Opinion dynamics, stubbornness and mean-field games

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Introduction: what is modeled?

Opinion propagation:
Dynamics that describe the evolution of the opinions in a large population as a result of repeated interactions on the individual level. The level of stubbornness amongst the individuals vary.

Phenomenon that occur in opinion propagation:
Herd behaviour - convergence towards one (consensus) or multiple (polarization/plurality) opinion values.
A set of populations is considered, each made up of uniform agents characterized by a given level of stubbornness.

Individuals are partially stubborn while interested in reaching a consensus with as many other agents as possible. In Sweden you need 4% of the election votes to get seats in the parliament.
Introduction: main contribution

Affine controls preserve the Gaussian distribution of population under the considered model.
Model set-up: dynamics

State process for a generic member of population $i \in I$ follows:

\[
\begin{aligned}
\left\{
\begin{array}{l}
    dx_i = u_i dt + \xi_i dW_i \\
    x_i(0) = x_{0i}
\end{array}
\right.
\end{aligned}
\]

where $u_i$ is a control function which may depend on time $t$, state $x_i$ and the density $m(x, t)$ and $\xi_i$ is a real number.
Model set-up: cost

Player $i$ wants to maximize the functional

$$J_i(u_i, x_{0i}) = \mathbb{E} \left[ \int_0^\infty e^{-\rho t} c_i(x_i, u_i, m) dt \right]$$

where

$$c_i(x_i, u_i, m) = (1 - \alpha_i) \sum_{j \in I} (\nu_j \ln(m_j(x_i(t)), t))$$

$$- \alpha_i (x_i(t) - \bar{m}_{0i})^2$$

$$- \beta u_i^2.$$
Model set-up: cost

Lets look at the terms in the running cost...

- $\sum_{j \in I} \nu_j \ln (m_j(x_i(.), .))$: player $i$ wants to share its opinion with as many other players as possible.

- $(x_i(.) - \bar{m}_{0i})^2$: player $i$ dislikes departing from the initial mean of its population.

- $\beta u_i^2$: the usual energy penalization.

The parameter $\alpha_i$ is determining the level of stubbornness of the players in population $i$. 
The problem to solve, in $u_i$, for each population is

Maximize $J_i(u_i; x_{0i}) = \mathbb{E} \left[ \int_0^\infty e^{-\rho t} c_i(x_i, u_i, m) dt \right]$ 

Subject to $dx_i = u_i dt + \xi_i dW_i$

$x_i(0) = x_{0i}$

$u_i$ admissible
The mean-field equations

**Theorem 1**
The mean-field system corresponding to \((P_i)\) is described by the equations: for all \(i \in I\),

\[
\begin{align*}
\partial_t v_i(x_i, t) &+ (1 - \alpha_i) \sum_{j \in I} \nu_j \ln(m_j(x_i, t)) - \alpha_i (x_i - \bar{m}_0i)^2 \\
+ &\frac{1}{2\beta} (\partial_x v_i(x_i, t))^2 + \frac{\xi_i^2}{2} \partial_{xx}^2 v_i(x_i, t) - \rho v_i(x_i, t) = 0
\end{align*}
\]

(1)

\[
\begin{align*}
\partial_t m_i(x_i, t) + \partial_x \left[ m_i(x_i, t) \left( -\frac{1}{2\beta} \partial_x v_i(x_i, t) \right) \right] \\
- \frac{\xi_i^2}{2} \partial_{xx}^2 m_i(x_i, t) = 0
\end{align*}
\]

(2)

\[m_i(x_i, 0) = m_{0i}\]

(3)
The mean-field equations

The optimal control to in \((P_i)\) is given by

\[
u_i^*(x_i, t) = -\frac{1}{2\beta} \partial_x v_i(x_i, t)
\]

So far so good. What happens if we restricts ourselves to linear strategies only?
Inverse Fokker-Planck problem

Consider the Fokker-Planck problem in one dimension:

\[ \partial_t m_i(x_i, t) - \frac{\xi_i^2}{2} \partial_{xx} m_i(x_i, t) + \partial_x (u_i(x_i, t)m_i(x_i, t)) . \]

If we assume that \( m_i : S \to \mathbb{R} \), where \( S \subset \mathbb{R}^2 \), that \( m_i \in C^2(S) \) and that \( m_i \) is a probability density function for each \( t \) which is positive for all \((x, t) \in S\). Then \( u_i \) is the solution to

\[ u_i(x_i, t) = \frac{1}{m_i(x_i, t)} \left( C(t) + \frac{\xi_i^2}{2} \partial_x m_i(x_i, t) - \int_{x_0}^{x_i} \partial_t m_i(x, t) dx \right) \]

where \( C(t) \) is an arbitrary function.
Gaussian Distribution Preserving Strategies

Assume that population $i$ has initial density

$$m_i(x_i, 0) = \frac{1}{\sigma_0 i \sqrt{2\pi}} e^{-\frac{(x_i - \mu_0 i)^2}{2\sigma_0^2 i}}$$

and that the agents in this population implement a linear control strategy

$$u_i(x, t) = \hat{p}_i(t)x + \hat{q}(t), \quad \hat{q}, \hat{p} \in C^1(\mathbb{R}_+)$$

Then

$$x_i(t) = e^{\hat{P}_i(t)} \left( x_{0i} + \int_0^t e^{-\hat{P}_i(\tau)} q_i(\tau) d\tau + \xi_i \int_0^t e^{-\hat{P}_i(\tau)} dW_i \right)$$

where $\hat{P}_i(t) = \int_0^t \hat{p}_i(\tau) d\tau$. 
Gaussian Distribution Preserving Strategies

Furthermore, the density of population $i$ is for each time $t$ equal to

$$m_i(x_i, t) = \frac{1}{\sigma_i(t)\sqrt{2\pi}} e^{-\frac{(x_i - \mu_i(t))^2}{2\sigma_i^2(t)}}$$

where

$$\sigma_i^2(t) = e^{2\hat{P}_i(t)} \left(\sigma_{0i}^2 + \xi_i^2 \int_0^t e^{-2\hat{P}_i(\tau)} d\tau\right),$$

$$\mu_i(t) = e^{\hat{P}_i(t)} \left(\mu_{0i} + \int_0^t e^{-\hat{P}_i(\tau)} q_i(\tau) d\tau\right)$$

Note:
The implemented $u_i$ is not necessarily optimal.
Gaussian Distribution Preserving Strategies

\[ m_0 = \frac{1}{\sigma_0 \sqrt{2\pi}} e^{-\frac{(x - \mu_0)^2}{2 \sigma_0^2}} \]

\[ u = \hat{p}(t)x + \hat{q}(t) \]

\[ x \text{ satisfies } \begin{cases} dx = ud(t) + q(dW) \\ x(0) = x_0 \end{cases} \]

\[ \sigma_t^2 = e^{2\hat{p}(t)} \left( \sigma_0^2 + \int_0^t d\tau \right) \]

\[ \mu_t = e^{\hat{p}(t)} \left( \mu_0 + \int_0^t d\tau \right) \]
Gaussian Distribution Preserving Strategies

\[ m \text{ solves } \frac{\partial m}{\partial t} - \frac{1}{2} \int^2 \frac{\partial^2}{\partial x^2} m + \frac{\partial}{\partial x} (um) = 0 \]

\[ u = \frac{1}{m} \left( C(t, x) + \frac{1}{2} \int^2 \frac{\partial^2}{\partial x^2} m + \sum_{i=1}^{\infty} \int_{x_{0i}}^x \frac{\partial m}{\partial x} \, dx \right) \]

\[ m_0 = \frac{1}{\sigma_0 \sqrt{2\pi}} e^{-\frac{(x - \mu_0)^2}{2\sigma_0^2}} \]

\[ x(t) = e^{P(t)} x_0 + \int_0^t \sigma_0 e^{\hat{P}(t)} \left( \sigma_0^2 + \int_0^t \sigma_\tau \, d\tau \right) \, dW \]

\[ u = \hat{p}(t) x + q(t) \]

\[ \text{Itô} \]

\[ m_t = \frac{1}{\sigma_t \sqrt{2\pi}} e^{-\frac{(x - \mu_t)^2}{2\sigma_t^2}} \]

\[ \sigma_t^2 = e^{P(t)} \left( \sigma_0^2 + \int_0^t \sigma_\tau \, d\tau \right) \]

\[ \mu_t = e^{P(t)} \left( \mu_0 + \int_0^t \mu_\tau \, d\tau \right) \]
Gaussian Distribution Preserving Strategies

\[ m \text{ solves } \frac{d}{dt}m - \frac{1}{2} \frac{d^2}{dx^2}m + \frac{x}{x_m} \left( \frac{d^2}{dx^2}m \right) = 0 \]

\[ U = \frac{1}{m} \left( C(t) + \frac{1}{2} \frac{x^2}{x_m^2} \right) + \sum_{i=1}^{x} \frac{x_i}{x_{oi}} \]

\[ m_0 = \frac{1}{\sigma_0 \sqrt{2\pi}} e^{-\frac{(x_0 - \mu_0)^2}{2 \sigma_0^2}} \]

\[ U = \hat{P}(t) x + \hat{q}(t) \]

\[ x(t) = e^{\hat{P}(t)} \left( x_0 + \int_0^t \int_0^t d\tau \right) \]

\[ m_t = \frac{1}{\sigma_t \sqrt{2\pi}} e^{-\frac{(x - \mu_t)^2}{2 \sigma_t^2}} \]

\[ \sigma_t^2 = e^{2\hat{P}(t)} \left( \sigma_0^2 + \int_0^t \int_0^t d\tau \right) \]

\[ \mu_t = e^{\hat{P}(t)} \left( \mu_0 + \int_0^t \int_0^t d\tau \right) \]

\[ dx = u dt + \sigma dw \]

\[ x(0) = x_0. \]
Gaussian Distribution Preserving Strategies

\[ m \text{ solves } \partial_t m - \frac{1}{2} \nabla^2 \partial_x m + \partial_x (m m) = 0 \]

\[ U = \frac{1}{m} \left( C(t) + \frac{1}{2} \sigma^2 \right) \partial_x m + \int_{x_0}^x \partial_x m \, dx \]

\[ m_0 = \frac{1}{\sigma_0 \sqrt{2\pi}} e^{-\frac{(x - \mu_0)^2}{2 \sigma_0^2}} \]

\[ X(t) = e^{\frac{\hat{P}(t)}{2}} \left( x_0 + \int_0^t \partial \tau \right) + \int_0^t \partial \tau \, dW \]

\[ U = \hat{p}(t) x + \hat{q}(t) \]

\[ X \text{ satisfies } \partial_x x = u \partial_t x + g \partial W, \quad x(0) = x_0. \]

\[ m_t = \frac{1}{\sigma_t \sqrt{2\pi}} e^{-\frac{(x - \mu_t)^2}{2 \sigma_t^2}} \]

\[ \sigma_t^2 = \frac{2 \hat{P}(t)}{\partial \tau} \left( \sigma_0^2 + \int_0^t \partial \tau \right) \]

\[ \mu_t = e^{\hat{P}(t)} \left( \mu_0 + \int_0^t \partial \tau \right) \]
Gaussian Distribution Preserving Strategies

\[
\begin{align*}
  \mathbf{m} & \text{ solves } \mathbf{\partial}_t \mathbf{m} - \frac{1}{2} \mathbf{\Sigma}^{\mathbf{m}} \mathbf{\partial}_x \mathbf{m} + \mathbf{\partial}_x (\mathbf{m} \mathbf{m}) = 0 \\
m_0 &= \frac{1}{\sigma_0 \sqrt{2\pi}} e^{-\frac{(x - \mu_0)^2}{2\sigma_0^2}} \\
U &= \frac{1}{m} \left( \mathbf{C}(t) + \frac{1}{2} \mathbf{\Sigma}^{\mathbf{m}} \mathbf{\partial}_x \mathbf{m} + \int_{x_{oi}}^{x_i} \mathbf{\partial}_x \mathbf{m} \, dx \right) \\
x(t) &= e^{\hat{\mathbf{p}}(t)} \left( x_0 + \int_0^t \sigma^2 \, dt + \int_0^t \mathbf{dW} \right) \\
\hat{\mathbf{m}}_t &= \frac{1}{\sigma_t \sqrt{2\pi}} e^{-\frac{(x - \mu_t)^2}{2\sigma_t^2}} \\
\sigma_t &= e^{\hat{\mathbf{p}}(t)} \left( \sigma_0^2 + \int_0^t \sigma^2 \, dt \right) \\
\mu_t &= e^{\hat{\mathbf{p}}(t)} \left( \mu_0 + \int_0^t \mathbf{dW} \right) \\
x(s) &= x_0 \\
dx =\mathbf{u} \, dt + \mathbf{\gamma} \, d\mathbf{W} \\
\]

\[
\mathbf{x}(s) = x_0 
\]
Two crowd seeking populations

A detailed example with two populations.

If the agents of two populations apply linear strategies then the utility function becomes:

\[
J_i(x_{0i}) = \sup_{u_i} \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \left\{ (1 - \alpha_i) \right. \right.
\]
\[
= \sum_{j=1}^{2} - \frac{v_j}{2} \left( \ln(2\pi \sigma_j^2(t)) + \frac{(x_i(t) - \bar{m}_j(t))^2}{\sigma_j^2(t)} \right)
\]
\[
= -\alpha_i (x_i(t) - \bar{m}_{0i})^2 - \beta u_i^2 \} dt \right] 
\]

The populations follow the same dynamics as previously.
Two crowd seeking populations

Two assumptions are made on the crowds:

A1. At time 0 the two populations have a Gaussian distribution

A2. The agents adopt linear strategies that track a weighted sum:

\[ u_i(x, t) = d_i(a_i \bar{m}_i(t) + b_i \bar{m}_j(t) + c_i \bar{m}_0 - x_i) \]  

\[ d_i > 0 \]
\[ 1 = a_i + b_i + c_i \]

Note:
Assumption 1 together with the dynamics of the population implies that the strategies (4) are GDPS.
By introducing the dynamics of $\bar{m}_i, i = 1, 2$, into the model it is possible to characterize an optimal control. The extended state space equations are: for $i = 1, 2, j \neq i$,

$$dx_i = u_i dt + \xi_i dW_i$$

$$x_i(0) = x_{0i}$$

$$\dot{\bar{m}}_i(t) = d_i(b_i \bar{m}_j(t) + c_i \bar{m}_0i - (b_i + c_i) \bar{m}_i(t))$$

$$\bar{m}_i(0) = \mu_{0i}$$
Extending the state space

Time for some rewriting...

Denote by $\gamma_i = -d_i(b_i + c_i)$. Then the state space equations can be written as

$$
\begin{bmatrix}
    dx_i \\
    d\bar{m}_i \\
    d\bar{m}_j
\end{bmatrix} =
\begin{bmatrix}
    0 & 0 & 0 \\
    0 & \gamma_i & d_ib_i \\
    0 & d_jb_j & \gamma_j
\end{bmatrix}
\begin{bmatrix}
    x_i \\
    \bar{m}_i \\
    \bar{m}_j
\end{bmatrix}
dt
+ \begin{bmatrix}
    1 & 0 & 0 \\
    0 & c_i & 0 \\
    0 & 0 & c_j
\end{bmatrix}
\begin{bmatrix}
    u_i \\
    \bar{m}_{0i} \\
    \bar{m}_{0j}
\end{bmatrix}
dt + \begin{bmatrix}
    \xi_i \\
    0 \\
    0
\end{bmatrix}
dW_i
$$
Extending the state space

An even more compact notation is

\[
\dot{\bar{m}}(t) = M \begin{bmatrix} -\gamma_i & d_i b_i \\ d_j b_j & \gamma_j \end{bmatrix} \begin{bmatrix} \bar{m}_i(t) \\ \bar{m}_j(t) \end{bmatrix} + C \begin{bmatrix} c_i & 0 \\ 0 & c_j \end{bmatrix} \begin{bmatrix} \bar{m}_{0i} \\ \bar{m}_{0j} \end{bmatrix}
\]

Adding the constant vector \( \bar{m}_0 \) to the state vector, the problem becomes

\[
\sup_u \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \bar{c}_i(x, u, m, \theta) dt \right]
\]

\[
\begin{bmatrix} dx_i \\ d\bar{m} \\ d\bar{m}_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & M & C \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_i \\ \bar{m} \\ \bar{m}_0 \end{bmatrix} dt + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_i dt + \begin{bmatrix} \xi_i \\ 0 \\ 0 \end{bmatrix} dW_i
\]
Extending the state space

Finally, by letting \( X_i = [x_i \ \tilde{m} \ \tilde{m}_0]^T \) we get the LQ problem

\[
\inf_u \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \left( X_i^T \tilde{Q} X_i + \beta u_i^2 \right) dt \right],
\]

\[dX_i = (FX_i + GU_i) \, dt + LdW_i.\]

The solution to an LQ problem of this kind is well known. Consider the new value function \( V_i(X_i, t) \) that solves

\[
\partial_t V_t(X_i, t) + H(X_i, \partial_{X_i} V_i(X_i, t)) + \frac{1}{2} \partial_i^2 \partial_{xx}^2 V_i(X_i, t) = 0.
\]

If we assume that the value function is quadratic
\( V_i(X_i, t) = X_i^T P(t) X_i \), then \( P(t) \) is the solution to the Riccati equation

\[
\dot{P}(t) - \rho P(t) + P(t) F + F^T P(t) - P(t) \left( GR^{-1} G^T \right) P(t) + \tilde{Q} + W = 0.
\]
Extending the state space

If $P$ solves the Riccati equation then the optimal control is given by

$$\hat{u}_i^*(t) = -\frac{1}{\beta} G^T P(t) X_i$$

$$= \frac{1}{\beta} (P_{11}(t)x_i(t) + P_{12}(t)\bar{m}_i + P_{13}(t)\bar{m}_j$$

$$+ P_{14}(t)\bar{m}_0i + P_{15}(t)\bar{m}_0j)$$

Conclusion:
The extended state space model allows us to characterize an optimal control under assumptions (A1)-(A2).
Model behavior

The mean $\bar{m}_i(t)$ converges to a finite value.

The variance $\sigma_i^2(t)$ converges exponentially to $\frac{\xi^2_i}{2d_i}$. The term

$\ln(2\pi\sigma_j^2(t)) + \frac{(x_i(t) - \mu_j^2(t))}{2\sigma_j^2(t)}$ therefore grows to infinity if there is no disturbance in population $j$, i.e. $\xi_j = 0$. A consequence is that an optimal strategy (in the no disturbance case) must satisfy $\rho - 2d_i > 0$ or guarantee that all states converge to a single consensus.
Recall that under A2, \( u_i = d_i(a_i \bar{m}_i(t) + b_i \bar{m}_j(t) + c_i \bar{m}_0i - x_i) \) where \( a_i + b_i + c_i = 1 \).

The mean of population \( i \) converges to

\[
\bar{m}_{si} = \frac{(c_i c_j + b_j c_i) \bar{m}_{0i} + b_i c_j \bar{m}_{0j}}{c_i c_j + b_j c_i + b_i c_j}
\]

**Note:** If \( c_i, c_j \neq 0 \) then \( \bar{m}_{si} \neq \bar{m}_{sj} \).
Model behavior

A population with $a_i = b_i = 0$ is called hard core stubborn. A population with $c_i = 0$ is called most gregarious. Some extreme cases:

- If population $i$ is hard core stubborn, then $\bar{m}_i(t) = \bar{m}_{0i}$. If $b_i = 0$ but $a_i \neq 0$, $\bar{m}_{si} = \bar{m}_{0i}$.
- If both populations are most gregarious, consensus is reached at
  
  $$\bar{m}_{si} = \bar{m}_{sj} = \frac{b_i d_i \bar{m}_j + b_j d_j \bar{m}_i}{b_i d_i + b_j d_j}.$$  

- If population $i$ is most gregarious and population $j$ is not, then $\bar{m}_{si} = \bar{m}_{sj} = \bar{m}_{0j}$. However, if population $j$ is not hard core stubborn then $\bar{m}_j(t) \neq \bar{m}_{0j}$.  

Conclusions

Multi-population scenario for mean-field game model of opinion and stubbornness.

State space extension technique gives possibility to study mean-field equilibria under different stubbornness levels.

Stuff not in the paper:
What is the approximation error for a finite number of players?
Suggestion on numerical scheme for the equilibrium computation.