Some aspects of mean-field type modeling of pedestrian crowd dynamics

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Based on joint work with Boualem Djehiche

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March to TELE2 Arena, Stockholm
Music for the Royal Fireworks, KTH Courtyard
Empirical studies of human crowds have been conducted since the '50s\(^2\).

Basic guidelines for pedestrian behavior: will to reach specific targets, repulsion from other individuals and deterministic if the crowd is sparse but partially random if the crowd is dense\(^3\).

Humans motion is decision-based.

<table>
<thead>
<tr>
<th>Classical particles</th>
<th>&quot;Smart agents&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>▶ Robust - interaction only through collisions</td>
<td>▶ Fragile - avoidance of collisions and obstacles</td>
</tr>
<tr>
<td>▶ Blindness - dynamics ruled by inertia</td>
<td>▶ Vision - dynamics ruled at least partially by decision</td>
</tr>
<tr>
<td>▶ Local - interaction is pointwise</td>
<td>▶ Nonlocal - interaction at a distance</td>
</tr>
<tr>
<td>▶ Isotropy - all directions equally influential</td>
<td>▶ Anisotropy - some directions more influential than others</td>
</tr>
</tbody>
</table>


Pedestrian crowd motion: mathematical modeling approaches

Microscopic
S Okazaki. “A study of pedestrian movement in architectural space, part 1: Pedestrian movement by the application on of magnetic models”. In: Trans. AIJ 283 (1979), pp. 111–119

Macroscopic

Mesoscopic/Kinetic
G Albi et al. “Mean field control hierarchy”. In: Applied Mathematics & Optimization 76.1 (2017), pp. 93–135

Mean-field games:
a macroscopic approximation of a microscopic model

Mean-field type games/control:
a macroscopic approximation of a microscopic model
or
a distribution dependent microscopic model
Pedestrian crowd modeling: heuristics of the mean-field approach

- The dynamics of a pedestrian is given by
  - \( \text{change in position} = \text{velocity} + \text{noise} \)
  The pedestrian controls its velocity.

- The pedestrian controls its velocity rationally, it minimizes
  - \( \text{Expected cost} \)
    \[= \mathbb{E} \left[ \int_0^T f (\text{energy use}(t), \text{interaction}(t)) \, dt + \text{deviation from final target} \right] \]

- The interaction is assumed to depend on an aggregate of distances to other pedestrians:
  - \( \text{Lots of pedestrians in my neighborhood - congestion cost} \)
  - \( \text{Seeking the company of others - social gain} \)

- To evaluate its interaction cost, the pedestrian anticipates the movement of other pedestrians via the distribution of the crowd.

Many possible extensions:
controlled noise, multiple interacting crowds, fast exit times, interaction with the environment, common noise, hard congestion.
Pedestrian crowd motion: mean-field models

Early works
C Dogbé. “Modeling crowd dynamics by the mean-field limit approach”. In: Mathematical and Computer Modelling 52.9-10 (2010), pp. 1506–1520

Aversion and congestion
A Lachapelle and M-T Wolfram. “On a mean field game approach modeling congestion and aversion in pedestrian crowds”. In: Transportation research part B: methodological 45.10 (2011), pp. 1572–1589
Y Achdou and M Laurière. “Mean field type control with congestion”. In: Applied Mathematics & Optimization 73.3 (2016), pp. 393–418

Fast exits (evacuation)
M Burger et al. “On a mean field game optimal control approach modeling fast exit scenarios in human crowds”. In: Decision and Control (CDC), 2013 IEEE 52nd Annual Conference on. IEEE. 2013, pp. 3128–3133
M Burger et al. “Mean field games with nonlinear mobilities in pedestrian dynamics”. In: Discrete and Continuous Dynamical Systems-Series B (2014)

Multi-population
M Cirant. “Multi-population mean field games systems with Neumann boundary conditions”. In: Journal de Mathématiques Pures et Appliquées 103.5 (2015), pp. 1294–1315
Y Achdou, M Bardi, and M Cirant. “Mean field games models of segregation”. In: Mathematical Models and Methods in Applied Sciences 27.01 (2017), pp. 75–113
Another model categorization: *level of rationality*\(^2\).

<table>
<thead>
<tr>
<th>Rationality level</th>
<th>Information structure</th>
<th>Area of application</th>
</tr>
</thead>
<tbody>
<tr>
<td>Irrational</td>
<td>-</td>
<td>Panic situations</td>
</tr>
<tr>
<td>Basic</td>
<td>Destination and environment</td>
<td>Movement in large unfamiliar environments</td>
</tr>
<tr>
<td>Rational</td>
<td>Current position of other pedestrians</td>
<td>Movement in small and well-known environment</td>
</tr>
<tr>
<td>Highly rational</td>
<td>Forecast of other pedestrians movement</td>
<td>Movement in small and well-known environment</td>
</tr>
<tr>
<td>Optimal</td>
<td>Omniscient central planner</td>
<td>&quot;Soldiers&quot;</td>
</tr>
</tbody>
</table>

Mean field games can model highly rational pedestrians.

Mean-field type control can model optimal pedestrians.

Lachapelle & Wolfram (2011) studies a game between two crowds. Non-local interactions can be included (vision), and an arbitrary number of crowds can take part in the game (KTH courtyard).

Let there be $M$ crowds. Each crowd has its own target region, modeled by $\Psi_j$, and preference towards averting the other crowds, $\{\lambda_{jk}\}_{k=1}^M$. The pedestrians in crowd $j$ cooperates, they observes the other crowds and replies jointly. The equilibrium is given by

$$J^j(\hat{a}^1, \ldots, \hat{a}^M) \leq J^j(\hat{a}^j, \ldots, \hat{a}^{j-1}, \alpha, \hat{a}^{j+1}, \ldots, \hat{a}^M), \quad j = 1, \ldots, M, \quad \forall \alpha \in \mathcal{A},$$

where the crowd cost is

$$J^j(a^j; a^{-j}) := \int_{\mathbb{R}^d} \int_0^T \left[ \frac{1}{2} |a^j(t, x)|^2 m_j(t, x) 
+ \sum_{k=1}^M \lambda_{jk} \left( \int_{\mathbb{R}^d} \phi_r(x - y) m_k(t, y) dy \right) m_j(t, x) \right] dt dx + \int_{\mathbb{R}^d} \Psi_j(x) m_j(T, x) dx,$$

and the crowd dynamics is

$$\partial_t m_j = \frac{1}{2} Tr(\nabla^2 \sigma \sigma^T m_j) - \nabla \cdot (b(t, x, a^j) m_j), \quad m_j(0, x) = m_{j,0}(x).$$

---

Mean-field type game of crowds

Let $\beta = (\beta^1, \ldots, \beta^M)$ for $\beta \in \{|a|^2, m, G, \Psi\}$ and consider the optimization problem

$$
\begin{aligned}
\min_{a \in A^M} \quad & J(a) = \int_{\mathbb{R}^d} \int_0^T \left[ \frac{1}{2} |a(t, x)|^2 \cdot m(t, x) + G[m](t, x) \bar{\Lambda} m(t, x) \right] dt dx \\
+ & \int_{\mathbb{R}^d} \Psi(x) \cdot m(T, x) dx,
\end{aligned}
$$

s.t. $\partial_t m_j = \frac{1}{2} \text{Tr}[\nabla^2 \sigma \sigma^T m_j] - \nabla \cdot (b(t, x, a^i)m_j),$

$m_j(0, x) = m_{j,0}(x), \quad j = 1, \ldots, M,$

where $\bar{\Lambda} + \bar{\Lambda}^T - \text{diag}(\bar{\Lambda}) : = \Lambda$ and $\Lambda = (\lambda_{jk})_{j,k=1}^M$ contains the crowd aversion preferences.

**Theorem**

The control $\hat{a}$ solves (4) if and only if $\hat{a}$ is an equilibrium control for the game between crowds.
Mean-field optimization of a multiple crowd model

Let \( \hat{a} \) be admissible and \( \hat{m} \) be the corresponding solution to the PDE constraint. Let

\[
H(t, x, a, m, p) := \frac{1}{2} |a|^2 \cdot m + G[m]^T \bar{\Lambda} m + \sum_{j=1}^{M} b(t, x, a^j(t, x)) m_j \cdot \nabla p_j(t, x),
\]

where \( p \) solves

\[
\begin{cases}
\partial_t p = -\left( \frac{1}{2} |\hat{a}|^2 + 2 G[\hat{m}]^T \bar{\Lambda} + (\hat{b} \cdot \nabla p_1, \ldots, \hat{b} \cdot \nabla p_M) + \frac{1}{2} \text{Tr}(\hat{\sigma} \hat{\sigma}^T \nabla^2 p) \right), \\
p(T, x) = \Psi(x).
\end{cases}
\]

Theorem

If \( (a, m) \mapsto \int_{\mathbb{R}^d} H(t, x, a, m, p) \, dx \) is convex for all \( t \in [0, T] \) and for all admissible control vectors \( (\alpha^1, \ldots, \alpha^M) \),

\[
\int_{\mathbb{R}^d} \int_0^T D_{ai} H(t, x, \hat{a}, \hat{m}, p) \cdot \alpha^j \, dt \, dx = 0, \quad j = 1, \ldots, M,
\]

then \( \hat{a} \) solves the mean-field control problem (4).

The convexity assumption holds if and only if

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi_r(x - y) (m(t, y) - m'(t, y))^T \bar{\Lambda} (m(t, x) - m'(t, x)) \, dy \, dx \geq 0
\]

for all densities \( m, m' \) and \( t \in [0, T] \).
Tagged pedestrian motion: control of mean-field BSDEs

Stochastic dynamics with initial condition cannot model motion that has to terminate in a target location at time horizon $T$, such as:

- Guards moving to a security threat
- Medical personnel moving to a patient
- Fire-fighters moving to a fire
- Deliveries

Control of mean-field BSDEs can be a tool for centrally planned decision-making for pedestrian groups, who are forced to reach a target position.

Recall, mean-field control is suitable for pedestrian crowd modeling when

- the central planner is rational and has the ability to anticipate the behaviour of other pedestrians
- aggregate effects are considered
The motion of our representative agent is described by a BSDE,
\[
\begin{aligned}
    dY^u_t &= b(t, Y^u_t, \mathbb{P}Y^u_t, Z^u_t, u_t)dt + Z^u_t dB_t, \\
    Y^u_T &= y_T.
\end{aligned}
\] (8)

The central planner faces the optimization problem
\[
\begin{aligned}
    \min_{(u_t)_{t \in [0,T]} \in \mathcal{U}[0,T]} \mathbb{E} \left[ \int_0^T f(t, Y^u_t, \mathbb{P}Y^u_t, u_t)dt + h(Y^u_0, \mathbb{P}Y^u_0) \right] \\
    \text{s.t. } (Y^u_t, Z^u_t)_{t \in [0,T]} \text{ solves (8).}
\end{aligned}
\] (9)

From a modeling point of view, the tagged pedestrian uses two controls:

- \((u_t)_{t \in [0,T]}\) - picked by an optimization procedure to reduce energy use, movement in densely crowded areas

- \((Z_t)_{t \in [0,T]}\) - to predict the best path to \(y_T\) given \((u_t)_{t \in [0,T]}\), given implicitly by the martingale representation theorem.

A spike pertubation technique leads to a Pontryagin type maximum principle\(^2\).

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Assumptions: i) $u \mapsto b(\cdot, \cdot, \cdot, \cdot, u)$ is Lipschitz and its $y$-, $z$- and $\mu$-derivatives are bounded ii) $b(\cdot, 0, \delta_0, 0, u)$ is square-integrable for all $u \in U$ iii) $y_T \in L^2(F_T)$ iv) admissible controls $(U[0, T])$ take values in the compact set $U$ and are square-integrable.

**Theorem - necessary conditions**

Suppose that $(\hat{Y}, \hat{Z}, \hat{\nu})$ is solves the control problem. Let $H$ be the Hamiltonian

$$H(t, y, \mu, z, u, p) := b(t, y, \mu, z, u)p - f(t, y, \mu, u),$$

and let $(p_t)_{t \in [0, T]}$ solve the adjoint equation

$$\begin{aligned}
dp_t &= - \left\{ \partial_y H(t, \hat{Y}_t, \mathbb{P}_{\hat{Y}_t}, \hat{Z}_t, \hat{u}_t, p_t) + \mathbb{E} \left[ \partial_\mu H(t, \hat{Y}_t, (\mathbb{P}_{\hat{Y}_t})^*, \hat{Z}_t, \hat{u}_t, p_t) \right] \right\} dt \\
&\quad - p_t \partial_z b(t, \hat{Y}_t, \mathbb{P}_{\hat{Y}_t}, \hat{Z}_t, \hat{u}_t) dB_t,
\end{aligned}$$

$$p_0 = \partial_y h(\hat{Y}_0, \mathbb{P}_{\hat{Y}_0}) + \mathbb{E} \left[ \partial_\mu h(\hat{Y}_0, (\mathbb{P}_{\hat{Y}_t})^*) \right].$$

Then for a.e. $t$, $\mathbb{P}$-a.s.,

$$\hat{u}_t = \arg \max_{u \in U} H(t, \hat{Y}_t, \mathbb{P}_{\hat{Y}_t}, \hat{Z}_t, u, p_t).$$

**Theorem - sufficient conditions**

Suppose that $H$ is concave in $(y, \mu, z, u)$, $h$ is convex in $(y, \mu)$ and $(\hat{u}_t)_{t \in [0, T]}$ satisfies (12) $\mathbb{P}$-a.s. for a.e. $t$. Then $(\hat{Y}, \hat{Z}, \hat{u})$ solves the control problem.
Tagged pedestrian motion: control of mean-field BSDEs

\[
\begin{align*}
\min_{(u_t)_{t \in [0,1]} \in U} & \quad \frac{1}{2} \mathbb{E} \left[ \int_0^1 \lambda_1 u_t^2 + \lambda_2 (Y_t - \mathbb{E}[Y_t])^2 dt + \lambda_3 (Y_0 - [0.2, 0.2]^T)^2 \right], \\
\text{s.t.} & \quad dY_t = (u_t + B_t) dt + Z_t dB_t, \quad Y_1 = [2, 2]^T.
\end{align*}
\] (13)

Upper row: \((\lambda_1, \lambda_2, \lambda_3) = (50, 50, 10)\).
Lower row: \((\lambda_1, \lambda_2, \lambda_3) = (50, 0, 10)\).
Simulations based on the least-square Monte Carlo method\(^2\).

Thank you!