Mean field Capital Accumulation

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The paper

This presentation addresses the material in the paper

Mean Field Capital Accumulation with Stochastic Depreciation

By Minyi Huang & Son Luu Nguyen.

Goal & Main Idea

Goal

The paper focuses on continuous time MF modelling for stochastic growth optimization, taking into account uncertainty caused by stochastic depreciation.

Main Idea

Compute the solution of the mean field game by a set of ODEs.

Outline

1. Problem formulation
2. MFG with finite horizon
3. Connection to the course
4. MFG with infinite horizon
5. MFG with non-linear dynamics

The original problem

A finite population model of N players is described. Denote the capital stock of player i by X_t^i. Its dynamics is modelled by the SDE

$$dX_t^i = \{F(X_t^{(N-1)}, X_t^i) - \delta X_t^i\}dt - C_t^i dt - \sigma X_t^i dW_t^i, \qquad (1)$$

where

- F is the production output
- C_t^i is the consumption rate
- $(\delta dt + \sigma dW_t^i)$ is the stochastic capital depreciation rate
- $X_t^{(N-1)} = \frac{1}{N-1}\sum_{j=1}^{N-1} X_t^j, \forall i \neq j$ is the mean field coupling term
- $\{W^i, i = 1, \ldots, N\}$ are i.i.d. standard Brownian motions.
- $\{X_0^i, i = 1, \ldots, N\}$ are i.i.d. initial states independent of N Brownian motions.

The original problem

The Hyperbolic Absolute Risk Aversion Utility Functional (HARA)for player i is given by

$$J(C^i) = E\left[\int_0^T e^{-\rho t} U(C_t^i)dt + e^{-\rho T}\lambda(X_T^{N-1})S(X_T^i)\right], \qquad (2)$$

where λ is a positive decreasing function on $[0, \infty)$, and

$$U(C_t^i) = \frac{(C_t^i)^\gamma}{\gamma}, \quad S(X_T^i) = \frac{\eta(X_T^i)^\gamma}{\gamma}.$$

<u>Note</u>

The terminal Payoff depends on X_T^{N-1}. i.e. when the aggregate capital level X_T^{N-1} is high, X_T^i is low.

Mean Field Game Problem Formulation

Consider an infinite population and a representative player. The mean field coupling term,defined previously,is approximated by the average m_t which represents the aggregate capital level. Thus the original problem will be reformulated as

$$dX_t = \{F(m_t, X_t) - \delta X_t\}dt - C_t dt - \sigma X_t dW_t, \tag{3}$$

with the performance functional

$$J(C) = E\left[\int_0^T e^{-\rho t} U(C_t)dt + e^{-\rho T}\lambda(m_T)S(X_T)\right]. \tag{4}$$

Mean field game for linear dynamics and finite horizon

Consider a generalized AK model for the production dynamics. Suppose X_t is given by

$$dX_t = A(m_t)X_t dt - C_t dt - \sigma X_t dW_t, \quad t \geq 0. \tag{5}$$

In other words, we take $F(m_t, X_t) = [A(m_t) + \sigma]X_t$ which is linear in X_t.

Furthermore, assume that

$$A(\cdot) \text{ is a continuous strictly decreasing function on } [0, \infty). \tag{A1}$$

The utility functional will be the same as in (4).

Solution to the mean field problem

The solution to the mean field game $(4) - (5)$ is given by a set of ODEs. The procedure is described as follows

1. Fix m_t and solve the ODE

$$D'_t = \left[\frac{\rho}{1-\gamma} - \frac{A(m_t)\gamma}{1-\gamma} + \frac{\gamma\sigma^2}{2}\right] D_t - 1. \qquad (6)$$

2. Given the solution D_t, solve the ODE

$$dm_t = \left[A(m_t) - \frac{1}{D_t}\right] m_t dt \qquad (7)$$

Iterate until the fixed point is achieved (matching argument)

Comments on step 1

The ODE (6) is derived from the HJB equation.

Define the value function by

$$V(t,x) = \sup_{C(\cdot)} \left[E_{x,t} \int_t^T e^{-\rho(s-t)U(C_s)} ds + e^{-\rho(T-t)} \lambda(m_T) S(X_T) \right],$$

the HJB equation, then , has the form

$$\rho V = V_t + \frac{(\sigma x)^2}{2} V_{xx} + \sup_c \left[A(m_t)x - c + \frac{c^\gamma}{\gamma} \right]$$

$$V(T,x) = \frac{\eta \lambda(m_T) x^\gamma}{\gamma}$$

Note: The supremum is attained at $\hat{C} = V_x^{\frac{1}{\gamma-1}}$

Comments on step 1

Now, make the Ansatz $V(t,x) = \frac{D_t^{1-\gamma} x^\gamma}{\gamma}$, then take derivatives. The HJB equation leads to equation (6).

For simplification, denote

$$B(m_t) = \frac{1}{1-\gamma}\left[\rho - A(m_t)\gamma + \frac{\sigma^2\gamma(1-\gamma)}{2}\right]$$

Then (6) can be rewritten as $D_t' = B(m_t)D_t - 1$

Comments on step 1

Lemma 1: The ODE (6) has a unique solution $d_t > 0, 0 \leq t \leq T$ and if (m_t, C_t) is a mean field solution to the mean field game (4)-(5). Then the optimal response C_t is given by

$$C_t = D_t^{-1} X_t. \tag{8}$$

The solution is proved to be

$$D_t = (\eta\lambda(m_T))^{\frac{1}{1-\gamma}} e^{-\int_t^T B(m_s)ds} + \int_t^T e^{-\int_t^s B(m_\tau)d\tau} ds \quad > 0 \tag{9}$$

Comments on step 2-matching

Theorem 1 : If (m_t, C_t) is mean field solution to the mean field game (4)-(5), then m_t satisfies the following equation

$$dm_t = [A(m_t) - \frac{1}{D_t}]m_t dt$$

<u>Proof:</u>
Plug (8) in the dynamics equation (5). We obtain

$$dX_t = \left[A(m_t) - D^{-1}\right] X_t d_t - \sigma X_t dW_t,$$

take expectation on both sides of the above equation, the mean field dynamics equation follows.

Connection to the course

We formulated the Hamiltonian associated with the optimal control problem

$$\min_{C}\{-J(C_t)\}$$

, as

$$H(t,X,C,p,q) = (A(m_t)X - C)\,p + \sigma X q + e^{-\rho t}\frac{C^{\gamma}}{\gamma}.$$

The dynamics of X_t will be the same as in (5) and the adjoint process (q,p) solves

$$\begin{cases} dp(t) = -\left[(A(m_t))\,P(t) + \sigma q(t)\right]dt + q(t)dW_t, \\ p(T) = e^{-\rho T}\lambda(m_T)\eta\hat{x}_T^{\gamma-1}. \end{cases} \tag{10}$$

The Maximum Principal

The Hamiltonian is concave in X and C, set the derivative w.r.t C to 0. we obtain

$$\hat{C}_t = \left(e^{\rho t} p(t)\right)^{\frac{1}{\gamma-1}}. \tag{11}$$

Now, make the Ansatz $p(t) = a(t)X^{\gamma-1}$ and apply Ito formula. This leads eventually to the ODE

$$\begin{cases} \dot{a}(t) = a(t)\left((\gamma-1)\left(-A(m_t) - \frac{\sigma^2}{2}(\gamma-2) + \sigma^2\right) - A(m_t)\right) \\ +a(t)^{\frac{\gamma}{\gamma-1}} e^{\frac{\rho t}{\gamma-1}}, \\ a(T) = e^{-\rho T}\lambda(m_T)\eta. \end{cases} \tag{12}$$

This is Bernoulli's equation which can be solved explicitly.

Solving the adjoint equation

The solution to Bernoulli's equation will give the adjoint process and thus the optimal control $\hat{C}$ can be obtained by equation (10).

MFG with infinite horizon

Goal: Identify a constant value $\hat{m}$ s.t the infinite horizon optimal control problem solvable and $E[X_t] = \hat{m}$ for all t.
Motivation: the study of $\hat{m}$ is very useful to understand the dynamic properties of the mean field system.

Consider the dynamics

$$dX_t = A(\hat{m})X_t dt - C_t - \sigma X_t dW_t, \tag{13}$$

where the initial state $X_0 > 0$. The infinite horizon utility functional is

$$J(C) = E\left[\int_0^\infty e^{-\rho t}\frac{C_t^\gamma}{\gamma}dt\right]. \tag{14}$$

Stability condition

<u>Lemma 2:</u> If $\hat{m}$ is given and satisfies

$$\rho - A(\hat{m}\gamma + \frac{\sigma^2\gamma(1-\gamma)}{2} > 0, \tag{15}$$

the optimal problem has a finite optimal utility and the optimal control is given by

$$C_t = \left[\frac{\rho - A(\hat{m}\gamma + \frac{\sigma^2\gamma(1-\gamma)}{2}}{1-\gamma} \right] X_t. \tag{16}$$

Stability of the stationary MF solution- Results

The out-of-equilibrium behavior of the MF dynamics was examined, by setting $E[X_0] \neq \hat{m}$ for the MF solution which is assumed to exist.

Results: The stationary mean field solution has the ability to asymptotically restore itself even if the i.i.d. initial states have their mean away from $\hat{m}$.

MFG with non-linear dynamics

Let x_t denote the capital of the representative agent which has the dynamics

$$dX_t = F(m_t, X_t)dt - \delta X_t dt - C_t dt - \sigma X_t dW_t, t \geq 0,$$

where $X_0 > 0$ and $E[X_0] < \infty$ Furthermore some assumptions are made:

1. F is continuous function of (m,x) where x,m ≥ 0.
2. For each fixed x, F is a decreasing function of m.
3. For each fixed m, F is increasing concave function of $x \in (0.\infty)$.

Now, we take $F(m_t, X_t) = A(m_t)x^{1-\gamma}$
The utility (cost)functional is

$$J(C) = E\left[\int_0^T e^{-\rho t} U(C_t)dt + e^{-\rho T} S(m_T, X_T)\right]. \qquad (17)$$

Summary of the steps

1. Solve the HJB for fixed m_t.
2. Solve the ODE system.

$$\begin{cases} \dot{p}(t) = \left[\rho + \frac{\sigma^2\gamma(1-\gamma)}{2} + \delta\gamma\right] p(t) - (1-\gamma)p^{\frac{\gamma}{\gamma-1}}(t) \\ \dot{h} = ph(t) - A(m_t)\gamma p(t). \end{cases}$$

3. By theorem 7, in the paper, we obtain the optimal response which take the feedback form

$$C_t = P^{\frac{1}{\gamma-1}} X_t.$$

4. Get the unique strong solution $X_t, t \in (0, T]$, which exists by theorem 8.

Solution for the MF game

The solution system is determined from the ODE of p_t together with

$$\begin{cases} dZ_t = \gamma A(m_t) - \left[\gamma\delta - \gamma p^{\frac{1}{\gamma-1}}(t)X_t\right] dt - \sigma X_t dW_t, \\ m_t = E(Z_t^{\frac{1}{\gamma}}) \;\; (= E[X_t]). \end{cases} \tag{18}$$

<u>Note 1:</u> dZ_t is derived from the dynamics of X_t by setting $Z_t = X_t^{\gamma}$ and applying Ito formula.

<u>Note 2:</u> $Z_t$can be solved explicitly to derive its probability density function of $(m, p^{\frac{1}{\gamma-1}} = \Phi)$, and subsequently m_t can be given as functional equation of $(m, \Phi) \in C[0, T]$:

$$m_t = F(m, \Phi).$$

Solution to the mean field game

Thus, the solution to the MFG is determined by an ODE for Φ and a functional equation for m_t, and the two equations are coupled together.