# Pontryagin's Maximum Principle

## an introduction

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Deterministic optimal control problem:

Minimize or maximize

$$J(u) = \int_0^T f(x(t), u(t))dt + h(x(T)), \tag{1}$$

with respect to $u : [0, T] \longrightarrow U$, subject to

$$\begin{cases} \dot{x}(t) = b(x(t), u(t)), & 0 < t \leq T, \\ x(0) = x_0. \end{cases} \tag{2}$$

where, $U$ is a given set of controls.

Stochastic optimal control problem:

Minimize or maximize

$$J(u) = E\left[\int_0^T f(x(t), u(t))dt + h(x(T))\right], \tag{3}$$

with respect to $u : [0, T] \longrightarrow U$, subject to

$$\begin{cases} dx(t) = b(x(t), u(t))dt + \sigma(x(t), u(t))dW(t), \\ x(0) = x_0. \end{cases} \tag{4}$$

Stochastic optimal control problem for systems of mean-field type

Minimize or maximize

$$J(u) = E\left[\int_0^T f(x(t), E[x(t)], u(t))dt + h(x(T), E[x(T)])\right], \quad (5)$$

with respect to $u : [0, T] \longrightarrow U$, subject to

$$\begin{cases} dx(t) = b(x(t), E[x(t)], u(t))dt + \sigma(x(t), E[x(t)], u(t))dW(t), \\ x(0) = x_0. \end{cases} \quad (6)$$

Example:

$$J(u) = Var(x(T)) = E[x^2(T)] - (E[x(T)])^2 .$$

Optimal control theory tries to answer two questions:

- Existence of a minimum/maximum of the performance functional $J$.
- Explicit computation/characterization of such a minimum/maximum.
  - The Bellman principle which yields the Hamilton-Jacobi-Bellman equation (HJB) for the value function;
  - Pontryagin's maximum principle which yields the Hamiltonian system for "the derivative" of the value function.

Features of the Bellman principle and the HJB equation

- The Bellman principle is based on the "law of iterated conditional expectations".
  - It does not apply for dynamics of mean-filed type:

$$J(u) = E\left[\int_0^T f(x(t), E[x(t)], u(t))dt + h(x(T), E[x(T)])\right].$$

- The HJB equation is a nonlinear PDE, valid only for "Markovian systems", where the coefficients $b, \sigma, h$ and $f$ are deterministic functions of $(t, x)$.
- An eventual optimal control should be of "feedback form": $\bar{u}(t) = v(t, X_t)$, where $v$ is a deterministic function of $(t, x)$.

Features of the Pontryagin's maximum principle

- Pontryagin's principle is based on a "perturbation technique" for the control process, that does not put "structural" restrictions on the dynamics of the controlled system.
- It seems well suited for
  - Non-Markovian systems. i.e. where the coefficients $b, \sigma, h$ and $f$ are random and not necessarily deterministic functions of $(t, x)$.
  - Systems of mean-field type.

# A heuristic derivation of the Pontryagin's principle

Minimize

$$J(u) = \int_0^T f(x(t), u(t))dt + h(x(T)), \tag{7}$$

with respect to $u : [0, T] \longrightarrow U$, subject to

$$\begin{cases} \dot{x}(t) = b(x(t), u(t)), & 0 < t \leq T, \\ x(0) = x_0. \end{cases} \tag{8}$$

Using the Lagrangian multiplier method, we minimize the Lagrangian functional

$$\begin{aligned} \mathcal{L}(x, u, p) := & \int_0^T (f(x(t), u(t)) + p(t) \cdot (\dot{x}(t) - b(x(t), u(t)) \, dt \\ & + h(x(T)), \end{aligned} \tag{9}$$

Introducing the so-called Hamiltonian

$$H(x, u, p) := p \cdot b(x, u) - f(x, u), \tag{10}$$

the Lagrangian functional becomes

$$\mathcal{L}(x, u, p) := \int_0^T (-H(x(t), u(t), p(t)) + p(t) \cdot \dot{x}(t))\, dt + h(x(T)).$$

If $(\bar{x}, \bar{u}, p)$ is a minimizer of $\mathcal{L}$, performing a first-order Taylor expansion with perturbation $(\delta x, \delta u, \delta p)$, such that $\delta x(0) = 0$:

$$\begin{aligned} \delta \mathcal{L} \quad &:= \mathcal{L}(\bar{x} + \delta x, \bar{u} + \delta u, p + \delta p) - \mathcal{L}(\bar{x}, \bar{u}, p) \\ &\approx \int_0^T \left(-H_x \cdot \delta x - H_u \cdot \delta u - H_p \cdot \delta p + p \cdot \tfrac{d}{dt}(\delta x) + \delta p \cdot \dot{\bar{x}}\right) dt \\ &+ h_x \cdot \delta x(T), \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned} \delta \mathcal{L} \approx \int_0^T &\left((-H_x - \dot{p}) \cdot \delta x - H_u \cdot \delta u + \left(-H_p + \dot{\bar{x}}\right) \cdot \delta p + \tfrac{d}{dt}(p \cdot \delta x)\right) dt \\ &+ h_x \cdot \delta x(T). \end{aligned}$$

Requiring $\delta\mathcal{L} = 0$ and noting that $\delta x(0) = 0$, yields

$$0 = \int_0^T \left((-H_x - \dot{p}) \cdot \delta x - H_u \cdot \delta u + \left(-H_p + \dot{\bar{x}}\right) \cdot \delta p\right) dt \\ +(h_x + p(T)) \cdot \delta x(T).$$

This in turn suggests that the extremal $(\bar{x}, \bar{u}, p)$ satisfies the so-called *Hamiltonian system* associated with the optimal control problem (7)-(8):

$$\begin{cases} \bar{x}(0) = x_0, & \text{(Initial value)} \\ \dot{\bar{x}}(t) = H_p(\bar{x}(t), p(t), \bar{u}(t)), & \text{(Controlled dynamics)} \\ \dot{p}(t) = -H_x(\bar{x}(t), p(t), \bar{u}(t)), & \text{(Adjoint equation)} \\ p(T) = -h_x(\bar{x}(T)), & \text{(Adjoint terminal value)} \\ H_u(\bar{x}(t), \bar{u}(t), p(t)) = 0, & (\bar{u}\text{ extremal for } H) \end{cases}$$

**The adjoint equation**

$$\begin{cases} \dot{p}(t) = -H_x(\bar{x}(t), p(t), \bar{u}(t)), & 0 \le t < T, \\ p(T) = -h_x(\bar{x}(T)). \end{cases} \tag{11}$$

**Pontryagin's Maximum Principle.** If $(\bar{x}, \bar{u})$ is an optimal solution of the control problem (7)-(8), then there exists a function $p$ solution of the adjoint equation (11) for which

$$\bar{u}(t) = \arg\max_{u \in U} H(\bar{x}(t), u, p(t)), \quad 0 \le t \le T. \quad \text{(Maximum Principle)}$$

This result says that $\bar{u}$ is not only an extremal for the Hamiltonian $H$. It is in fact a maximum.

- This perturbation method, implicitly assumes that the set $U$ of control is *linear* i.e. if $u \in U$ then also $u + \delta u \in U$.
- But, in many practical examples, the set/space $U$ is rather convex or a general metric space.
- Device a new perturbation method which should be compatible with the structure of the set of controls.

If the set of controls $U$ is convex, then a natural perturbation method would be a convex perturbation of a given optimal control $\bar{u}$:

$$\delta u := \epsilon(u - \bar{u}), \quad u \in U,$$

where, $\epsilon$ will tend to 0. Hence, if $\bar{u}$ is optimal for $J$, we have

$$J(\bar{u} + \epsilon(u - \bar{u})) - J(\bar{u}) \geq 0,$$

and thus, we can write

$$(J'(\bar{u}), u - \bar{u}) \geq 0, \qquad \forall u \in U,$$

provided that $J$ is Gateaux differentiable.

# A local maximum principle

Mimicking the previous formal computations we would expect the following (local) version of the maximum principle.

$$\frac{dH}{du}(\bar{x}(t), \bar{u}(t), p(t)) \cdot (u - \bar{u}(t)) \leq 0, \qquad 0 \leq t \leq T, \quad \forall u \in U.$$

If the set $U$ is not necessarily convex, but a *general separable metric space*,

Pontryagin's approach suggests the following perturbation method called *spike variation* would be appropriate.

For $\epsilon > 0$, pick a subset $E_\epsilon \subset [0, T]$ such that $|E_\epsilon| = \epsilon$. The control process $u^\epsilon$ is a spike variation of $u$ if

$$u^\epsilon(t) := \begin{cases} u(t), & t \in E_\epsilon, \\ \bar{u}(t), & t \in E_\epsilon^c, \end{cases} \tag{12}$$

where, $u \in U$ is an arbitrary control.

Using the spike variation technique, Pontryagin's main contribution is the following key relation between the performance functional $J$ and the associated Hamiltonian $H$:

$$J(u^\epsilon) - J(\bar{u}) = -\int_0^T \delta H(s) I_{E_\epsilon}(s)\, ds + \circ(\epsilon), \tag{13}$$

for arbitrary $u \in U$, where,

$$\delta H(s) = H(\bar{x}(s), u(s), p(s)) - H(\bar{x}(s), \bar{u}(s), p(s)).$$

We can choose it of the form $E_\epsilon := [\bar{t}, \bar{t} + \epsilon]$, for arbitrarily chosen $\bar{t} \in [0, T]$. This yields

$$0 \leq J(u^\epsilon) - J(\bar{u}) = -\int_{\bar{t}}^{\bar{t}+\epsilon} \delta H(t)\, dt + \circ(\epsilon).$$

Dividing by $\epsilon$ and then sending $\epsilon$ to zero together with the separability of $U$ we obtain

$$\bar{u}(t) = \arg\max_{u \in U} H(\bar{x}(t), u, p(t)), \quad 0 \leq t \leq T. \quad \text{(Maximum Principle)}$$

# Relation to the Hamilton-Jacobi-Bellman's equation

The Hamilton-Jacobi-Bellman equation is a nonlinear backward PDE

$$\begin{cases} \frac{\partial V}{\partial t}(t,x) + \max_u H(x, V_x(t,x), u) = 0, \\ V(x,T) = h(x). \end{cases} \tag{14}$$

satisfied by the value function $V(x,t)$ defined by

$$V(t,x) = \min_{u:x(t)=x} \int_t^T f(x(s), u(s))dt + h(x(T)) \tag{15}$$

The associated optimal control is of *feedback* type:

$$u^*(t,x) = \arg\max_u H(x, V_x(t,x), u),$$

which depends on $V_x(t,x)$ but not on $V$.

An informal comparison between the adjoint equation and the HJB equation suggests that

$$p(t) = V_x(t, x),$$

provided that the HJB equation admits a smooth solution $V$. Unfortunately, smooth solutions of HJB equations are obtained in very few cases. Most of real world problems exhibit non-smooth solutions of the HJB equation.

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