# Opinion dynamics, stubbornness and mean-field games

Alexander Aurell

October 29, 2015

# Introduction: what is modeled?

Opinion propagation:
Dynamics that describe the evolution of the opinions in a large population as a result of repeated interactions on the individual level. The level of stubbornness amongst the individuals vary.

Phenomenon that occure in opinion propagation:
Herd behaviour - convergence towards one (consensus) or multiple (polarization/plurality) opinion values.

# Introduction: model set-up

A set of populations is considered, each made up of uniform agents characterized by a given level of stubbornness.

Individuals are partially stubborn while interested in reaching a consensus with as many other agents as possible. In Sweden you need 4% of the election votes to get seats in the parliament.

# Introduction: main contribution

Affine controls preserve the Gaussian distribution of population under the considered model.

# Model set-up: dynamics

State process for a generic member of population $i \in I$ follows:

$$\begin{cases} dx_i = u_i dt + \xi_i dW_i \\ x_i(0) = x_{0i} \end{cases}$$

where $u_i$ is a control function which may depend on time $t$, state $x_i$ and the density $m(x,t)$ and $\xi_i$ is a real number.

# Model set-up: cost

Player $i$ wants to maximize the functional

$$J_i(u_i, x_{0i}) = \mathbb{E}\left[\int_0^\infty e^{-\rho t} c_i(x_i, u_i, m)dt\right]$$

where

$$\begin{aligned} c_i(x_i, u_i, m) = &(1-\alpha_i)\sum_{j\in I}(\nu_j \ln(m_j(x_i(t)), t)) \\ &-\alpha_i\left(x_i(t) - \bar{m}_{0i}\right)^2 \\ &-\beta u_i^2. \end{aligned}$$

# Model set-up: cost

Lets look at the terms in the running cost...

- $\sum_{j \in I} \nu_j \ln \left( m_j(x_i(.), .) \right)$: player $i$ wants to share its opinion with as many other players as possible.
- $(x_i(.) - \bar{m}_{0i})^2$: player $i$ dislikes departing from the initial mean of its population.
- $\beta u_i^2$: the usual energy penalization.

The parameter $\alpha_i$ is determining the level of stubbornness of the players in population $i$.

# Model set-up: problem statement

The problem to solve, in $u_i$, for each population is

$$\begin{aligned} \text{Maximize} \quad & J_i(u_i; x_{0i}) = \mathbb{E}\left[\int_0^\infty e^{-\rho t} c_i(x_i, u_i, m) dt\right] \qquad (\mathrm{P}_i) \\ \text{Subjec to} \quad & dx_i = u_i dt + \xi_i dW_i \\ & x_i(0) = x_{0i} \\ & u_i \text{ admissible} \end{aligned}$$

# The mean-field equations

Theorem 1

The mean-field system corresponding to $(\mathrm{P}_i)$ is described by the equations: for all $i \in I$,

$$\partial_t v_i(x_i,t) + (1-\alpha_i)\sum_{j\in I}\nu_j \ln(m_j(x_i,t)) - \alpha_i(x_i - \bar{m}_{0i})^2$$
$$+ \ \frac{1}{2\beta}(\partial_x v_i(x_i,t))^2 + \frac{\xi_i^2}{2}\partial^2_{xx} v_i(x_i,t) - \rho v_i(x_i,t) = 0 \qquad (1)$$

$$\partial_t m_i(x_i,t) + \partial_x\left[m_i(x_i,t)\left(-\frac{1}{2\beta}\partial_x v_i(x_i,t)\right)\right]$$
$$- \ \frac{\xi_i^2}{2}\partial^2_{xx} m_i(x_i,t) = 0 \qquad (2)$$
$$m_i(x_i,0) = m_{0i} \qquad (3)$$

# The mean-field equations

The optimal control to in $(\mathrm{P}_i)$ is given by

$$u_i^*(x_i, t) = -\frac{1}{2\beta}\partial_x v_i(x_i, t)$$

So far so good. What happens if we restrics ourselves to linear strategies only?

# Inverse Fokker-Planck problem

Consider the Fokker-Planck problem in one dimension:

$$\partial_t m_i(x_i,t) - \frac{\xi_i^2}{2}\partial_{xx}^2 m_i(x_i,t) + \partial_x \left(u_i(x_i,t)m_i(x_i,t)\right).$$

If we assume that $m_i : S \to \mathbb{R}$, where $S \subset \mathbb{R}^2$, that $m_i \in C^2(S)$ and that $m_i$ is a probability density function for each $t$ which is positive for all $(x,t) \in S$. Then $u_i$ is the solution to

$$u_i(x_i,t) = \frac{1}{m_i(x_i,t)}\left(C(t) + \frac{\xi_i^2}{2}\partial_x m_i(x_i,t) - \int_{x_{0i}}^{x_i} \partial_t m_i(x,t)dx\right)$$

where $C(t)$ is an arbitrary function.

# Gaussian Distribution Preserving Strategies

Assume that population $i$ has initial density

$$m_i(x_i, 0) = \frac{1}{\sigma_{0i}\sqrt{2\pi}} e^{-\frac{(x_i - \mu_{0i})^2}{2\sigma_{0i}^2}}$$

and that the agents in this population implement a linear control strategy

$$u_i(x,t) = \hat{p}_i(t)x + \hat{q}(t), \quad \hat{q}, \hat{p} \in C^1(\mathbb{R}_+)$$

Then

$$x_i(t) = e^{\hat{P}_i(t)} \left( x_{0i} + \int_0^t e^{-\hat{P}_i(\tau)} q_i(\tau) d\tau + \xi_i \int_0^t e^{-\hat{P}_i(\tau)} dW_i \right)$$

where $\hat{P}_i(t) = \int_0^t \hat{p}_i(\tau) d\tau$.

# Gaussian Distribution Preserving Strategies

Furthermore, the density of population $i$ is for each time $t$ equal to

$$m_i(x_i, t) = \frac{1}{\sigma_i(t)\sqrt{2\pi}} e^{-\frac{(x_i - \mu_i(t))^2}{2\sigma_i^2(t)}}$$

where

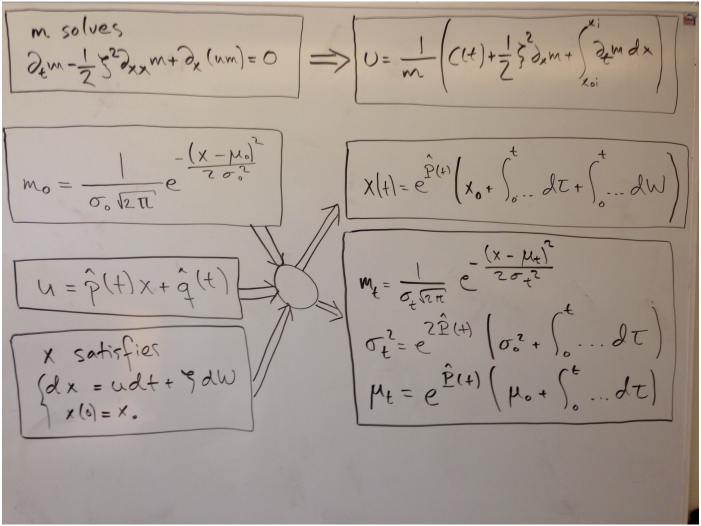
$$\sigma_i^2(t) = e^{2\hat{P}_i(t)} \left( \sigma_{0i}^2 + \xi_i^2 \int_0^t e^{-2\hat{P}_i(\tau)} d\tau \right),$$

$$\mu_i(t) = e^{\hat{P}_i(t)} \left( \mu_{0i} + \int_0^t e^{-\hat{P}_i(\tau)} q_i(\tau) d\tau \right)$$

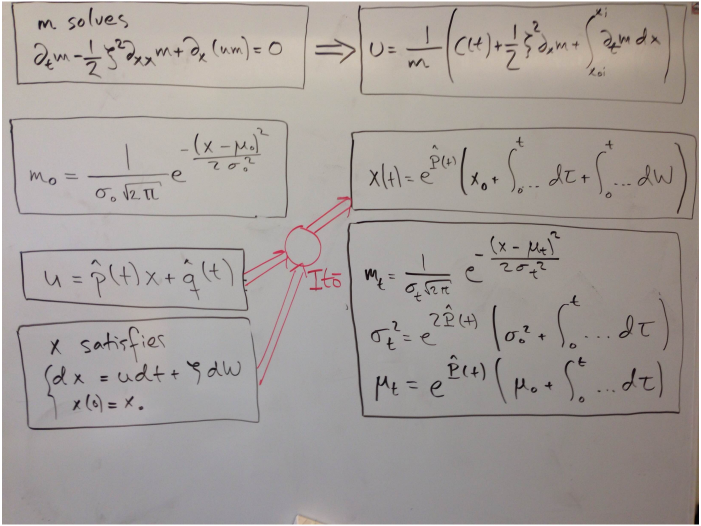
Note:

The implemented $u_i$ is not necessarily optimal.

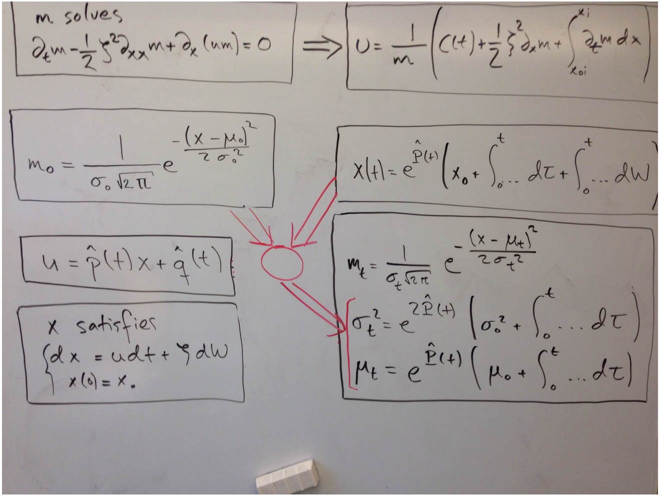
# Gaussian Distribution Preserving Strategies



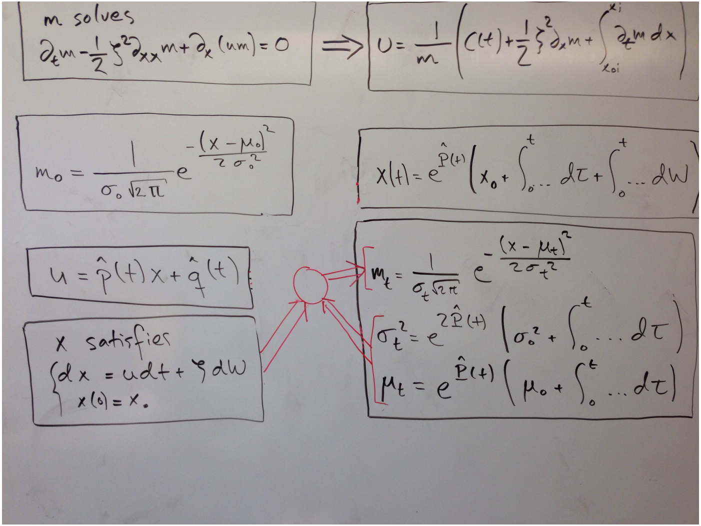
# Gaussian Distribution Preserving Strategies



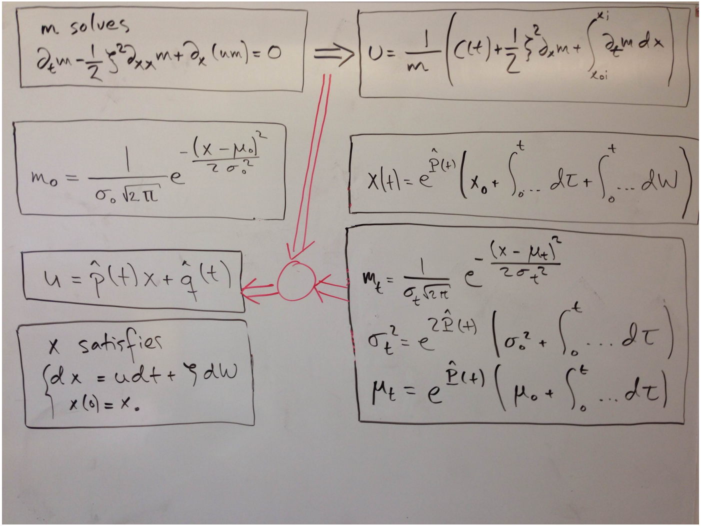
# Gaussian Distribution Preserving Strategies



# Gaussian Distribution Preserving Strategies



# Gaussian Distribution Preserving Strategies



# Two crowd seeking populations

A detailed example with two populations.

If the agents of two populations apply linear strategies then the utility function becomes:

$$J_i(x_{0i}) = \sup_{u_i} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\Big\{(1-\alpha_i)\right.$$

$$= \sum_{j=1}^{2} -\frac{v_j}{2}\left(\ln(2\pi\sigma_j^2(t)) + \frac{(x_i(t)-\bar{m}_j(t))^2}{\sigma_j^2(t)}\right)$$

$$\left. = -\alpha_i(x_i(t)-\bar{m}_{0i})^2 - \beta u_i^2\Big\}dt\right]$$

The populations follow the same dynamics as previously.

# Two crowd seeking populations

Two assuptions are made on the crowds:

A1. At time 0 the two popilations have a Gaussian distribution

A2. The agens adopt linear strategies that track a weighted sum:

$$\begin{aligned} u_i(x,t) &= d_i(a_i\bar{m}_i(t) + b_i\bar{m}_j(t) + c_i\bar{m}_{0i} - x_i) \\ d_i &> 0 \\ 1 &= a_i + b_i + c_i \end{aligned} \tag{4}$$

<u>Note:</u>
Assumption 1 together with the dynamics of the population implies that the strategies (4) are GDPS.

# Extending the state space

By introducing the dynamics of $\bar{m}_i, i = 1, 2$, into the model it is possible to characterize an optimal control. The extended state space equations are: for $i = 1, 2$, $j \neq i$,

$$
\begin{aligned}
dx_i &= u_i dt + \xi_i dW_i \\
x_i(0) &= x_{0i} \\
\dot{\bar{m}}_i(t) &= d_i(b_i \bar{m}_j(t) + c_i \bar{m}_{0i} - (b_i + c_i)\bar{m}_i(t)) \\
\bar{m}_i(0) &= \mu_{0i}
\end{aligned}
$$

# Extending the state space

Time for some rewriting...

Denote by $\gamma_i = -d_i(b_i + c_i)$. Then the state space equations can be written as

$$\begin{bmatrix} dx_i \\ d\bar{m}_i \\ d\bar{m}_j \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \gamma_i & d_i b_i \\ 0 & d_j b_j & \gamma_j \end{bmatrix} \begin{bmatrix} x_i \\ \bar{m}_i \\ \bar{m}_j \end{bmatrix} dt + \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_i & 0 \\ 0 & 0 & c_j \end{bmatrix} \begin{bmatrix} u_i \\ \bar{m}_{0i} \\ \bar{m}_{0j} \end{bmatrix} dt + \begin{bmatrix} \xi_i \\ 0 \\ 0 \end{bmatrix} dW_i$$

# Extending the state space

An even more compact notation is

$$\dot{\bar{m}}(t) = \underbrace{\begin{bmatrix} -\gamma_i & d_i b_i \\ d_j b_j & \gamma_j \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \bar{m}_i(t) \\ \bar{m}_j(t) \end{bmatrix}}_{\bar{m}} + \underbrace{\begin{bmatrix} c_i & 0 \\ 0 & c_j \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \bar{m}_{0i} \\ \bar{m}_{0j} \end{bmatrix}}_{\bar{m}_0}$$

Adding the constant vector $\bar{m}_0$ to the state vector, the problem becomes

$$\sup_u \ \mathbb{E}\left[\int_0^\infty e^{-\rho t}\widetilde{c}_i(x,u,m,\theta)dt\right]$$

$$\begin{bmatrix} dx_i \\ d\bar{m} \\ d\bar{m}_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & M & C \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_i \\ \bar{m} \\ \bar{m}_0 \end{bmatrix} dt + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_i dt + \begin{bmatrix} \xi_i \\ 0 \\ 0 \end{bmatrix} dW_i$$

# Extending the state space

Finally, by letting $X_i = \begin{bmatrix} x_i & \bar{m} & \bar{m}_0 \end{bmatrix}^\mathsf{T}$ we get the LQ problem

$$\inf_u \quad \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(X_i^\mathsf{T}\widetilde{Q}X_i + \beta u_i^2\right)dt\right],$$

$$dX_i = (FX_i + GU_i)\,dt + LdW_i.$$

The solution to an LQ problem of this kind is well known. Consider the new value function $V_i(X_i, t)$ that solves

$$\partial_t V_t(X_i, t) + H(X_i, \partial_{X_i} V_i(X_i, t)) + \frac{1}{2}\partial_i^2\partial_{xx}^2 V_i(X_i, t) = 0.$$

If we assume that the value function is quadratic $V_i(X_i, t) = X_i^\mathsf{T} P(t) X_i$, then $P(t)$ is the solution to the Riccati equation

$$\dot{P}(t) - \rho P(t) + P(t)F + F^\mathsf{T}P(t) - P(t)\left(GR^{-1}G^\mathsf{T}\right)P(t) + \widetilde{Q} + W = 0$$

# Extending the state space

If $P$ solves the Riccati equation then the optimal control is given by

$$\begin{aligned}\hat{u}_i^*(t) &= -\frac{1}{\beta} G^\top P(t) X_i \\ &= \frac{1}{\beta}\left(P_{11}(t)x_i(t) + P_{12}(t)\bar{m}_i + P_{13}(t)\bar{m}_j\right. \\ &\quad \left. + P_{14}(t)\bar{m}_{0i} + P_{15}(t)\bar{m}_{0j}\right)\end{aligned}$$

<u>Conclusion:</u>
The extended state space model allows us to characterize an optimal control under assumptions (A1)-(A2).

# Model behavior

The mean $\bar{m}_i(t)$ converges to a finite value.

The variance $\sigma_i^2(t)$ converges exponentially to $\frac{\xi_i^2}{2d_i}$. The term $\ln(2\pi\sigma_j^2(t)) + \frac{(x_i(t)-\mu_j^2(t))}{2\sigma_j^2(t)}$ therefore grows to infinity if there is no disturbance in population $j$, i.e. $\xi_j = 0$. A consequence is that an optimal strategy (in the no disturbance case) must satisfy $\rho - 2d_i > 0$ or guarantee that all states converge to a single consensus.

# Model behavior

Recall that under A2, $u_i = d_i(a_i \bar{m}_i(t) + b_i \bar{m}_j(t) + c_i \bar{m}_{0i} - x_i)$ where $a_i + b_i + c_i = 1$.

The mean of population $i$ converges to

$$\bar{m}_{si} = \frac{(c_i c_j + b_j c_i)\bar{m}_{0i} + b_i c_j \bar{m}_{0j}}{c_i c_j + b_j c_i + b_i c_j}$$

Note: If $c_i, c_j \neq 0$ then $\bar{m}_{si} \neq \bar{m}_{sj}$.

# Model behavior

A population with $a_i = b_i = 0$ is called hard core stubborn. A population with $c_i = 0$ is called most gregarious. Some extreme cases:

- If population $i$ is hard core stubborn, then $\bar{m}_i(t) = \bar{m}_{0i}$. If $b_i = 0$ but $a_i \neq 0$, $\bar{m}_{si} = \bar{m}_{0i}$.
- If both populations are most gregarious, consensus is reached at
$$\bar{m}_{si} = \bar{m}_{sj} = \frac{b_i d_i \bar{m}_j + b_j d_j \bar{m}_i}{b_i d_i + b_j d_j}.$$
- If population $i$ is most gregarious and population $j$ is not, then $\bar{m}_{si} = \bar{m}_{sj} = \bar{m}_{0j}$. However, if population $j$ is not hard core stubborn then $\bar{m}_j(t) \neq \bar{m}_{0j}$.

# Conclusions

Multi-population scenario for mean-field game model of opinion and stubbornness.

State space extension technique gives possibility to study mean-field equilibria under different stubbornness levels.

Stuff not in the paper:
What is the approximation error for a finite number of players?
Suggestion on numerical scheme for the equilibrium computation.