Some aspects of mean-field type modeling of pedestrian crowd dynamics

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Based on joint work with Boualem Djehiche

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March to TELE2 Arena, Stockholm

Music for the Royal Fireworks, KTH Courtyard

Pedestrian crowd motion: quick facts

Empirical studies of human crowds have been conducted since the '50s[2].

Basic guidelines for pedestrian behavior: will to reach specific targets, repulsion from other individuals and deterministic if the crowd is sparse but partially random if the crowd is dense[3].

Humans motion is decision-based.

Classical particles

- Robust - interaction only through collisions
- Blindness - dynamics ruled by inertia
- Local - interaction is pointwise
- Isotropy - all directions equally influential

"Smart agents"

- Fragile - avoidance of collisions and obstacles
- Vision - dynamics ruled at least partially by decision
- Nonlocal - interaction at a distance
- Anisotropy - some directions more influential than others

2 BD Hankin and R Wright. "Passenger flow in subways". In: *Journal of the Operational Research Society* 9.2 (1958), pp. 81–88.

3 E Cristiani, B Piccoli, and A Tosin. "Modeling self-organization in pedestrians and animal groups from macroscopic and microscopic viewpoints". In: *Mathematical modeling of collective behavior in socio-economic and life sciences*. Springer, 2010, pp. 337–364.

Pedestrian crowd motion: mathematical modeling approaches

Microscopic

D Helbing and P Molnar. "Social force model for pedestrian dynamics". In: *Physical review E* 51.5 (1995), p. 4282
A Schadschneider. "Cellular automaton approach to pedestrian dynamics-theory". In: *Pedestrian and Evacuation Dynamics* (2002), pp. 75–85
S Okazaki. "A study of pedestrian movement in architectural space, part 1: Pedestrian movement by the application on of magnetic models". In: *Trans. AIJ* 283 (1979), pp. 111–119

Macroscopic

LF Henderson. "The statistics of crowd fluids". In: *Nature* 229.5284 (1971), p. 381
R Hughes. "The flow of human crowds". In: *Annual review of fluid mechanics* 35.1 (2003), pp. 169–182
S Hoogendoorn and P Bovy. "Pedestrian route-choice and activity scheduling theory and models". In: *Transportation Research Part B: Methodological* 38.2 (2004), pp. 169–190

Mesoscopic/Kinetic

C Dogbe. "On the modelling of crowd dynamics by generalized kinetic models". In: *Journal of Mathematical Analysis and Applications* 387.2 (2012), pp. 512–532
G Albi et al. "Mean field control hierarchy". In: *Applied Mathematics & Optimization* 76.1 (2017), pp. 93–135

Mean-field games:
a macroscopic approximation of a microscopic model

Mean-field type games/control:
a macroscopic approximation of a microscopic model
or
a distribution dependent microscopic model

Pedestrian crowd modeling: heuristics of the mean-field approach

- The dynamics of a pedestrians is given by
  - *change in position = velocity + noise*

  The pedestrian controls it's velocity.

- The pedestrian controls it's velocity rationally, it minimizes
  - *Expected cost*
    $= \mathbb{E}\left[\int_0^T f\,(\text{energy use}(t), \text{interaction}(t))\,dt + \text{deviation from final target}\right]$

- The interaction is assumed to depend on an aggregate of distances to other pedestrians:
  - *Lots of pedestrians in my neighborhood - congestion cost*
  - *Seeking the company of others - social gain*

- To evaluate its interaction cost, the pedestrian anticipates the movement of other pedestrians via the distribution of the crowd.

Many possible extensions:
controlled noise, multiple interacting crowds, fast exit times, interaction with the environment, common noise, hard congestion.

Pedestrian crowd motion: mean-field models

Early works

S Hoogendoorn and P Bovy. "Pedestrian route-choice and activity scheduling theory and models". In: *Transportation Research Part B: Methodological* 38.2 (2004), pp. 169–190

C Dogbé. "Modeling crowd dynamics by the mean-field limit approach". In: *Mathematical and Computer Modelling* 52.9-10 (2010), pp. 1506–1520

Aversion and congestion

A Lachapelle and M-T Wolfram. "On a mean field game approach modeling congestion and aversion in pedestrian crowds". In: *Transportation research part B: methodological* 45.10 (2011), pp. 1572–1589

Y Achdou and M Laurière. "Mean field type control with congestion". In: *Applied Mathematics & Optimization* 73.3 (2016), pp. 393–418

Fast exits (evacuation)

M Burger et al. "On a mean field game optimal control approach modeling fast exit scenarios in human crowds". In: *Decision and Control (CDC), 2013 IEEE 52nd Annual Conference on*. IEEE. 2013, pp. 3128–3133

M Burger et al. "Mean field games with nonlinear mobilities in pedestrian dynamics". In: *Discrete and Continuous Dynamical Systems-Series B* (2014)

B Djehiche, A Tcheukam, and H Tembine. "A Mean-Field Game of Evacuation in Multilevel Building". In: *IEEE Transactions on Automatic Control* 62.10 (2017), pp. 5154–5169

Multi-population

E Feleqi. "The derivation of ergodic mean field game equations for several populations of players". In: *Dynamic Games and Applications* 3.4 (2013), pp. 523–536

M Cirant. "Multi-population mean field games systems with Neumann boundary conditions". In: *Journal de Mathématiques Pures et Appliquées* 103.5 (2015), pp. 1294–1315

Y Achdou, M Bardi, and M Cirant. "Mean field games models of segregation". In: *Mathematical Models and Methods in Applied Sciences* 27.01 (2017), pp. 75–113

Pedestrian crowd motion: rationality

Another model categorization: *level of rationality*[2].

Rationality level	**Information structure**	**Area of application**
Irrational	-	Panic situations
Basic	Destination and environment	Movement in large unfamiliar environments
Rational	Current position of other pedestrians	Movement in small and well-known environment
Highly rational	Forecast of other pedestrians movement	Movement in small and well-known environment
Optimal	Omnipotent central planner	"Soldiers"

Mean field games can model highly rational pedestrians.

Mean-field type control can model optimal pedestrians.

[2] E Cristiani, F Priuli, and A Tosin. "Modeling rationality to control self-organization of crowds: an environmental approach". In: *SIAM Journal on Applied Mathematics* 75.2 (2015), pp. 605–629.

Mean-field type game of crowds

Lachapelle & Wolfram (2011) studies a game between two crowds. Non-local interactions can be included (*vision*), and an arbitrary number of crowds can take part in the game (*KTH courtyard*)[2].

Let there be M crowds. Each crowd has its own target region, modeled by Ψ_j, and preference towards averting the other crowds, $\{\lambda_{jk}\}_{k=1}^M$. The pedestrians in crowd j cooperates, they observes the other crowds and replies jointly. The equilibrium is given by

$$J^j(\hat{a}^1, \dots, \hat{a}^M) \leq J^j(\hat{a}^j, \dots, \hat{a}^{j-1}, \alpha, \hat{a}^{j+1}, \dots, \hat{a}^M), \quad j = 1, \dots, M, \ \forall \alpha \in \mathcal{A}, \qquad (1)$$

where the crowd cost is

$$\begin{aligned} J^j(a^j; a^{-j}) := \int_{\mathbb{R}^d} \int_0^T \Bigg[\frac{1}{2} |a^j(t,x)|^2 m_j(t,x) \\ + \sum_{k=1}^M \lambda_{jk} \underbrace{\left(\int_{\mathbb{R}^d} \phi_r(x-y) m_k(t,y) dy \right)}_{=: G^k[m](t,x)} m_j(t,x) \Bigg] dtdx + \int_{\mathbb{R}^d} \Psi_j(x) m_j(T,x) dx, \end{aligned} \qquad (2)$$

and the crowd dynamics is

$$\partial_t m_j = \frac{1}{2} Tr(\nabla^2 \sigma \sigma^T m_j) - \nabla \cdot (b(t,x,a^j) m_j), \quad m_j(0,x) = m_{j,0}(x). \qquad (3)$$

[2] A Aurell and B Djehiche. “Mean-field type modeling of nonlocal crowd aversion in pedestrian crowd dynamics”. In: *SIAM Journal on Control and Optimization* 56.1 (2018), pp. 434–455.

Mean-field type game of crowds

Let $\beta = (\beta^1, \dots, \beta^M)$ for $\beta \in \{|a|^2, m, G, \Psi\}$ and consider the optimization problem

$$\begin{cases} \min\limits_{a \in \mathcal{A}^M} & J(a) = \int_{\mathbb{R}^d} \int_0^T \left[\frac{1}{2} |a(t,x)|^2 \cdot m(t,x) + G[m]^T(t,x) \bar{\Lambda} m(t,x) \right] dtdx \\ & \quad + \int_{\mathbb{R}^d} \Psi(x) \cdot m(T,x) dx, \\ \text{s.t.} & \partial_t m_j = \frac{1}{2} \mathrm{Tr}[\nabla^2 \sigma \sigma^T m_j] - \nabla \cdot (b(t,x,a^j) m_j), \\ & m_j(0,x) = m_{j,0}(x), \quad j = 1, \dots, M, \end{cases} \tag{4}$$

where $\bar{\Lambda} + \bar{\Lambda}^T - \mathrm{diag}(\bar{\Lambda}) :\, = \Lambda$ and $\Lambda = (\lambda_{jk})_{j,k=1}^M$ contains the crowd aversion preferences.

Theorem

The control $\hat{a}$ solves (4) if and only if $\hat{a}$ is an equilibrium control for the game between crowds.

Mean-field optimization of a multiple crowd model

Let $\hat{a}$ be admissible and $\hat{m}$ be the corresponding solution to the PDE constraint. Let

$$H(t,x,a,m,p) := \frac{1}{2}|a|^2 \cdot m + G[m]^T \bar{\Lambda} m + \sum_{j=1}^{M} b(t,x,a^j(t,x)) m_j \cdot \nabla p_j(t,x), \qquad (5)$$

where p solves

$$\begin{cases} \partial_t p = -\left(\frac{1}{2}|\hat{a}|^2 + 2G[\hat{m}]^T \bar{\Lambda} + \left(\hat{b} \cdot \nabla p_1, \dots, \hat{b} \cdot \nabla p_M \right) + \frac{1}{2} \mathrm{Tr}(\hat{\sigma}\hat{\sigma}^T \nabla^2 p) \right), \\ p(T,x) = \Psi(x). \end{cases} \qquad (6)$$

Theorem

If $(a,m) \mapsto \int_{\mathbb{R}^d} H(t,x,a,m,p) dx$ is convex for all $t \in [0,T]$ and for all admissible control vectors $(\alpha^1, \dots, \alpha^M)$,

$$\int_{\mathbb{R}^d} \int_0^T D_{a^j} H(t,x,\hat{a},\hat{m},p) \cdot \alpha^j \, dt dx = 0, \quad j = 1, \dots, M,$$

then $\hat{a}$ solves the mean-field control problem (4).

The convexity assumption holds if and only if

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi_r(x-y)(m(t,y) - m'(t,y))^T \bar{\Lambda} (m(t,x) - m'(t,x)) dy dx \geq 0 \qquad (7)$$

for all densities m, m' and $t \in [0,T]$.

Tagged pedestrian motion: control of mean-field BSDEs

Stochastic dynamics with initial condition cannot model motion that *has to terminate in a target location at time horizon T*, such as:

- Guards moving to a security threat
- Medical personnel moving to a patient
- Fire-fighters moving to a fire
- Deliveries

Control of mean-field BSDEs can be a tool for *centrally planned decision-making for pedestrian groups*, who are forced to reach a target position.

Recall, mean-field control is suitable for pedestrian crowd modeling when

- the central planner is rational and has the ability to anticipate the behaviour of other pedestrians
- aggegate effects are considered

Tagged pedestrian motion: control of mean-field BSDEs

The motion of our representative agent is described by a BSDE,

$$\begin{cases} dY_t^u = b(t, Y_t^u, \mathbb{P}_{Y_t^u}, Z_t^u, u_t)dt + Z_t^u dB_t, \\ Y_T^u = y_T. \end{cases} \tag{8}$$

The central planner faces the optimization problem

$$\begin{cases} \min\limits_{(u_t)_{t\in[0,T]}\in\mathcal{U}[0,T]} & \mathbb{E}\left[\int_0^T f(t, Y_t^u, \mathbb{P}_{Y_t^u}, u_t)dt + h(Y_0^u, \mathbb{P}_{Y_0^u})\right] \\ \text{s.t.} & (Y_t^u, Z_t^u)_{t\in[0,T]} \text{ solves (8).} \end{cases} \tag{9}$$

From a modeling point of view, the tagged pedestrian uses two controls:

- $(u_t)_{t\in[0,T]}$ - picked by an optimization procedure to reduce energy use, movement in densely crowded areas
- $(Z_t)_{t\in[0,T]}$ - to predict the best path to y_T given $(u_t)_{t\in[0,T]}$, given implicitly by the martingale representation theorem.

A spike pertubation technique leads to a Pontryagin type maximum principle[2].

[2] A Aurell and B Djehiche. "Modeling tagged pedestrian motion: a mean-field type control approach". In: *arXiv preprint arXiv:1801.08777v2* (2018).

Tagged pedestrian motion: control of mean-field BSDEs

Assumptions: i) $u \mapsto b(\cdot,\cdot,\cdot,\cdot,u)$ is Lipschitz and its y-,z- and μ-derivatives are bounded ii) $b(\cdot, 0, \delta_0, 0, u)$ is square-integrable for all $u \in U$ iii) $y_T \in L^2(F_T)$ iv) admissible controls ($\mathcal{U}[0,T]$) take values in the compact set U and are square-integrable.

Theorem - necessary conditions

Suppose that $(\hat{Y}, \hat{Z}, \hat{v})$ is solves the control problem. Let H be the Hamiltonian

$$H(t,y,\mu,z,u,p) := b(t,y,\mu,z,u)p - f(t,y,\mu,u), \tag{10}$$

and let $(p_t)_{t\in[0,T]}$ solve the adjoint equation

$$\begin{cases} dp_t = -\left\{\partial_y H(t,\hat{Y}_t,\mathbb{P}_{\hat{Y}_t},\hat{Z}_t,\hat{u}_t,p_t) + \mathbb{E}\left[\partial_\mu H(t,\hat{Y}_t,(\mathbb{P}_{\hat{Y}_t})^*,\hat{Z}_t,\hat{u}_t,p_t)\right]\right\} dt \\ \qquad - p_t\partial_z b(t,\hat{Y}_t,\mathbb{P}_{Y_t},\hat{Z}_t,\hat{u}_t)dB_t, \\ p_0 = \partial_y h(\hat{Y}_0,\mathbb{P}_{\hat{Y}_0}) + \mathbb{E}\left[\partial_\mu h(\hat{Y}_0,(\mathbb{P}_{\hat{Y}_t})^*)\right]. \end{cases} \tag{11}$$

Then for a.e. t, $\mathbb{P}$-a.s.,

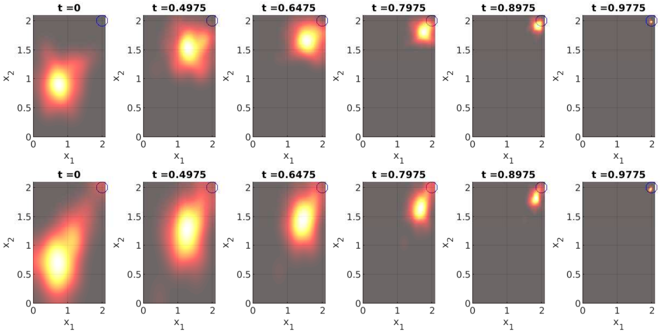
$$\hat{u}_t = \underset{u\in U}{argmax}\ H(t,\hat{Y}_t,\mathbb{P}_{\hat{Y}_t},\hat{Z}_t,u,p_t). \tag{12}$$

Theorem - sufficient conditions

Suppose that H is concave in (y,μ,z,u), h is convex in (y,μ) and $(\hat{u}_t)_{t\in[0,T]}$ satisfies (12) $\mathbb{P}$-a.s. for a.e. t. Then $(\hat{Y},\hat{Z},\hat{u})$ solves the control problem.

Tagged pedestrian motion: control of mean-field BSDEs

$$
\begin{cases} \min\limits_{(u_t)_{t\in[0,1]}\in\mathcal{U}} & \frac{1}{2}\mathbb{E}\left[\int_0^1 \lambda_1 u_t^2 + \lambda_2(Y_t - \mathbb{E}[Y_t])^2 dt + \lambda_3(Y_0 - [0.2, 0.2]^T)^2\right], \\ \text{s.t.} & dY_t = (u_t + B_t)dt + Z_t dB_t,\ \ Y_1 = [2,2]^T. \end{cases} \quad (13)
$$



Upper row: $(\lambda_1, \lambda_2, \lambda_3) = (50, 50, 10)$.
Lower row: $(\lambda_1, \lambda_2, \lambda_3) = (50, 0, 10)$.
Simulations based on the least-square Monte Carlo method[2].

[2] C Bender and J Steiner. "Least-squares Monte Carlo for backward SDEs". In: *Numerical methods in finance.* Springer, 2012, pp. 257–289.

Thank you!