H symmetric $\Rightarrow$ ran$(H) \perp$ ker(H)

Setup
Let x and y be vectors in $\mathbb{R}^n$ and let H be a matrix in $\mathbb{R}^{n\times n}$. We say that x is in the set $\ker(H)$ if $Hx = 0$ and we say that y is in the set $\operatorname{ran}(H)$ if there exists a vector z in $\mathbb{R}^n$ such that $Hz = y$. We say that H is symmetric if $H = H^{\mathrm{T}}$.

The following result is a special case of Theorem 25.1 in the course litterature.

Proposition 1 *If H is a symmetric matrix then ran$(H) \perp$ ker(H).*

Proof Let $x \in \ker(H)$ and let $y \in \operatorname{ran}(H)$. Then there exists a vector z such that $Hz = y$ and

$$x^{\mathrm{T}}y = x^{\mathrm{T}}Hz.$$

Since H is symmetric

$$x^{\mathrm{T}}Hz = x^{\mathrm{T}}H^{\mathrm{T}}z = (Hx)^{\mathrm{T}}z.$$

But $x \in \ker(H)$ so $Hx = 0$. Thus $x^{\mathrm{T}}y = 0$ which implies that x and y are orthogonal to each other.
Since x was arbitrary in $\ker(H)$ and y was arbitrary in $\operatorname{ran}(H)$ the result holds for all vectors in these sets and we conclude that $\ker(H) \perp \operatorname{ran}(H)$.

□