On the mean-field type approach to crowd dynamics: the behavior of pedestrians near walls

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(Based on joint work with Boualem Djehiche (KTH))
Pedestrian crowds in confined domains
Experimental results show that average pedestrian speed in a cross-section of a corridor can be **higher in the center than near the walls**\(^2\), but also **higher near the walls**\(^3\), depending on the circumstances (congestion, etc).

![Graph showing speeds as function of lateral position](image)

**Fig. 5.** Speeds as function of the lateral position in a cross-section upstream of the bottleneck during congestion.

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\(^3\)Francesco Zanlungo, Tetsushi Ikeda, and Takayuki Kanda. “A microscopic social norm model to obtain realistic macroscopic velocity and density pedestrian distributions”. In: *PloS one 7.12* (2012), e50720.
## Treatment of walls in pedestrian crowd models

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Neumann/no-flux boundary conditions on the pedestrian density correspond to *reflection*. 
Pedestrian crowds in confined domains: the mean-field approach

Disutility

C Dogbé. “Modeling crowd dynamics by the mean-field limit approach”. In: Mathematical and Computer Modelling 52.9-10 (2010), pp. 1506–1520

Neumann/No-flux
A Lachapelle and M-T Wolfram. “On a mean field game approach modeling congestion and aversion in pedestrian crowds”. In: Transportation research part B: methodological 45.10 (2011), pp. 1572–1589

M Burger et al. “On a mean field game optimal control approach modeling fast exit scenarios in human crowds”. In: Decision and Control (CDC), 2013 IEEE 52nd Annual Conference on. IEEE. 2013, pp. 3128–3133

M Burger et al. “Mean field games with nonlinear mobilities in pedestrian dynamics”. In: Discrete and Continuous Dynamical Systems-Series B (2014)

M Cirant. “Multi-population mean field games systems with Neumann boundary conditions”. In: Journal de Mathématiques Pures et Appliquées 103.5 (2015), pp. 1294–1315

Y Achdou, M Bardi, and M Cirant. “Mean field games models of segregation”. In: Mathematical Models and Methods in Applied Sciences 27.01 (2017), pp. 75–113
In this talk we will introduce sticky reflected SDEs of mean-field type with boundary diffusion as an alternative approach to wall modeling in the mean-field approach to crowd dynamics.

Outline

1. Sticky reflected SDEs of mean-field type with boundary diffusion
2. Weak optimal control of sticky reflected SDEs of mean-field type with boundary diffusion
3. Particle picture
4. Example: Unidirectional pedestrian flow in a tight corridor
Consider the SDE system

\[
\begin{align*}
    dX_t &= \frac{1}{2} d\ell^0_t(X) + 1_{\{x_t > 0\}} dB_t, \quad X_0 = x_0, \\
    1_{\{x_t = 0\}} dt &= \frac{1}{2\gamma} d\ell^0_t(X),
\end{align*}
\]

where

\begin{itemize}
    \item $x_0 \in \mathbb{R}_+$,
    \item $\gamma \in (0, \infty)$ is a given constant,
    \item $\ell_0(X)$ is the local time of $X$ at 0,
    \item $B$ is a standard Brownian motion.
\end{itemize}

Engelberg and Peskir (2014)\textsuperscript{2}:

System (1) has no strong solution but a unique weak solution, called a reflected Brownian motion $X$ in $\mathbb{R}_+$ sticky at 0.

Grothaus and Vosshall (2017)² extend the result to a bounded domain $\mathcal{D} \subset \mathbb{R}^d$ with sticky $C^2$-smooth boundary $\partial \mathcal{D}$.

To write down the sticky reflected SDE with boundary diffusion system, let

- $n(x)$ be the outward normal of $\partial \mathcal{D}$ at $x$,
- $\pi(x) := E - n(x)(n(x))^*$, the orthogonal projection on the tangent space of $\partial \mathcal{D}$ at $x$,
- $\kappa(x) := (\pi(x) \nabla) \cdot n(x)$, the mean curvature of $\partial \mathcal{D}$ at $x$.

These quantities are uniformly bounded over $\partial \mathcal{D}$.

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Furthermore, let

- \( \Omega := C([0, T]; \mathbb{R}^d) \) be path space,
- \( \mathcal{F} \) the Borel \( \sigma \)-field over \( \Omega \),
- \( X_t(\omega) = \omega(t) \) the coordinate process,
- \( \mathcal{F} \) the \( m \in \mathcal{P}(\Omega) \)-completed filtration generated by \( X \).

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Furthermore, let

- $\Omega := C([0, T]; \mathbb{R}^d)$ be path space,
- $\mathcal{F}$ the Borel $\sigma$-field over $\Omega$,
- $X_t(\omega) = \omega(t)$ the coordinate process,
- $\mathcal{F}$ the $m \in \mathcal{P}(\Omega)$-completed filtration generated by $X$.

There exists a unique probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ under which

\[
\begin{align*}
  dX_t &= 1_D(X_t)dB_t + 1_{\partial D}(X_t) \left( dB_{t}^{\partial D} - \frac{1}{2\gamma} n(X_t)dt \right), \\
  dB_{t}^{\partial D} &= \pi(X_t) \circ dB_t = -\frac{1}{2} \kappa(X_t) n(X_t)dt + \pi(X_t)dB_t,
\end{align*}
\]

$B$ standard Brownian motion in $\mathbb{R}^d$, $X_0 = x_0 \in \bar{D}$, $\gamma > 0$,

and $X$ is $C([0, T]; \bar{D})$-valued $\mathbb{P}$-a.s. (in particular, $X$ is $\mathbb{P}$-a.s. uniformly bounded).\(^2\)

Sticky reflected SDEs of mean-field type with boundary diffusion

\[ dX_t = (1_D(X_t) + 1_{\partial D}(X_t)\pi(X_t)) dB_t - 1_{\partial D}(X_t) \frac{1}{2} \left( \kappa(X_t) + \frac{1}{\gamma} \right) n(X_t) dt \]

The sticky reflected SDE with boundary diffusion is composed of

- **interior diffusion** \( 1_D(X_t) dB_t \),
- **boundary diffusion** \( 1_{\partial D}(X_t) dB_{\partial D} \)
- **normal sticky reflection** \(-1_{\partial D}(X_t) \frac{1}{2\gamma} n(X_t) dt\)

From now on, we abbreviate

\[ dX_t =: \sigma(X_t) dB_t + a(X_t) dt. \]

\[ \sigma(X_t) := 1_D(X_t) + 1_{\partial D}(X_t)\pi(X_t), \quad a(X_t) := -1_{\partial D}(X_t) \frac{1}{2} \left( \kappa(X_t) + \frac{1}{\gamma} \right) n(X_t). \]

are bounded.
The stickiness level $\gamma$

$\gamma$ represents the level of stickiness of $\partial D$.

Let

- $\lambda$ be the Lebesgue measure on $\mathbb{R}^d$,
- $s$ be the surface measure on $\partial D$,
- $\rho := 1_D \alpha \lambda + 1_{\partial D} \alpha' s$, $\alpha, \alpha' \in \mathbb{R}$.

Choosing

$$\alpha = \bar{\alpha} / \lambda(D), \quad \alpha' = (1 - \bar{\alpha}) / s(\partial D), \quad \bar{\alpha} \in [0, 1],$$

$\rho$ becomes a probability measure on $\mathbb{R}^d$ with full support on $\bar{D}$.

The measure $\rho$ is in fact the invariant distribution of $X_t$ whenever

$$\frac{1}{\gamma} = \frac{\bar{\alpha}}{(1 - \bar{\alpha})} \frac{s(\partial D)}{\lambda(D)}.$$

$\bar{\alpha} \to 1$ as $\gamma \to 0$, and the invariant distribution $\rho$ concentrates on $D$

$\bar{\alpha} \to 0$ as $\gamma \to \infty$, and the invariant distribution $\rho$ concentrates on $\partial D$
Sticky reflected SDEs of mean-field type with boundary diffusion

Interaction and control is introduced via Girsanov transformation (Dominated case).

Let

- \(|x|_t := \sup_{0 \leq s \leq t} |x_s|, 0 \leq t \leq T,\)
- \(U \subset \mathbb{R}^d\) be compact and \(\mathcal{U} =: \{u : [0, T] \times \Omega \to U \mid u \text{ F-prog.meas.}\},\)
- \(Q(t) := Q \circ X^{-1}_t\) denote the \(t\)-marginal distribution of \(X\) under \(Q \in \mathcal{P}(\Omega),\)
- \(\beta : [0, T] \times \Omega \times \mathcal{P}(\mathbb{R}^d) \times U \to \mathbb{R}^d\) be a measurable function such that

| | (A) \((\beta(t, X, Q(t), u_t))_{t \leq T}\) is F-prog.meas. for every \(Q \in \mathcal{P}(\Omega)\) and \(u \in \mathcal{U}.\) |
| (B) For every \(t \in [0, T], \omega \in \Omega, u \in U,\) and \(\mu \in \mathcal{P}(\mathbb{R}^d),\) |
| \(|\beta(t, x, \mu, u)| \leq C \left(1 + |x|_T + \int_{\mathbb{R}^d} |y| \mu(dy)\right),\) |
| (C) For every \(t \in [0, T], \omega \in \Omega, u \in U,\) and \(\mu, \mu' \in \mathcal{P}(\mathbb{R}^d),\) |
| \(|\beta(t, \omega, \mu, u) - \beta(t, \omega, \mu', u)| \leq C \cdot d_{TV}(\mu, \mu')\) |
Given $Q \in \mathcal{P}(\Omega)$ and $u \in \mathcal{U}$, let

$$L_{t}^{u,Q} := \mathcal{E}_{t} \left( \int_{0}^{t} \beta(s, X, Q(s), u_{s}) dB_{s} \right).$$

**Lemma 1**

The positive measure $\mathbb{P}^{u,Q}$ defined by $d\mathbb{P}^{u,Q} = L_{t}^{u,Q} d\mathbb{P}$ on $\mathcal{F}_{t}$, for all $t \in [0, T]$, is a probability measure on $\Omega$. Moreover, under $\mathbb{P}^{u,Q}$ the coordinate process satisfies

$$X_{t} = x_{0} + \int_{0}^{t} \left( \sigma(X_{s}) \beta(s, X, Q(s), u_{s}) + a(X_{s}) \right) ds + \int_{0}^{t} \sigma(X_{s}) dB_{s}^{u,Q},$$

where $B_{s}^{u,Q}$ is a standard $\mathbb{P}^{u,Q}$-Brownian motion.
Step 1. If \( \varphi \) is a process such that \( \mathbb{P}^{\varphi} \), defined by \( d\mathbb{P}^{\varphi} = L_t^{\varphi}d\mathbb{P} \) on \( \mathcal{F}_T \) where 
\[ L_t^{\varphi} := \mathcal{E}_t(\int_0^t \varphi_s dB_s), \]
is a probability measure on \( \Omega \), the coordinate process under \( \mathbb{P}^{\varphi} \) satisfies

\[ dX_t = (\sigma(X_t)\varphi_t + a(X_t)) \, dt + \sigma(X_t)dB_t^{\varphi}, \]

where \( B^{\varphi} \) is a \( \mathbb{P}^{\varphi} \)-Brownian motion. Smoothness of \( \partial \mathcal{D} \) together with Burkholder-Davis-Gundy's inequality yields

\[
E^{\varphi}[|X|^p_T] \leq CE^{\varphi} \left[ |x_0|^p + \int_0^T |\sigma(X_s)\varphi_s + a(X_s)|^p \, ds + \left| \int_0^T \sigma(X_s)dB_s^{\varphi} \right|^p_T \right]
\]

\[
\leq C \left( 1 + \int_0^T E^{\varphi}[|\varphi_s|^p] \, ds \right),
\]

where \( E^{\varphi} \) denotes expectation taken under \( \mathbb{P}^{\varphi} \).
Proof of Lemma 1

Step 2. Consider the measure $\mathbb{P}^u_n, Q$ given (on $\mathcal{F}_t$) by

$$
d\mathbb{P}^u_n, Q = \mathcal{E}_t \left( \int_0^T \beta(s, X, Q(s), u_s) 1\{ |X| \leq n \} dB_s \right) d\mathbb{P}.
$$

Use TV-distance to show that $\mathbb{P}^u_n, Q \in \mathcal{P}(\Omega)$. By Step 1, (B), and (C),

$$
E_n^{u, Q}[|X|^p_T] \leq C \left( 1 + \int_0^T E_n^{u, Q}[|\beta(s, X, Q(s), u_s)|^p] \, ds \right)
\leq C \left( 1 + d_{TV}(Q(s), \mathbb{P}(s))^p + \int_0^T E_n^{u, Q}[|\beta(s, X, \mathbb{P}(s), u_s)|^p] \, ds \right)
\leq C \left( 1 + \int_0^T E_n^{u, Q}[C \left( 1 + |X|^p_s + E^\mathbb{P}[|X|^p_s] \right)] \, ds \right)
\leq C \left( 1 + \int_0^T E_n^{u, Q}[|X|^p_s] \, ds \right).
$$

By Gronwall’s inequality $E_n^{u, Q}[|X|^p_T] \leq C_p$, where $C_p$ depends only on $p$, $T$, the Lipschitz and linear growth constant of $\beta$, and $|x_0|^p$. 
Proof of Lemma 1

Step 3. By the same lines as the proof of Proposition (A.1) in El-Karoui & Hamadène (2003)\(^2\) (see also Benes (1971)\(^3\)), the likelihood \(L^{u,Q}\) is a martingale for every \(Q \in \mathcal{P}(\Omega)\) and \(u \in \mathcal{U}\), hence \(\mathbb{P}^{u,Q} \in \mathcal{P}(\Omega)\).

Step 4. By Girsanov’s theorem the coordinate process under \(\mathbb{P}^{u,Q}\) satisfies

\[
X_t = x_0 + \int_0^t \left( \sigma(X_s)\beta(s, X, Q(s), u_s) + a(X_s) \right) ds + \int_0^t \sigma(X_s)dB^{Q}_s.
\]

\(\blacksquare\)

---


For a given $u \in \mathcal{U}$, consider the map
\[ \Phi : \mathcal{P}(\Omega) \ni Q \mapsto \mathbb{P}^u, Q \in \mathcal{P}(\Omega). \]

**Proposition 1**

The map $\Phi$ is well-defined and admits a unique fixed point. Moreover, for every $p \geq 2$, the fixed point, denoted $\mathbb{P}^u$, belongs to $\mathcal{P}_p(\Omega)$, i.e.
\[ E^u [\|X\|^p_T] \leq C_p < \infty, \]
where the constant $C_p$ depends only on $p$, $T$, the Lipschitz and the linear-growth constant of $\beta$, and $|x_0|^p$. 
**Step 1.** By Lemma 1, the map is well-defined.

**Step 2.** Given $Q, \tilde{Q} \in \mathcal{P}(\Omega)$, by Csiszár-Kullback-Pinsker’s inequality and the fact that $\int_0^T (dB_s - \beta_s^Q ds)$ is a martingale under $\Phi(Q)$,

$$D_T^2(\Phi(Q), \Phi(\tilde{Q})) \leq E^{\Phi(Q)} \left[ \log \left( \frac{L_T^Q}{L_T^{\tilde{Q}}} \right) \right]$$

$$= E^{\Phi(Q)} \left[ \int_0^T (\beta_s^Q - \beta_s^{\tilde{Q}}) dB_s - \frac{1}{2} \int_0^T (\beta_s^Q)^2 - (\beta_s^{\tilde{Q}})^2 ds \right]$$

$$= E^{\Phi(Q)} \left[ \int_0^T (\beta_s^Q - \beta_s^{\tilde{Q}}) \beta_s^Q - \frac{1}{2} (\beta_s^Q)^2 + \frac{1}{2} (\beta_s^{\tilde{Q}})^2 ds \right]$$

$$= \frac{1}{2} \int_0^T E^{\Phi(Q)} \left[ (\beta_s^Q - \beta_s^{\tilde{Q}})^2 \right] ds$$

$$\leq C \int_0^T d_{TV}^2(Q(s), \tilde{Q}(s)) ds \leq C \int_0^T D_s^2(Q, \tilde{Q}) ds.$$
Proof of Proposition 1

Step 3. Iterating the inequality, we obtain for every $N \in \mathbb{N}$,

$$D_T^2(\Phi^N(Q), \Phi^N(\tilde{Q})) \leq \frac{C^N T^N}{N!} D_T^2(Q, \tilde{Q}),$$

where $\Phi^N$ denotes the $N$-fold composition of $\Phi$. Hence $\Phi^N$ is a contraction for $N$ large enough, thus admitting a unique fixed point.
**Step 3.** Iterating the inequality, we obtain for every $N \in \mathbb{N}$,

$$D^2_T(\Phi^N(Q), \Phi^N(\tilde{Q})) \leq C^N T^N N! D^2_T(Q, \tilde{Q}),$$

where $\Phi^N$ denotes the $N$-fold composition of $\Phi$. Hence $\Phi^N$ is a contraction for $N$ large enough, thus admitting a unique fixed point.

**Step 4.** Under $\mathbb{P}^u$, the fixed point of $\Phi$ given $u \in \mathcal{U}$, the coordinate process satisfies

$$dX_t = (\sigma(X_t) \beta(t, X_t, \mathbb{P}^u(t), u_t) + a(X_t)) \, dt + \sigma(X_t) dB_t^u,$$

where $B^u$ is a $\mathbb{P}^u$-Brownian motion. Following the calculations of Lemma 1, we get the estimate

$$\|\mathbb{P}^u\|^p = E^u[|X|_T^p] \leq C_p \left(1 + E^u \left[\int_0^T |X|^p_s \, ds\right]\right),$$

where $C_p$ depends only on $p$, $T$, the Lipschitz and the linear growth constant of $\beta$, and $|x_0|^p$. Gronwall’s inequality then yields $E^u[|X|_T^p] \leq C_p < \infty$. 

\[\blacksquare\]
Theorem 2

Under (A)-(C) there exists for each $u \in \mathcal{U}$ a unique weak solution $(\mathbb{P}^u)$ to the sticky reflected SDE of mean-field type with boundary diffusion

$$dX_t = \sigma(X_t)dB_t^u + \left(a(X_t) + \sigma(X_t)\beta(t, X_t, \mathbb{P}^u(t), u_t)\right)dt$$

Under $\mathbb{P}^u$ the $t$-marginal distribution of $X$ is $\mathbb{P}^u(t)$ for $t \in [0, T]$ and $X$ is almost surely $C([0, T]; \bar{D})$-valued. Furthermore, $\mathbb{P}^u \in \mathcal{P}_p(\Omega)$. 
Let
\[ f : [0, T] \times \Omega \times \mathcal{P}(\mathbb{R}^d) \times U \to \mathbb{R}, \]
\[ g : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}. \]

Consider the following finite time-horizon problem:
\[
\left\{ \begin{array}{l}
\min_{u \in U} \ J(u) = E^u \left[ \int_0^T f(t, X, \mathbb{P}^u(t), u_t) \, dt + g(X_T, \mathbb{P}^u(T)) \right] \\
\end{array} \right.
\]
Let
\[ f : [0, T] \times \Omega \times \mathcal{P}(\mathbb{R}^d) \times U \rightarrow \mathbb{R}, \]
\[ g : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \rightarrow \mathbb{R}. \]

Consider the following finite time-horizon problem:

\[
\begin{aligned}
\min_{u \in U} J(u) &= E^u \left[ \int_0^T f(t, X(t), \mathbb{P}^u(t), u(t)) dt + g(X_T, \mathbb{P}^u(T)) \right] \\
&= E \left[ \int_0^T L_t^u f(t, X(t), \mathbb{P}^u(t), u(t)) dt + L_T^u g(X_T, \mathbb{P}^u(T)) \right] \\
\text{s.t. } &dL_t^u = L_t^u \beta(t, X(t), \mathbb{P}^u(t), u(t)) dB_t, \quad L_0^u = 1,
\end{aligned}
\]

\[ X \text{ is the coordinate process,} \]

Problem (2) is a weak form mean-field type control problem. The probability space is controlled via the likelihood \( L^u \).
Additional assumptions on $\beta$, $f$, and $g$:

**D** For $\phi \in \{\beta, f\}$,

$$
\phi^u_t = \phi(t, X, E^u[r\phi(X_t)], u_t) = \phi(t, X, E[L^u_t r\phi(X_t)], u_t),
$$

and $g^u_T = g(X_T, E[L^u_T r_g(X_T)])$, where $r_\beta, r_f, r_g : \mathbb{R}^d \to \mathbb{R}^d$.

**E** The functions $(t, x, y, u) \mapsto (f, \beta)(t, x, y, u)$ and $(x, y) \mapsto g(x, y)$ are twice continuously differentiable with respect to $y$. Moreover, $\beta, f$ and $g$ and all their derivatives up to second order with respect to $y$ are continuous in $(y, u)$, and bounded.

(D)-(E) can be relaxed, current form used for the sake of technical simplicity.
In view of (A)-(E) Pontryagin’s type stochastic maximum principle is available\(^2\).

**Theorem 3**

Assume that \((\hat{u}, L\hat{u})\) is an optimal solution to the mean-field type control problem (2). Then for all \(v \in U\) and a.e. \(t \in [0, T]\) it holds \(\mathbb{P}\)-a.s. that

\[
\mathcal{H}(L_t\hat{u}, v, p_t, q_t) - \mathcal{H}(L_t\hat{u}, \hat{u}_t, p_t, q_t) + \frac{1}{2} \left[\delta(L\beta)(t)\right]^T P_t [\delta(L\beta)(t)] \leq 0,
\]

where

\[
\mathcal{H}(L_t u, u_t, p_t, q_t) := L_t u_t p_t - L_t f_t u_t,
\]

\[
\delta(L\beta)(t) := L_t\beta(t, X, E[L_t\beta_r(X_t)], v) - \beta_t\hat{u},
\]

\[
\begin{aligned}
    dp_t &= -\left(q_t\beta_t + E \left[q_t L_t \nabla y \beta_t\right] r_\beta(X_t) - f_t - E \left[L_t \nabla y f_t\right] r_f(X_t)\right) dt + q_t dB_t, \\
    p_T &= -g_T - E \left[L_T \nabla y g_T\right] r_g(X_T), \\
    dP_t &= -\left(\beta_t + E[L_t \nabla y \beta_t] r_\beta(X_t)\right)^2 P_t + 2 \left(\beta_t + E[L_t \nabla y \beta_t] r_\beta(X_t)\right) Q_t \\
    &\quad + E[q_t \nabla y \beta_t] r_\beta(X_t) - E[\nabla y f_t] r_f(X_t)\right) dt + Q_t dB_t, \\
    P_T &= 0,
\end{aligned}
\]

\(^2\)Rainer Buckdahn, Boualem Djehiche, and Juan Li. “A general stochastic maximum principle for SDEs of mean-field type”. In: *Applied Mathematics & Optimization* 64.2 (2011), pp. 197–216.
Identifying optimal controls when $U$ is convex.

Whenever $U$ is convex, the optimality condition simplifies to

$$\mathcal{H}(L_t^{\hat{u}}, v, p_t, q_t) - \mathcal{H}(L_t^{\hat{u}}, \hat{u}_t, p_t, q_t) \leq 0, \quad \forall v \in U; \ \mathbb{P}\text{-a.s., a.e.-} t \in [0, T].$$

Assume that $\hat{u}$ is optimal. A matching argument yields

$$q_t = -\nabla_x \phi(X_t, t) \sigma(X_t),$$

where $\phi(X_T, T)$ is the terminal condition for $p$,

$$\phi(X_t, t) := g(X_t, E^{\hat{u}}[r_g(X_t)]) + E^{\hat{u}} \left[ \nabla_y g(X_t, E^{\hat{u}}[r_g(X_t)]) \right] r_g(X_t),$$

and the optimality condition (variation of $\mathcal{H}$) relates $\hat{u}$ to $q$,

$$q_t \nabla_u \beta_t^{\hat{u}} = \nabla_u f_t^{\hat{u}}, \quad \mathbb{P}\text{-a.s., a.e.-} t \in [0, T].$$
Experimental results show that average pedestrian speed in a cross-section of a corridor can be higher in the center than near the walls\textsuperscript{2}, but also higher near the walls\textsuperscript{3}, depending on the circumstances (congestion, etc).

\textbf{Fig. 5.} Speeds as function of the lateral position in a cross-section upstream of the bottleneck during congestion.


\textsuperscript{3}Francesco Zanlungo, Tetsushi Ikeda, and Takayuki Kanda. “A microscopic social norm model to obtain realistic macroscopic velocity and density pedestrian distributions”. In: PloS one 7.12 (2012), e50720.
Example: Unidirectional pedestrian flow

Let $D$ be a long narrow corridor with exit $x_T$ and entrance $x_0$ in opposite ends.

\[
\min_{u \in U} \frac{1}{2} E \left[ \int_0^1 L_t^u f(t, X_t, E[L_t^u r_f(X_t)], u_t) dt + L_T^u |X_T - x_T|^2 \right],
\]

s.t. $dL_t^u = L_t^u u_t dB_t$, $L_0^u = 1$.

$f$ is a congestion-type running cost:

\[
f(t, X_t, E[L_t^u r_f(X_t)], u_t) = C(X_t) \{1 + h(t, X_t, E^u[r_f(X_t)])\} |u_t|^2,
\]

where

- $|u|^2$, $c_f > 0$, is the cost of moving in free space;
- $h|u|^2$ is the additional cost to move in congested areas;
- $C(X_t) := \xi 1_{\Gamma}(X_t) + 1_D(X_t)$, $\xi > 0$, monitors $f$ on the boundary $\partial D$.

Lower $\xi$ yields lower overall cost of moving on $\partial D$ and vice versa.
Assuming $U$ is convex, an optimal control satisfies

$$
\hat{u}_t = \frac{\sigma(X_t)(X_t - x_T)}{C(X_t)(1 + h(t, X_t, E[\hat{u}[r_f(X_t)]]))}, \quad \mathbb{P}\text{-a.s., a.e.} - t \in [0, T].
$$

$\hat{u}$ implements the following strategy:

- Move towards the exit $x_T$, but scale the speed according to the local congestion.
Example: Unidirectional pedestrian flow

\[
\hat{u}_t = \frac{\sigma(X_t)(X_t - x_T)}{C(X_t)(1 + h(t, X, E[\hat{u}[r_f(X_t)]]))}.
\]

We will compare two congestion costs

- **friendly**
  
  \[
  h = h_1 := |X_2(t) - E[\hat{u}[X_2(t)]]|
  \]

- **averse**
  
  \[
  h = h_2 := \frac{1}{|X_2(t) - E[\hat{u}[X_2(t)]]|}
  \]

In both cases,

- \( r_f((x_1, x_2)) = x_2 \)
- \( X_2(t) \) is the \( y \)-component of \( X_t \) (perpendicular to the corridor walls).
Example: Unidirectional pedestrian flow

Estimated cross-section mean speed profiles

(a) Congestion friendly \((h = h_1)\).

(b) Congestion averse \((h = h_2)\).

- Boundary movement speed is indeed monitored through \(\xi\).
Consider $N \in \mathbb{N}$ (non-transformed, independent) sticky reflected SDEs with boundary diffusion

\[
\begin{aligned}
    dX_t^i &= a(X_t^i)dt + \sigma(X_t^i)dB_t^i, \\
    X_0^i &= x_i, \quad i = 1, \ldots, N.
\end{aligned}
\]  

(3)

Grothaus and Vosshall\(^2\) (2017):

There exists a unique probability measure $\mathbb{P}^N$ on $(\Omega, \mathcal{F})$, where $\Omega := C([0, T]; \mathbb{R}^{Nd})$ and $\mathcal{F}$ is the corresponding filtration. Under $\mathbb{P}^N$, $(X^1, \ldots, X^N)$ satisfies (3) and is $C([0, T]; \overline{D}^N)$-valued $\mathbb{P}^N$-a.s.

Weak interaction and control can be introduced in the particle system\textsuperscript{2}

Given \( u := (u^1, \ldots, u^N) \in \mathcal{U}^N \), let \( \mu^N(t) := \frac{1}{N} \sum_{i=1}^{N} \delta_{X^i_t} \) and

\[
dL_{i,t}^u = L_{i,t}^u \beta(t, X^i_t, \mu^N(t), u^i_t) dB^i_t, \quad L_{i,0}^u = 1, \quad i = 1, \ldots, N.
\]

\[
L_{t}^{N,u} := \prod_{i=1}^{N} L_{i,t}^u.
\]

\( L_{t}^{N,u} \) defines a Girsanov transformation of \( \mathbb{P}^N \) to \( \mathbb{P}^{N,u} \).

Under \( \mathbb{P}^{N,u} \) the coordinate process is \( C([0, T]; \bar{D}) \)-valued a.s. and satisfies

\[
\begin{aligned}
&dX^i_t = (\sigma(X^i_t) \beta(t, X^i_t, \mu^N(t), u^i_t) + a(X^i_t))dt + \sigma(X^i_t)dB^i_{t,u}, \\
&X^i_0 = x^i_0, \quad i = 1, \ldots, N,
\end{aligned}
\]

where \( B^i_{u} \) is a \( \mathbb{P}^{N,u} \)-Brownian motion. Also, \( \mathbb{P}^{N,u} \in \mathcal{P}_p((C([0, T]; \bar{D})^N)). \)

Social cost for the particle system:

\[ J_N(u) := \frac{1}{N} \sum_{i=1}^{N} E^{N,u} \left[ \int_0^T f(t, X^i, \mu^N(t), u^i_t)dt + g(X^i_T, \mu^N(T)) \right] \]

Minimization of \( J_N(u) \) is a cooperative scenario.

Mean-field type optimal control is \( \epsilon(N) \)-optimal for the collaborative social cost minimization, where \( \epsilon(N) \to 0 \) as \( N \to \infty \). Based on results concerning convergence properties of relaxed controls.

Main references: El Karoui, Huu Nguyen and Jean-Blanc (1988)\(^2\) (controlled standard SDEs), Ökschläger (1984)\(^3\) (mean-field SDEs without control), Lacker (2017)\(^4\) (controlled mean-field SDEs).


Conclusions

- Mean-field approach to crowd dynamics
  - congestion, crowd aversion, etc.
  - decision-based modeling with anticipating agents
  - correspondence between micro- and macroscopic picture

- Sticky reflected SDEs of mean-field type with boundary diffusion
  - as an alternative to reflective boundary conditions in confined domains
  - pedestrians no longer “bounce” at the boundary
  - pedestrians may interact and take actions while spending time at the boundary
  - preserves a micro-macro correspondence for crowds in confined domains

Thank you!
Examples: Convex and compact $U$

Assume that $(\hat{u}, \hat{L})$ is an optimal solution for the mean-field type control problem. Recall the first order adjoint equation,

$$
\begin{align*}
    dp_t &= -\left(q_t \beta_t^\hat{u} + E \left[q_t L_t^\hat{u} \nabla_y \beta_t^\hat{u}\right] r\beta(X_t) \\
    &\quad - f_t^\hat{u} - E \left[L_t^\hat{u} \nabla_y f_t^\hat{u}\right] r_f(X_t)\right) dt + q_t dB_t, \\
    p_T &= -g_T^\hat{u} - E \left[L_T^\hat{u} \nabla_y g_T^\hat{u}\right] r_g(X_T).
\end{align*}
$$

Rewriting $E[L_t^\hat{u} Y_t] = E^\hat{u}[Y_t]$ and changing measure to $P^\hat{u}$,

$$
\begin{align*}
    dp_t &= -\left(E^\hat{u} \left[q_t \nabla_y \beta_t^\hat{u}\right] r\beta(X_t) - f_t^\hat{u} - E^\hat{u} \left[\nabla_y f_t^\hat{u}\right] r_f(X_t)\right) dt + q_t dB_t^\hat{u}, \\
    p_T &= -g_T^\hat{u} - E^\hat{u} \left[\nabla_y g_T^\hat{u}\right] r_g(X_T).
\end{align*}
$$
Assume that $(\hat{u}, \hat{L})$ is an optimal solution for the mean-field type control problem. Recall the first order adjoint equation,

$$
\begin{cases}
    dp_t = -\left( q_t \beta_t^{\hat{u}} + E \left[ q_t L_t^{\hat{u}} \nabla y \beta_t^{\hat{u}} \right] r \beta(X_t) \\
    \quad - f_t^{\hat{u}} - E \left[ L_t^{\hat{u}} \nabla y f_t^{\hat{u}} \right] r_f(X_t) \right) dt + q_t dB_t, \\
    p_T = -g_T^{\hat{u}} - E \left[ L_T^{\hat{u}} \nabla y g_T^{\hat{u}} \right] r_g(X_T).
\end{cases}
$$

Rewriting $E[L_t^{\hat{u}} Y_t] = E^{\hat{u}}[Y_t]$ and changing measure to $\mathbb{P}^{\hat{u}}$,

$$
\begin{cases}
    dp_t = -\left( E^{\hat{u}} \left[ q_t \nabla y \beta_t^{\hat{u}} \right] r \beta(X_t) - f_t^{\hat{u}} - E^{\hat{u}} \left[ \nabla y f_t^{\hat{u}} \right] r_f(X_t) \right) dt + q_t dB_t^{\hat{u}}, \\
    p_T = -g_T^{\hat{u}} - E^{\hat{u}} \left[ \nabla y g_T^{\hat{u}} \right] r_g(X_T).
\end{cases}
$$

Whenever $U$ is convex, the optimality condition simplifies to

$$
\mathcal{H}(\hat{L}_t, v, p_t, q_t) - \mathcal{H}(\hat{L}_t, \hat{u}_t, p_t, q_t) \leq 0, \quad \forall v \in U; \ \mathbb{P}\text{-a.s.}, \ \text{a.e.-}t \in [0, T].
$$
$p$ part of the solution to a BSDE so it is the conditional expectation

$$p_t = -E^\hat{\theta} \left[ \phi (X_T, T) \mid \mathcal{F}_t \right] + E^\hat{\theta} \left[ \int_t^T (\ldots) ds \mid \mathcal{F}_t \right], \quad (5)$$

where as before

$$\phi (X_t, t) := g \left( X_t, E^\hat{\theta} [r_g (X_t)] \right) + E^\hat{\theta} \left[ \nabla_y g \left( X_t, E^\hat{\theta} [r_g (X_t)] \right) \right] r_g (X_t).$$
Example: Convex and compact $U$

$p$ part of the solution to a BSDE so it is the conditional expectation

$$p_t = -E^{\hat{\mathcal{U}}} [\phi (X_T, T) \mid \mathcal{F}_t] + E^{\hat{\mathcal{U}}} \left[ \int_t^T (\ldots) ds \mid \mathcal{F}_t \right],$$

(5)

where as before

$$\phi (X_t, t) := g \left( X_t, E^{\hat{\mathcal{U}}} [r_g (X_t)] \right) + E^{\hat{\mathcal{U}}} \left[ \nabla_y g \left( X_t, E^{\hat{\mathcal{U}}} [r_g (X_t)] \right) \right] r_g (X_t).$$

By Dynkin’s formula,

$$E^{\hat{\mathcal{U}}} [\phi (X_T, T) \mid \mathcal{F}_t] = \phi (X_t, t) + \int_t^T E^{\hat{\mathcal{U}}} [(\ldots)(s) \mid \mathcal{F}_t] ds.$$
Example: Convex and compact $U$

$p$ part of the solution to a BSDE so it is the conditional expectation

$$p_t = -E^\hat{\mathcal{U}} [\phi (X_T, T) \mid \mathcal{F}_t] + E^\hat{\mathcal{U}} \left[ \int_t^T (\ldots) ds \mid \mathcal{F}_t \right], \quad (5)$$

where as before

$$\phi (X_t, t) := g \left( X_t, E^\hat{\mathcal{U}} [r_g (X_t)] \right) + E^\hat{\mathcal{U}} \left[ \nabla_y g \left( X_t, E^\hat{\mathcal{U}} [r_g (X_t)] \right) \right] r_g (X_t).$$

By Dynkin’s formula,

$$E^\hat{\mathcal{U}} [\phi (X_T, T) \mid \mathcal{F}_t] = \phi (X_t, t) + \int_t^T E^\hat{\mathcal{U}} [(\ldots)(s) \mid \mathcal{F}_t] ds.$$ 

Itô-differentiating $p$ from (5) and matching the diffusion coefficients yeilds

$$q_t = -\nabla_x \phi (X_t, t) \sigma (X_t).$$
Example: Convex and compact $U$

$p$ part of the solution to a BSDE so it is the conditional expectation

$$p_t = -E^\hat{u}[\phi(X_T, T) \mid \mathcal{F}_t] + E^\hat{u}\left[\int_t^T (\ldots) ds \mid \mathcal{F}_t\right],$$

(5)

where as before

$$\phi(X_t, t) := g\left(X_t, E^\hat{u}[r_g(X_t)]\right) + E^\hat{u}\left[\nabla_y g\left(X_t, E^\hat{u}[r_g(X_t)]\right)\right] r_g(X_t).$$

By Dynkin’s formula,

$$E^\hat{u}[\phi(X_T, T) \mid \mathcal{F}_t] = \phi(X_t, t) + \int_t^T E^\hat{u}[\ldots](s) \mid \mathcal{F}_t]ds.$$

Itô-differentiating $p$ from (5) and matching the diffusion coefficients yeilds

$$q_t = -\nabla_x \phi(X_t, t) \sigma(X_t).$$

The optimality condition (variation of $\mathcal{H}$) relates $\hat{u}$ to $q$,

$$q_t \nabla_u \beta_t^\hat{u} = \nabla_u f_t^\hat{u}, \quad \mathbb{P}\text{-a.s., a.e.} - t \in [0, T].$$
Consider on some admissible domain $\mathcal{D} \subset \mathbb{R}^d$ the mean-field LQ problem of minimizing final variance

$$\min_{u \in \mathcal{U}} \frac{1}{2} \mathbb{E} \left[ \int_0^T L_t^u |u_t|^2 dt + L_T^u |X_T - E[L_T^u X_T]|^2 \right],$$

subject to

$$dL_t^u = L_t^u u_t dB_t, \quad L_0^u = 1,$$
Example: Mean-field LQ (convex and compact $U$)

Consider on some admissible domain $\mathcal{D} \subset \mathbb{R}^d$ the mean-field LQ problem of minimizing final variance

$$
\begin{align*}
\min_{u \in U} \frac{1}{2} E \left[ \int_0^T L_t^u |u_t|^2 dt + L_T^u |X_T - E[L_T^u X_T]|^2 \right], \\
\text{s.t.} \quad dL_t^u = L_t^u u_t dB_t, \quad L_0^u = 1,
\end{align*}
$$

The optimality condition says that $\hat{u}_t = q_t^*$ holds for an optimal control.
Example: Mean-field LQ (convex and compact $U$)

Consider on some admissible domain $D \subset \mathbb{R}^d$ the mean-field LQ problem of minimizing final variance

$$
\begin{aligned}
\min_{u \in U} & \frac{1}{2} \mathbb{E} \left[ \int_0^T L^u_t |u_t|^2 dt + L^u_T |X_T - E[L^u_T X_T]|^2 \right], \\
\text{s.t.} & \quad dL^u_t = L^u_t u_t dB_t, \quad L^u_0 = 1,
\end{aligned}
$$

The optimality condition says that $\hat{u}_t = q^*_t$ holds for an optimal control.

With $\nabla_x \phi(X_t, t) = (X_t - E[\hat{X}_t])^*$ we identify $q_t$ and get:

$$
\hat{u}_t = -(X_t - E[\hat{X}_t])^* \sigma(X_t), \quad \mathbb{P}\text{-a.s. for almost every } t \in [0, T].
$$
Example: Mean-field LQ (convex and compact $U$)

Consider on some admissible domain $D \subset \mathbb{R}^d$ the mean-field LQ problem of minimizing final variance

$$
\begin{align*}
&\min_{u \in U} \frac{1}{2} E \left[ \int_0^T L_t^u |u_t|^2 \, dt + L_T^u |X_T - E[L_T^u X_T]|^2 \right], \\
&\text{s.t. } dL_t^u = L_t^u u_t \, dB_t, \quad L_0^u = 1,
\end{align*}
$$

The optimality condition says that $\hat{u}_t = q_t^*$ holds for an optimal control.

With $\nabla_x \phi(X_t, t) = (X_t - E[\hat{u}[X_t]])^*$ we identify $q_t$ and get:

$$
\hat{u}_t = - (X_t - E[\hat{u}[X_t]])^* \sigma(X_t), \quad \mathbb{P}\text{-a.s. for almost every } t \in [0, T].
$$

$\hat{u}$ takes $\mathbb{P}$ to $\mathbb{P}^{\hat{u}}$ under which the coordinate process solves the non-linear SDE

$$
dX_t = \left( a(X_t) - \sigma(X_t)(X_t - E[\hat{u}[X_t]]) \right) \, dt + \sigma(X_t) dB_t^{\hat{u}}.
$$
Total variation distance on $\mathcal{P}(\Omega)$

For $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, the total variation distance is defined by the formula

$$d(\mu, \nu) = 2 \sup_{B \in \mathcal{B}(\mathbb{R}^d)} |\mu(B) - \nu(B)|.$$  \hspace{1cm} (6)

Define on $\mathcal{F}$ the total variation metric

$$d(P, Q) := 2 \sup_{A \in \mathcal{F}} |P(A) - Q(A)|.$$  \hspace{1cm} (7)

On the filtration $\mathbb{F}$,

$$D_t(Q, \tilde{Q}) := 2 \sup_{A \in \mathcal{F}_t} |Q(A) - \tilde{Q}(A)|, \quad 0 \leq t \leq T.$$  \hspace{1cm} (8)

It satisfies

$$D_s(Q, \tilde{Q}) \leq D_t(Q, \tilde{Q}), \quad 0 \leq s \leq t.$$  \hspace{1cm} (9)

For $Q, \tilde{Q} \in \mathcal{P}(\Omega)$ with time marginals $Q_t := Q \circ x_t^{-1}$ and $\tilde{Q}_t := \tilde{Q} \circ x_t^{-1}$, then

$$d(Q_t, \tilde{Q}_t) \leq D_t(Q, \tilde{Q}), \quad 0 \leq t \leq T.$$  \hspace{1cm} (10)

Endowed with the total variation metric $D_T$, $\mathcal{P}(\Omega)$ is a complete metric space. Moreover, $D_T$ carries out the usual topology of weak convergence.