Examples of quadratic functions

Consider the unconstrained quadratic program

$$(QP) \begin{cases} \text{minimize} & f(x) = \frac{1}{2}x^T Hx + c^T x, \\ \text{subject to} & x \in \mathbb{R}^n. \end{cases}$$

where H is a symmetric matrix. Note that $\nabla f(x) = 0$ if and only if $Hx + c = 0$ and $\nabla^2 f(x) = H$. There are four possible scenarios for (QP).

i) If H is positive definite there exists a *unique globally optimal* solution $\hat{x}$ to (QP) and it is given by $\hat{x} = -H^{-1}c$. A proof that H is invertible can be constructed using the LDL^T-factorization.

ii) If H is positive semidefinite and $-c \in \mathcal{R}(H)$ then exists *non-unique globally optimal* solutions $\hat{x}$ to (QP). All possible $\hat{x}$ are given by $H\hat{x} + c = 0$.

iii) If H is positive semidefinite and $-c \notin \mathcal{R}(H)$ then there exists *no globally optimal* solution to (QP).

iv) If H is indefinite there exists *no globally optimal* solution to (QP).

The four scenarios will now be showcased with the help of an example. Let

$$H = \begin{bmatrix} 1 & 2 \\ 2 & \alpha \end{bmatrix}.$$

$\alpha = 6$ and $c = [1\ 2]^T$

H is positive definite since $1 > 0$, $6 > 0$ and $1 \cdot 6 - 2 \cdot 2 = 2 > 0$. This is an example of scenario *i*), visualized in Figure 1. The unique globally optimal solution is $\hat{x} = [-1\ 0]^T$.

$\alpha = 4$ and $c = [1\ 2]^T$

H is positive semidefinite since $1 \geq 0$, $4 \geq 0$ and $1 \cdot 4 - 2 \cdot 2 = 0 \geq 0$. Since

$$H \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -c,$$

we have $-c \in \mathcal{R}(H)$. This is an example of scenario *ii*), visualized in Figure 2. There exists infinitely many globally optimal solutions,

$$\hat{x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} t$$

for $t \in \mathbb{R}$.

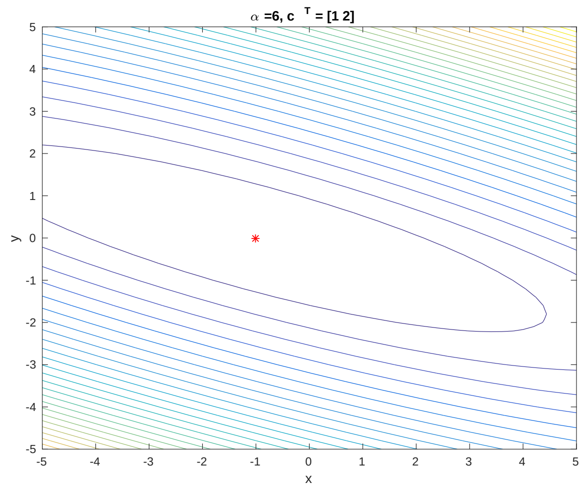

Figure 1: Contour-plot for $f(x)$ together with the optimum $\hat{x}$.

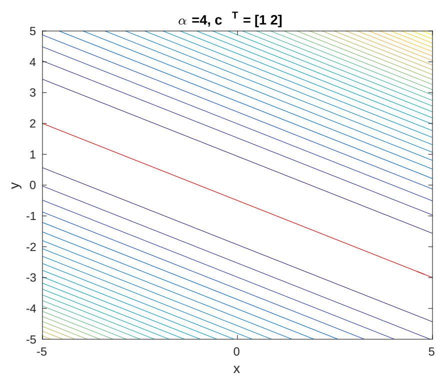

Figure 2: Contour-plot for $f(x)$ together with all the optima $\hat{x}$.

$\alpha = 4$ and $c = [1\ \ 0]^T$

H is positive definite but $-c \notin \mathcal{R}(H)$ so there exists no globally optimal solution. This is an example of scenarion iii), visualized in Figure 3.

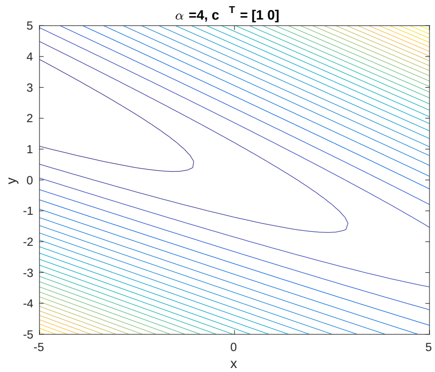

Figure 3: Contour-plot for $f(x)$.

$\alpha = 2$ and $c = [1\ 2]^T$

H is indefinite since $1 > 0$, $2 > 0$ and $1 \cdot 2 - 2 \cdot 2 = -2 < 0$ and there exists no globally optimal solution. This is an example of scenario iv), visualized in Figure 4. Note that $[-1\ 0]^T$ is a saddle point, it satisfies $\nabla f(x) = 0$, and is a locally optimal solution.

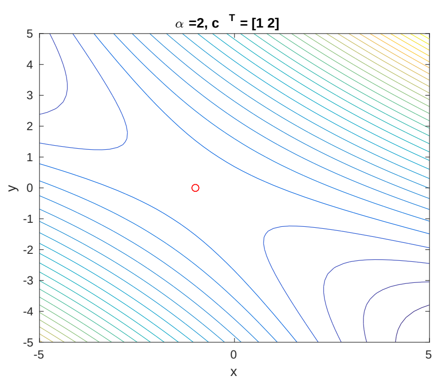

Figure 4: Contour-plot for $f(x)$ together with all the saddle point $[-1\ 0]^T$.