^{n}(1 - 1 + 1 - 1) = 0,$$
$$f(x^{(1)}) = \sum_{i=1}^{n}\sum_{j=1}^{4}\left(-\frac{3}{4}\right)^j < 0,$$

so we accept $t_0 = 1$ and the iteration is done.