Use the Simplex-method to solve (phase-1)!

Let $\tilde{x} = (x_1\ x_2\ x_3\ x_4\ x_5\ v_1\ v_2)^T$

$\tilde{x}_1 \ \dots \ \tilde{x}_6\ \tilde{x}_7$

Then

(phase-1) $\quad \min \tilde{c}^T \tilde{x}$

s.t. $\tilde{A}\tilde{x} = b$

$\tilde{x} \geq 0$

$$\tilde{A} = \begin{pmatrix} 1 & 2 & -1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & -1 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad \tilde{c} = (0\,0\,0\,0\,0\,1\,1)^T$$

Iteration 1 -(phase-1) $\quad \beta = (6,7) \quad \nu = (1,2,3,4,5)$

$$\tilde{A}_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \tilde{A}_\nu = \begin{pmatrix} 1 & 2 & -1 & -1 & 0 \\ 1 & -1 & 2 & 0 & -1 \end{pmatrix} = A$$

$$\tilde{c}_\beta = (1\ 1)^T \quad \tilde{c}_\nu = (0\ 0\ 0\ 0\ 0)^T$$

- $\tilde{A}_\beta \bar{b} = b \quad \Rightarrow \quad \bar{b} = b = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \geq 0$ ok!

- $\tilde{A}_\beta^T y = \tilde{c}_\beta \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} y = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad y = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- $r_\nu = \tilde{c}_\nu - \tilde{A}_\nu^T y$

$$\Rightarrow r_\nu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -1 & 2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \not\geq 0$$

$\nu_q = \nu_1 = 1 \quad \Rightarrow$ **$\tilde{x}_1$ will enter the basis** $\quad (x_1)$

- $\tilde{A}_\beta \bar{a}_1 = \tilde{a}_1 \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bar{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \bar{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- $t_{max} = \min\left\{ \frac{5}{1}, \frac{2}{1} \right\} = 2 \qquad \beta_p = \beta_2 = 7$

$\Rightarrow$ **$\tilde{x}_7$ will leave the basis** $\quad (v_2)$

Iteration 2 - (phase-1)

$\beta = (1,6)$ $\quad \nu = (2,3,4,5,7)$

$$\tilde{A}_\beta = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \tilde{A}_\nu = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 & 1 \end{pmatrix}$$

$$\tilde{c}_\beta = (0 \ 1)^T \quad \tilde{c}_\nu = (0\ 0\ 0\ 0\ 1)^T$$

- $\tilde{A}_\beta \bar{b} = b \quad \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \bar{b} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \Rightarrow \bar{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$
- $\tilde{A}^T y = \tilde{c}_\beta \quad \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} y = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow y = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
- $r_\nu = \tilde{c}_\nu - \tilde{A}_\nu^T y$

$$\Rightarrow r_\nu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ -1 & 2 \\ -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 1 \\ -1 \\ 2 \end{pmatrix} \ngeq 0$$

$\nu_q = \nu_1 = 2 \quad \Rightarrow$ **$\tilde{x}_2$ will enter the basis** $\quad (x_2)$

- $\tilde{A}_\beta \bar{a}_2 = \tilde{a}_2$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \bar{a}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \Rightarrow \bar{a}_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

- $t_{max} = \min_i \left\{ \frac{\bar{b}_i}{\bar{a}_{i,2}} : \bar{a}_{i,2} > 0 \right\} = \min\left\{ \times , \frac{3}{3} \right\} = 1$

$\beta_p = \beta_2 = 6 \quad \Rightarrow$ **$\tilde{x}_6$ will leave the basis** $\quad (v_1)$

Iteration 3 - (phase-1)

$\beta = (1,2)$ $\quad \nu = (3,4,5,6,7)$

$$\tilde{A}_\beta = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \quad \tilde{A}_\nu = \begin{pmatrix} -1 & -1 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 & 1 \end{pmatrix}$$

$$\tilde{c}_\beta = (0 \ 0)^T \quad \tilde{c}_\nu = (0\ 0\ 0\ 1\ 1)^T$$

- $\tilde{A}_\beta \bar{b} = b \quad \Rightarrow \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \bar{b} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \Rightarrow \bar{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$
- $\tilde{A}_\beta^T y = \tilde{c}_\beta \quad \Rightarrow \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} y = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
- $r_\nu = \tilde{c}_\nu - \tilde{A}_\nu^T y$

$$\Rightarrow r_\nu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \tilde{A}_\nu^T \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \geq 0$$

$\therefore$ We have **$r_\nu \geq 0$** $\Rightarrow \tilde{x} = (3\ 1\ 0\ 0\ 0\ 0\ 0)^T$ optimal solution to (phase-1)

Also $\tilde{c}^T\tilde{x} = (0\ 0\ 0\ 0\ 0\ 1\ 1)\begin{pmatrix}3\\1\\0\\0\\0\\0\\0\end{pmatrix} = 0$ optimal value zero!

$\Rightarrow$ we can use the basis from the optimal sol. of (phase-I) as our initial bfs to solve (LP) with Simplex

$\tilde{x} = (\tilde{x}_1\ \tilde{x}_2\ \tilde{x}_3\ \tilde{x}_4\ \tilde{x}_5\ \tilde{x}_6\ \tilde{x}_7)$

$\underbrace{x_1\ \ x_2}_{\beta}\ \underbrace{x_3\ \ x_4\ \ x_5}_{\nu}\ \ v_1\ \ v_2$

Solve (LP) with Simplex!

$$\min\ c^Tx \quad \text{s.t. } Ax = b,\ x \geq 0$$

$$A = \begin{pmatrix}1 & 2 & -1 & -1 & 0\\ 1 & -1 & 2 & 0 & -1\end{pmatrix} \qquad b = \begin{pmatrix}5\\2\end{pmatrix}$$

$$c = (6\ 2\ 3\ 0\ 0)^T$$

Iteration 1 $\beta = (1,2)$ $\nu = (3,4,5)$

$$A_\beta = \begin{pmatrix}1 & 2\\ 1 & -1\end{pmatrix} \qquad A_\nu = \begin{pmatrix}-1 & -1 & 0\\ 2 & 0 & -1\end{pmatrix}$$

$$c_\beta = (6\ \ 2)^T \qquad c_\nu = (3\ \ 0\ \ 0)^T$$

- $A_\beta \bar{b} = b \quad \Rightarrow \begin{pmatrix}1 & 2\\ 1 & -1\end{pmatrix}\bar{b} = \begin{pmatrix}5\\2\end{pmatrix} \quad \Rightarrow \bar{b} = \begin{pmatrix}3\\1\end{pmatrix}$

- $\tilde{A}_\beta^T y = c_\beta \quad \Rightarrow \begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix} y = \begin{pmatrix}6\\2\end{pmatrix} \quad \Rightarrow y = \begin{pmatrix}8/3\\10/3\end{pmatrix}$

- $r_\nu = c_\nu - A_\nu^T y \quad \Rightarrow r_\nu = \begin{pmatrix}3\\0\\0\end{pmatrix} - \begin{pmatrix}-1 & 2\\ -1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}8/3\\10/3\end{pmatrix} = \begin{pmatrix}-1\\8/3\\10/3\end{pmatrix} \not\geq 0$

$\therefore$ this basis is not optimal for (LP)! Keep going!

$r_q = r_1 = 3 \quad \Rightarrow$ **x_3 enters the basis**

- $A_\beta \bar{a}_3 = a_3 \quad \Rightarrow \begin{pmatrix}1 & 2\\ 1 & -1\end{pmatrix}\bar{a}_3 = \begin{pmatrix}-1\\2\end{pmatrix} \quad \Rightarrow \bar{a}_3 = \begin{pmatrix}1\\-1\end{pmatrix}$

- $t_{max} = \min\left\{\frac{3}{1}, \cancel{\frac{1}{-1}}\right\} = 3$, $\beta_p = \beta_1 = 1 \quad \Rightarrow$ **x_1 leaves the basis**

Iteration 2

$\beta = (2,3)$ $\quad\nu = (1,4,5)$

$$A_\beta = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad A_\nu = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$c_\beta = (2 \;\; 3)^T \qquad c_\nu = (6 \;\; 0 \;\; 0)^T$$

- $A_\beta \bar{b} = b \implies \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \bar{b} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \implies \bar{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

- $A_\beta^T y = c_\beta \implies \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} y = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \implies y = \begin{pmatrix} 7/3 \\ 8/3 \end{pmatrix}$

- $r_\nu = c_\nu - A_\nu^T y \implies r_\nu = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 7/3 \\ 8/3 \end{pmatrix} = \begin{pmatrix} 1 \\ 7/3 \\ 8/3 \end{pmatrix} \geq 0$

$\implies x = (0 \;\; 4 \;\; 3 \;\; 0 \;\; 0)^T$ is optimal to (LP)