Presentation of the Paper: *"Mean-field games and two-point boundry value problems"*

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Disclaimer

Any results without a source are to be interpreted as being from the paper or a direct consequences of the results in the paper.

Overview

Introduction

- A large population of indistinguishable agents regulates their states to low density ones.
- The problem is posed as a mean-field game, for which the solutions depend on two coupled PDE's (LQ, allows for closed form solution):
  - Hamilton-Jacobi-Bellman equation.
  - Fokker-Planck-Kolmogorov equation.
- The distribution of the agents is taken to be a sum of polynomials.
- The case when the value function is quadratic is considered.
- The dynamics of a single agent is given by a linear SDE and driven by a Brownian motion and under the influence of a control and **an adversarial disturbance**.

Results

Main Result

The main result of the paper is that the PDE's associated with the mean-field game are transformed into two sets of ODE's with two point boundry value conditions.

Secondary Result

A secondary result is, under some assumption, that the proposed mean-field equilibrium is exponentially stochastically stable.

Motivations

Problems of this type arise in different fields, however the main reason for studying this type of problem is in connection with opinion dynamics, in particular in social networks.
Crowd averse attitudes imply that the players tend to have different opinions, disensus if you will.

Player Dynamics

In a mean field game with a large number of players, let the individual state of a player be denoted by x. Further let x_0 be the initial state of the player, which is realized by the probability distribution for m_0, the initial player probability density.

$$\begin{cases} dx_t = [\alpha x_t + \beta u_t]dt + \sigma[x_t d\mathcal{B}_t + \zeta_t dt] \\ x(0) = x_0. \end{cases} \tag{1}$$

where $x_t \in \mathbb{R}$ is the state of the player, $u_t \in \mathbb{R}$ the control input of the player, $\mathcal{B}_t \in \mathbb{R}$ is a Brownian motion and $\zeta_t \in \mathbb{R}$ is an adversarial disturbance.
The constants $\alpha \in \mathbb{R}, \beta \in \mathbb{R}$ and $\sigma \in \mathbb{R}$ are model parameters.

Player Density

On a macroscopic level the game is obtained by looking at the probability density functions on the state space:

$$\begin{cases} m : \mathbb{R} \times [0, \infty) \to [0, \infty) \\ (x, t) \to m_t(x) \\ \int_{\mathbb{R}} m_t(x) dx = 1, \text{ for every t.} \end{cases} \tag{2}$$

Then the average state distribution at time t can be defined as

$$\bar{m}_t := \int_{\mathbb{R}} x m_t(x) dx. \tag{3}$$

First Assumption

Assume that the probability density distribution is polynomial in state. Then the density is given by

$$\begin{cases} m_t(x) = a_{0t} + \sum_{j=1}^{n} \frac{1}{j} a_{jt} x^j, & \text{in } \mathbb{R} \times [0, T] \\ m_0(x) = a_{00} + \sum_{j=1}^{n} \frac{1}{j} a_{j0} x^j \\ a_{j0} \text{ given for all } j = 0, \dots, n. \end{cases} \quad (4)$$

The sum of polynomial terms can in some sense be interpreted as the Taylor approximation of a general distribution $m_t(x)$.

Cost Functional

Each agent is given a cost functional which penalizes the final state $g(x_T)$, a stage cost function $c(x_t, u_t, m)$ and a quadratic penalty on the unknown disturbance,

$$J(x_0, u, m, \zeta) = \mathbb{E}\left[\int_0^T \left(c(x_t, u_t, \bar{m}_t) - \gamma^2 |\zeta_t|^2\right) dt + g(x_T)\right]. \quad (5)$$

Where

$$\begin{cases} c(x_t, u_t, m) = \underbrace{a_{0t} + a_{1t}x + \frac{1}{2}a_{2t}x^2}_{\text{mean-field term}} + \frac{b}{2}u_t^2, \quad b > 0 \\ g(x_T, m_T(x_T)) = a_{0T} + a_{1T}x + \frac{1}{2}a_{2T}x^2 \end{cases} \quad (6)$$

Robust Mean-Field Problem

Let $\mathcal{B}_t$ is a one-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$, a probability space with the natural filtration process $\mathcal{F}_t$. Let x_0 be independant of $\mathcal{B}$ and from density $m_0(x)$. Futher let m_t^* be the optimal mean-field trajectory. Then the robust mean-field problem in $\mathbb{R}$ and $(0, T]$ is given by

$$(P) \quad \begin{cases} \inf_{\{u_t\}_t} \sup_{\{\zeta_t\}_t} J(x, u, m^*, \zeta) \\ dx_t = [\alpha x_t + \beta u_t + \sigma \zeta_t] dt + \sigma x_t d\mathcal{B}_t. \end{cases} \tag{7}$$

Worst-Case Disturbance

Let $v_t(x)$ be the upper value of the robust optimization problem under worst-case disturbance, starting at t in state x. Considering a quadratic v.

$$\begin{cases} v_t(x) & = q_{0t} + q_{1t}x + \frac{1}{2}q_{2t}x^2, \quad \text{in } \mathbb{R} \times [0, T] \\ v_T(x) & = g(x_T, m_T(x_T)) \\ & = q_{0T} + q_{1T}x + \frac{1}{2}q_{2T}x^2 = a_{0T} + a_{1T}x + \frac{1}{2}a_{2T}x^2 \end{cases} \tag{8}$$

Expressing this concept in terms of the cost functional:

$$J(x, u, m, \zeta) = \mathbb{E}\left[\int_t^T \left(c(x_s, u_s, \bar{m}_s) - \gamma^2|\zeta_s|^2\right) ds + g(x_T)\right].$$

HJB Equation

HJB equation on (8) and for given (4)

$$\begin{cases} 0 & = \partial_t v_t + \left[-\frac{\beta^2}{2b} + \left(\frac{\sigma}{2\gamma} \right)^2 \right] (\partial_x v_t)^2 + \alpha x \partial_x v_t \\ & \quad + a_{0t} + a_{1t} x + \frac{1}{2} a_{2t} x^2 + \frac{1}{2} \sigma^2 x^2 \partial_{xx}^2 v_t, \quad \text{in } \mathbb{R} \times [0, T] \\ v_T(x) & = q_{0T} + q_{1T} x + \frac{1}{2} q_{2T} x^2, \qquad \text{in } \mathbb{R}. \end{cases} \tag{9}$$

forms the first part of the coupled PDE system.

FPK Equation

Using Fokker-Planck-Kolmogorov equation on (4) and (8)

$$
\begin{cases}
0 & = \partial_t m_t + \sum_{j=1}^{n} a_{jt} \Big[\left(1 + \frac{1}{j}\right) \left(\alpha - \frac{\beta^2}{b} q_{2t} + \frac{\sigma^2}{2\gamma^2} q_{2t}\right) x_t^j \\
& \quad + \left(-\frac{\beta^2}{b} q_{1t} + \frac{\sigma^2}{2\gamma^2} q_{1t}\right) x_t^{j-1} \Big] + a_{0t} \left(\alpha - \frac{\beta^2}{b} q_{2t} + \frac{\sigma^2}{2\gamma^2} q_{2t}\right) \\
& \quad - \frac{1}{2}\sigma^2 \partial_{xx}^2 (x^2 m_t), \qquad \text{in } \mathbb{R} \times [0, T] \\
m_0(x) & = a_{00} + \sum_{j=1}^{n} \frac{1}{j} a_{j0} x^j, \quad \text{in } \mathbb{R}.
\end{cases} \tag{10}
$$

forms the second part of the coupled system of PDE's.

Optimal Control and Worst Disturbance

The authors state, in the form of a theorem, that for the system generated by (9) and (10) the optimal control and the worst disturbance are given by:

$$\begin{cases} u_t^* & = \frac{-\beta}{b}\partial_x v_t = \frac{-\beta}{b}\left(q_{1t} + q_{2t}x_t\right) \\ \zeta_t^* & = \frac{\sigma}{2\gamma^2}\partial_x v_t = \frac{\sigma}{2\gamma^2}\left(q_{1t} + q_{2t}x_t\right) \end{cases} \quad (11)$$

This result means that to find the optimal control the two PDE's, (9) and (10), must be solved in terms of v and m with given boundry conditions.

Solving the PDE System

The solution process works as follows:

1. Fix m and solve (9)
2. Calculate the optimal u from (11)
3. Plug into (10)
4. repeat steps(1)-(3) until fixed point in v and m is found.

The existence of a solution in terms of (v, m) is guaranteed under some assumptions like m_0 being absolutely continuous with continuous density and a finite second moment. The proof of this is from the paper *"Mean field games"* by Lasry & Lions.

Connection to this Course

In this course we have focused on the maximum principle. To apply the maximum principle to the above given system first formulate the Hamiltonian:

$$H = \{\text{using course notation}\} = bp + \sigma q - f \tag{12}$$

where $f(x,u) = a_{0t} + a_{1t}x + \frac{1}{2}a_{2t}x^2 + \frac{b}{2}u^2 - \gamma^2\zeta^2$ and p is adjoint. Then

$$\begin{cases} \partial_u H = \beta p - bu = 0 \\ \partial_\zeta H = \sigma p + 2\gamma^2\zeta = 0 \end{cases} \implies \begin{cases} u_t^* = -\frac{\beta}{b}p \\ \zeta_t^* = \frac{\sigma}{2\gamma^2}p \end{cases} \tag{13}$$

with

$$\begin{cases} dp = -\hat{H}_x dt + q d\mathcal{B}_t \\ p(T) = -a_{1T} - a_{2T}\hat{x}_T \end{cases} \tag{14}$$

With some identification (13) corresponds to (11).

Main Result

Theorem (ODE Equivalence)

$$
\begin{cases}
\dot{q}_{0t} + \left[-\frac{\beta^2}{2b} + \left(\frac{\sigma}{2\gamma} \right)^2 \right] q_{1t}^2 + a_{0t} = 0 \\
\dot{q}_{1t} + \left[-\frac{\beta^2}{2b} + \left(\frac{\sigma}{2\gamma} \right)^2 \right] (2q_{1t}q_{2t}) + \alpha q_{1t} + a_{1t} = 0 \\
\frac{1}{2}\dot{q}_{2t} + \left[-\frac{\beta^2}{2b} + \left(\frac{\sigma}{2\gamma} \right)^2 \right] q_{2t}^2 + \alpha q_{2t} + \frac{\sigma^2}{2} q_{2t} + \frac{1}{2} a_{2t} = 0 \\
q_{jT} = a_{jT}, \quad j = 0, 1, 2 \\
\dot{a}_{0t} + a_{1t} \left(-\frac{\beta^2}{b} q_{1t} + \frac{\sigma^2}{2\gamma^2} q_{1t} \right) + a_{0t} \left(\alpha - \frac{\beta^2}{b} q_{2t} + \frac{\sigma^2}{2\gamma^2} q_{2t} \right) \\
-\frac{1}{2}\sigma^2 2 a_{0t} = 0 \\
\frac{1}{j}\dot{a}_{jt} + a_{jt} \left[\alpha(1 + \frac{1}{j}) + (-\frac{\beta^2}{b} + \frac{\sigma^2}{2\gamma^2})(1 + \frac{1}{j}) q_{2t} \right] + a_{j+1t}(-\frac{\beta^2}{b} + \frac{\sigma^2}{2\gamma^2}) q_{1t} \\
-\frac{1}{2}\sigma^2 \frac{(j+2)(j+1)}{j} a_{jt} = 0, \quad j = 1, 2, \ldots, n-1 \\
\frac{1}{n}\dot{a}_{nt} + a_{nt}(1 + \frac{1}{n}) \left(\alpha - \frac{\beta^2}{b} q_{2t} + \frac{\sigma^2}{2\gamma^2} q_{2t} \right) - \frac{1}{2}\sigma^2 \frac{(n+2)(n+1)}{n} a_{nt} = 0 \\
a_{j0} \textit{ given for all } j = 0, 1, \ldots, n
\end{cases}
$$

Main Result continued

Then the theorem (15) states that the optimal control and the worst disturbance are given by:

$$\begin{cases} \tilde{u}_t = -\frac{\beta}{b}(q_{2t}x_t + q_{1t}) \\ \tilde{\omega}_t = \frac{\sigma}{2\gamma^2}(q_{2t}x_t + q_{1t}) \end{cases} \tag{16}$$

Asymptotic Stability in Short

Plug the optimal control and worst case disturbance parameters into the dynamics for x_t, which then yields a closed loop system.
Furthermore under an assumption on $\kappa > 0$ a stability theorem is used in unison with a corollary then if $[\sigma^2 - 2\kappa] < 0$ then $\lim_{t \to \infty} x_t = 0 \quad a.s.$

Conclusions

- two coupled PDE's can be transformed to a system of ODE's with two point boundry.

References

T.Mylvaganam et al (2014)
Mean-field games and two-point boundry value problems
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The End